Covariant phase-space representation and overlapping distribution functions

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The Lorentz deformation property of the phase-space distribution function is studied for harmonic oscillators. The overlap effect of two distribution functions is discussed in detail. It is shown that the Lorentz deformation of the phase-space distribution is responsible for the polynomial cutoff behavior of the proton form factor in the harmonic-oscillator quark model.

I. INTRODUCTION

The phase-space representation of quantum mechanics[1] is of current interest.[2-4] We have shown in our previous papers[5,6] that the light-cone coordinate system is the natural language for the Lorentz-covariant phase space representation of quantum mechanics. The Lorentz-deformed phase-space distribution was discussed in detail for localized light waves[5] and harmonic oscillators.[6] It was shown in Ref. 6 that the covariant harmonic oscillator can illustrate the Lorentz-deformed phase-space distribution for a relativistic free particle with a space-time extension.

The purpose of the present paper is to study the overlapping phase-space distributions functions. We are of course interested in Lorentz-deformed distributions, and in how their effect manifests itself in the real world. For this purpose, we shall study the electromagnetic structure of nucleons. Because the radius of the proton is 10^5 times smaller than that of the hydrogen atom, the proton had long been thought to be a point particle. However, Hofstadter's discovery in 1955 clearly demonstrated that the proton has a spread-out charge distribution.[7]

Although there had been many attempts to understand the structure of nucleon since 1955, the first comprehensive approach to the probability distribution of hadronic matter was the quark model, in which the nucleon is a bound state of three quarks.[8] Among the mathematical models for this bound state, the harmonic-oscillator model gives a simple explanation for a wide range of hadronic phenomena observed in high-energy physics laboratories.[9-11] While the major strength of the oscillator model is its mathematical simplicity, its most useful property for our present purpose is that the oscillator model constitutes a representation of the Poincaré group.[12-16] The harmonic-oscillator model can be made covariant.

Another important property of the oscillator model is that it is the natural language for phase space.[2,6,17] Therefore the harmonic-oscillator model is an effective scientific language for the covariant description of phase space. We shall study in this paper overlapping distribution functions in the covariant phase-space formulation of harmonic oscillators.

In Sec. II we summarize the earlier works on overlapping phase-space distribution functions and their physical interpretations. Section III explains how the form factor is defined and why a Lorentz-covariant formalism is needed for studying the form factor. Section IV deals with the covariant phase-space distribution functions which are needed for calculating the form factors. In Sec. V we study how the overlap of two Lorentz-deformed phase-space distribution functions lead to the correct form-factor behavior in the harmonic-oscillator model for hadrons.

II. OVERLAPPING PHASE-SPACE DISTRIBUTION FUNCTIONS

If $\psi(x)$ is a solution of the time-independent Schrödinger equation, its phase-space distribution function is

$$W_{\psi}(x,p)=\frac{1}{\pi}\int \psi^{*}(x+y)\psi(x-y)e^{2ipy}dy \ . \tag{1}$$

For simplicity, we shall use the term "PSD function" for phase-space distribution function. This is a function of x and p which are c numbers. This function is real but is not necessarily positive everywhere in the two-dimensional phase space of x and p. We can, however, recover the positive distribution functions in the position and momentum coordinates by integrating the PSD function over p and x, respectively,[1,2]

$$\rho(x)=\int W_{\psi}(x,p)dp, \quad \sigma(p)=\int W_{\psi}(x,p)dx \ . \tag{2}$$

In this paper, we are interested in two overlapping PSD functions. Indeed, the overlap integral becomes the absolute square of the inner product of the two wave functions in the Schrödinger picture. If $W_{\psi}(x,p)$ and $W_{\phi}(x,p)$ are the PSD functions for $\psi(x)$ and $\phi(x)$, respectively, then[1,2]

$$\int W_{\psi}(x,p)W_{\phi}(x,p)dx\,dp=(1/2\pi)|(\phi(x),\psi(x))|^{2} \ . \tag{3}$$

This expression is non-negative, but can be zero if the two functions are orthogonal, indicating that the PSD functions are not always positive everywhere in phase space.

In studying the interaction of a photon with an atomic system, we often encounter the matrix element of the form

$$M_{fi}=(\phi,e^{ikx}\psi) . \tag{4}$$

This is the inner product of the wave functions ϕ and ψ' with $\psi'=e^{ikx}\psi$. Thus we have to construct the PSD function for ψ',

$$W_{\psi'}(x,p)=\frac{1}{\pi}\int \psi^*(x+y)\psi(x-y)e^{2i(p-k)y}dy . \tag{5}$$

This leads to

$$W_{\psi'}(x,p)=W_{\psi}(x,(p-k)) . \tag{6}$$

Therefore

$$|(\phi,e^{ikx}\psi)|^2=2\pi\int W_{\phi}(x,p)W_{\psi}(x,(p-k))dx\,dp . \tag{7}$$

We have thus far carried out the formalism for one-dimensional space. The generalization to the three-dimensional space is straightforward and has been discussed in the literature.[1,2]

III. FORM FACTORS

If electrons are scattered by a charged point particle, the scattering amplitude in the Born approximation is $f(\theta)=2me^2/(\mathbf{k}_f-\mathbf{k}_i)^2$. On the other hand, if the electron is scattered by a spread-out charge due to quantum probability distribution, the scattering amplitude is

$$f(\theta)=(2me^2/K^2)F(K^2) . \tag{8}$$

where $\mathbf{K}=\mathbf{k}_f-\mathbf{k}_i$, and $K^2=(\mathbf{k}_f-\mathbf{k}_i)^2$. $F(K^2)$ is called the form factor and takes the form

$$F(K^2)=(\psi_f,e^{-i\mathbf{K}\cdot\mathbf{x}}\psi_i)=\int[\psi_f(\mathbf{x})]^{\dagger}\psi_i(\mathbf{x})e^{-i\mathbf{K}\cdot\mathbf{x}}d^3x , \tag{9}$$

with $F(0)=1$, if the initial- and final-state wave functions are the same. If $\{[\psi_f(\mathbf{x})]^{\dagger}\psi_i(\mathbf{x})\}$ describes a point-charge distribution with $\delta(\mathbf{x})$, $F(K^2)=1$ for all values of K^2. According to Eq. (3), the form factor should take the form

$$|F(K^2)|^2=\int W_f(\mathbf{x},\mathbf{p})W_i(\mathbf{x},(\mathbf{p}-\mathbf{K}))d^3x\,d^3p . \tag{10}$$

This is a generalization of Eq. (7) to the three-dimensional space.

As the energy of incoming electrons becomes higher for the fixed nucleon target, K^2 becomes very large, and the problem becomes relativistic. For the electromagnetic interaction of point particles, we have to use quantum electrodynamics, where the scattering amplitude is expanded in a power series of the fine-structure constant $\alpha=e^2/4\pi$ in the Lorentz-Heaviside unit. The lowest nontrivial term in this expansion is essentially a relativistic version of the Born approximation.

In lowest order in α, we can describe the scattering of an electron by a proton using the diagram given in Fig. 1. The corresponding matrix element is given in many textbooks on elementary particle physics.[18] It is proportional to

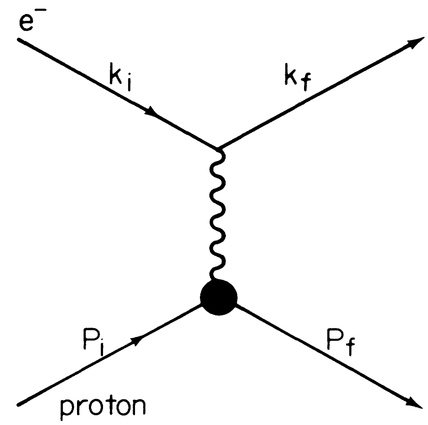

FIG. 1. Elastic electron-proton scattering. The electron behaves like a point charge. However, the proton has its hadronic structure, and has a spread-out charge distribution.

$$e^2[\bar{U}(P_f)\Gamma_{\mu}(P_f,P_i)U(P_i)](1/K^2)[\bar{U}(k_f)\gamma^{\mu}U(k_i)] , \tag{11}$$

where P_i, P_f, k_i, and k_f are the initial and final four-momenta of the proton and electron, respectively. $U(P_i)$ is the Dirac spinor for the initial proton. We use here the four-vector convention $x^{\mu}=(x,y,z,t)$. K^2 is the four-momentum transfer squared given by

$$K^2=\mathbf{K}^2-K_0^2=(P_f-P_i)^2=(k_i-k_f)^2 . \tag{12}$$

The $1/K^2$ factor in Eq. (11) comes from the virtual photon being exchanged between the electron and the proton. In the metric we use, the quantity is positive for physical values of the four-momenta for the particles involved in the scattering process.

The function $\Gamma_{\mu}(P_f,P_i)$ in Eq. (11) represents the closed circle in Fig. 1 and carries the effect of the nucleon structure. If the proton were a point charge, we would have $\Gamma_{\mu}=\gamma_{\mu}$. If the proton has an extended charge structure, we will be inclined to write it as $\Gamma_{\mu}=\gamma_{\mu}F(K^2)$. However, the proton and neutron have anomalous magnetic moments whose values are 2.79 and -1.91 in units of $e/2M$ for the proton and neutron, respectively, where M is the nucleon mass. If we include these observed anomalous magnetic moments, Γ_{μ} should be written as

$$\Gamma_{\mu}=\gamma_{\mu}F_1(K^2)+i(\sigma_{\mu\nu}K^{\nu}/2M)F_2(K^2) . \tag{13}$$

The form factors are scalar functions in the Lorentz-invariant variable K^2. When we compare $F_1(K^2)$ and $F(K^2)$ with experimental data, it is more convenient to use the following linear combinations:

$$\begin{aligned} G_M(K^2)&=F_1(K^2)+F_2(K^2) , \\ G_E(K^2)&=F_1(K^2)+(K^2/4M^2)F_2(K^2) . \end{aligned} \tag{14}$$

These form factors should be written for the proton and the neutron separately. We may use the superscripts p and n to distinguish them. When $K^2=0$,

$$G_M^p(0)=\mu_p=2.79,\quad G_E^p(0)=1\ \ (\text{proton})\ ;$$
$$G_M^n(0)=\mu_n=-1.91,\quad G_E^n(0)=0\ \ (\text{neutron})\ . \tag{15}$$

These numbers are the magnetic moments and electric charges of the proton and neutron, respectively.

Among the many attempts to understand the form factors, the quark model appears to be the most promising approach.[8,10,19] In this model, the nucleon consists of three nonstrange quarks. There are two nonstrange quarks called u (up) and d (down) which have electric charges $\frac{2}{3}$ and $-\frac{1}{3}$ respectively. The proton consists of two u quarks and one d quark, and the neutron is made up of one u quark and two d quarks. The neutron charge is therefore zero, while the proton charge is 1.

Indeed, one of the early successes of the quark model was the calculation of the magnetic moment ratio $\mu_n/\mu_p=-\frac{2}{3}$.[20] However, it is even more challenging to calculate the form factors for increasing values of K^2.[10,19] At present, we can make the following experimental observation. For the four form factors in the nucleonic system given in Eq. (14), the neutron charge form factor is zero at $K^2=0$, and remains small (not zero) for all values of K^2.[21] The three remaining form factors decrease like $1/(K^2)^2$ as K^2 increases beyond the value of the nucleon mass squared.[22] This behavior is usually called the dipole fit. We are interested in the question of whether each of the form factors can be written in terms of the single form factor $G(K^2)$, multiplied by a constant, where $G(K^2)$ is normalized as $G(0)=1$, and is proportional to $1/(K^2)^2$ for large values K^2, as is illustrated in Fig. 2. In the case of the neutron charge form factor, we have to multiply $G(K^2)$ by zero within the framework of the model in which the spin, unitary spin, and spatial wave functions are factorized.[21] The question is whether it is possible to calculate the above-mentioned dipole behavior of $G(K^2)$ using the wave functions obtained from the quark model.

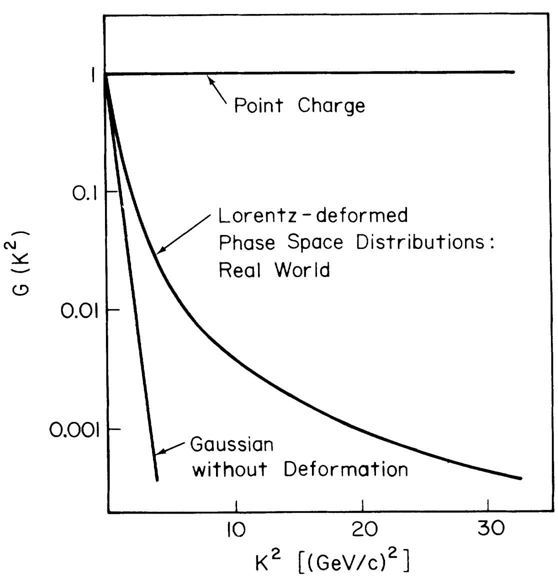

FIG. 2. Form-factor behaviors for increasing values of K^2. If the proton is a point charge, the form factor should be independent of K^2, as is illustrated by the horizontal line. If the charge distribution is Gaussian, the nonrelativistic calculation leads to an exponential cutoff in K^2. The relativistic calculation gives a reasonably accurate description of the real world. At present, the experimental data are available for $G_M^p(K^2)$, $G_E^p(K^2)$, and $G_M^n(K^2)$ from $K^2=0$ to 25, 15, and 7 $(\text{GeV}/c)^2$, respectively. They are all consistent with the relativistic calculation with Lorentz deformation.

Another important aspect of the quark model is that the forces between the quarks are like harmonic oscillators, and the hadronic mass spectrum is consistent with the equal mass-squared spacing predicted by the oscillator model.[9,16] In addition, the parton distribution shows a Gaussian shape in the region where the structure function can be measured accurately.[11]

It is therefore a reasonable approach to calculate the nucleon form factor assuming that the nucleon is in the ground state of a harmonic-oscillator system. However, the Gaussian distribution gives an exponential cutoff of the type $\exp(-K^2/4\Omega)$, where Ω is the spring constant of the oscillator system. This contradicts the experimental observation, as is indicated in Fig. 2. It is therefore interesting to see whether the effect of Lorentz deformation in phase space could transform this exponential decrease into a polynomial cutoff.

IV. COVARIANT PHASE-SPACE DISTRIBUTION FUNCTIONS

The covariant phase-space representation for harmonic oscillators has been discussed in Ref. 6. As in Ref. 6, we start with two quarks bound together by a harmonic-oscillator potential. Then the convenient coordinate variables are[15]

$$X=(x_a+x_b)/2,\quad x=(x_a-x_b)/2\sqrt{2}\ . \tag{16}$$

With these variables, the hadronic wave function takes the form

$$\varphi(X,x)=e^{\pm iPX}\psi(x)\ , \tag{17}$$

where P is the hadronic four-momentum. The wave function $\psi(x)$ describes the internal motion of the two-quark system. The preceding form as a representation space of the Poincaré group[12] for relativistic extended hadrons has been discussed in the literature.[16]

As for the four-momenta of the quarks p_a and p_b, we can combine them into the total four-momentum and momentum-energy separation between the quarks,[15]

$$P=p_a+p_b,\quad q=\sqrt{2}(p_a-p_b)\ , \tag{18}$$

where P is the hadronic four-momentum conjugate to X. The internal momentum-energy separation q is conjugate to x provided that there exist wave functions which can be Fourier transformed.

If the hadron moves along the z direction with velocity parameter β, the hadronic rest frame is important. In this frame, the coordinate variables are

$$x'=x,\quad y'=y\ ,$$
$$z'=(z-\beta t)/(1-\beta^2)^{1/2}\ , \tag{19}$$
$$t'=(t-\beta z)/(1-\beta^2)^{1/2}\ .$$

Since the x and y variables are not affected by boosts along the z direction, and since the harmonic-oscillator system is separable in the Cartesian coordinate system, we can drop these variables from the wave function. It is important to note that t and t' in Eq. (19) are the time-separation variables between the quarks. It was shown that the ground-state harmonic-oscillator function for the moving hadron takes the form

$$\psi_\beta^0(z,t)=\left[\frac{1}{\pi}\right]^{1/2}\exp[-\Omega(z'^2+t'^2)/2] . \tag{20}$$

Ω is the spring constant for the oscillator system. For simplicity, we can use the unit where $\Omega=1$, and restore this factor when we are ready to compare our calculation with experimental data.

This wave function can be written in the light-cone coordinate system, where the coordinate variables are

$$u=(t+z)/\sqrt{2}, \quad v=(t-z)/\sqrt{2} , \tag{21}$$

and

$$q_u=(q_z-q_0)/\sqrt{2}, \quad q_v=(q_z+q_0)/\sqrt{2} . \tag{22}$$

In this coordinate system, the Lorentz boost of Eq. (19) takes a form

$$\begin{aligned} u'&=\left[\frac{1-\beta}{1+\beta}\right]^{1/2}u, \quad v'=\left[\frac{1+\beta}{1-\beta}\right]^{1/2}v , \\ q_u'&=\left[\frac{1+\beta}{1-\beta}\right]^{1/2}q_u, \quad q_v'=\left[\frac{1-\beta}{1+\beta}\right]^{1/2}q_v . \end{aligned} \tag{23}$$

The wave function of Eq. (20) then becomes

$$\begin{aligned} \psi_\beta(z,t)&=\left[\frac{1}{\pi}\right]^{1/2}\exp[-(u'^2+v'^2)/2] \\ &=\left[\frac{1}{\pi}\right]^{1/2}\exp\left[-\frac{1}{2}\left[\frac{1-\beta}{1+\beta}u^2+\frac{1+\beta}{1-\beta}v^2\right]\right] . \end{aligned} \tag{24}$$

The Lorentz-deformation property of this wave function has been discussed in the literature.[16]

For the ground state, the PSD function can now be defined as[6]

$$\begin{aligned} W_\beta(u,q_u;v,q_v)=\left[\frac{1}{\pi}\right]^2\int&[\psi_\beta(u+x,v+y)]^* \\ &\times\psi_\beta(u-x,v-y) \\ &\times\exp[2i(q_ux+q_vy)] \\ &\times dx\,dy . \end{aligned} \tag{25}$$

After the evaluation of this integral, the PSD function becomes

$$W_\beta(u,q_u;v,q_v)=J_\beta(u,q_u)J_{-\beta}(v,q_v) , \tag{26}$$

with

$$J_\beta(u,q_u)=\frac{1}{\pi}\exp\left[-\left[\frac{1-\beta}{1+\beta}u^2+\frac{1+\beta}{1-\beta}q_u^2\right]\right] .$$

The PSD function $W_\beta(u,q_u;v,q_v)$ can be separated into two phase spaces consisting of (u,q_u) and (v,q_v), respectively. When the hadron is at rest with $\beta=0$, this PSD function is localized in the regions

$$(u^2+q_u^2)<1, \quad (v^2+q_v^2)<1 . \tag{27}$$

When the hadron moves, these regions undergo elliptic deformations.[6]

It is straightforward to generalized the preceding calculation to the three-quark system in the harmonic-oscillator regime.[15,16] If we let x_a, x_b, and x_c be the space-time coordinates of the quarks, it is more convenient to use the variables

$$\begin{aligned} &X=\tfrac{1}{3}(x_a+x_b+x_c) , \\ &r=\tfrac{1}{6}(x_a+x_b-2x_c), \quad s=\tfrac{1}{2}(x_b-x_a) , \end{aligned} \tag{28}$$

and their conjugate variables,

$$\begin{aligned} &P=p_a+p_b+p_c , \\ &q=p_a+p_b-2p_c, \quad k=\sqrt{3}(p_b-p_a) . \end{aligned} \tag{29}$$

In terms of these variables, the covariant harmonic-oscillator wave function for the three-particle bound system takes the form

$$\psi_\beta(r,s)=(1/\pi)\exp[-(\Omega/2)(r_z'^2+r_0'^2+s_z'^2+s_0'^2)] , \tag{30}$$

where, as in Eq. (20), the transverse components have been ignored. The primed coordinate variables are those in the hadronic rest frame. This function can also be written in terms of the light-cone coordinate variables, just as in the case of Eq. (24). It is then straightforward to construct the covariant phase-space distribution function.

V. CALCULATION OF THE FORM FACTOR

Let us now go back to the expression for the form factor in Eq. (10). Since the harmonic-oscillator model gives a reasonable description for the mass spectrum of non-strange baryons,[9] and since the shape of the proton structure function shows a Gaussian behavior where the experimental data are accurate,[11] we are compelled to calculate the nucleon form factors with the ground-state harmonic-oscillator wave function.

For a two-body bound state, the nonrelativistic calculation without the Lorentz-deformation effect gives the form factor of the form

$$g(K^2)=\exp(-K^2/4\Omega) . \tag{31}$$

We use $g(K^2)$, instead of $G(K^2)$, for the two-body bound state. This expression does not lead to a polynomial cutoff for large values of K^2, and therefore is not consistent with the real world as is described in Fig. 2. The story is the same for $G(K^2)$ for the three-body bound state.

We are interested in the question of whether the

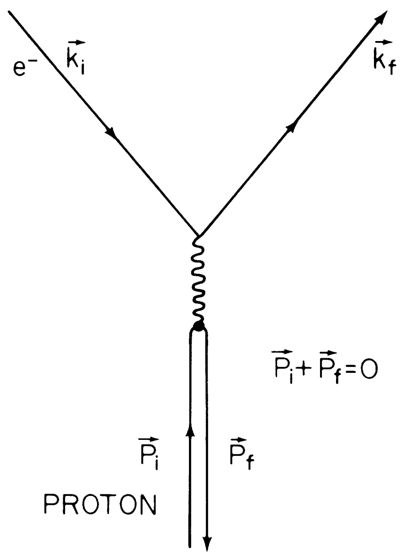

FIG. 3. Breit frame for electron-nucleon scattering. The momentum of the outgoing nucleon is equal in magnitude but opposite in direction to that of the incoming nucleon.

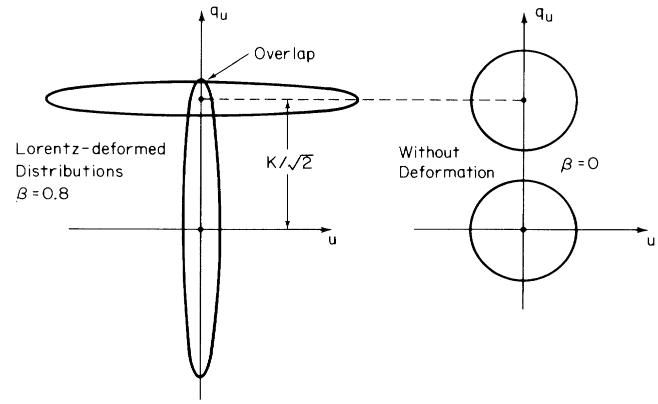

FIG. 4. Lorentz-deformed phase-space distribution functions and their overlaps. According to Eq. (27), the J_β function of Eq. (26) is localized within a circular or elliptic region. As the momentum transfer increases, the PSD functions become separated. Without Lorentz deformation, the PSD functions become completely separated in the overlap integral of Eq. (35). This lack of overlap is the cause of an unacceptable exponential cutoff in K^2. However, the Lorentz-deformed PSD functions maintain a small overlapping region as K^2 increases. This leads to a polynomial decrease of the form factor.

Lorentz effect on the Gaussian distribution will lead to a dipole fit. For this purpose, let us go to the Lorentz frame in which the momenta of the incoming and outgoing nucleons have an equal magnitude but opposite signs, as is described in Fig. 3. Then

$$\mathbf{P}_f+\mathbf{P}_i=0 \ . \tag{32}$$

The Lorentz frame in which this condition holds is usually called the Breit frame.[23] We assume that the proton comes in along the z direction and goes out along the negative z direction after the scattering process. In this frame, the four-vector $\mathbf{K}=(\mathbf{k}_f-\mathbf{k}_i)=(\mathbf{P}_i-\mathbf{P}_f)$ has no timelike component. Thus the exponential factor $\exp(-i\mathbf{K}\cdot\mathbf{x})$ can be replaced by the Lorentz-invariant form $\exp(-i\mathbf{K}\cdot\mathbf{x})$.

We can use the covariant harmonic-oscillator wave function discussed in Sec. IV for the proton. If we assume for simplicity that the proton is a bound state of two quarks, and the form factor should take the form

$$g(K^2)=\int \psi_f(x)\psi_i(x)e^{-i\mathbf{K}\cdot\mathbf{x}}d^4x \ . \tag{33}$$

Then the only difference between this form and the nonrelativistic cases is that the integral of Eq. (33) requires an integration over the timelike variable. This time-separation variable has been thoroughly discussed in the literature.[6,14,16] If β is the velocity parameter for the incoming proton, ψ_i and ψ_f in Eq. (33) should be replaced by ψ_β and $\psi_{-\beta}$ of Eq. (24), respectively. The form-factor integral in the Breit frame takes the form

$$g(K^2)=\int \psi^*_{-\beta}(z,t)\psi_\beta(z,t)e^{-iKz}dz\,dt \ . \tag{34}$$

In terms of the light-cone variables,

$$g(K^2)=\int \psi^*_{-\beta}(u,v)\psi_\beta(u,v)e^{-iK(u+v)/\sqrt{2}}du\,dv \ , \tag{35}$$

where K is the magnitude of the vector $\mathbf{K}$. The form factor can then be computed from the overlap integral of two Lorentz-deformed PSD functions,

$$\begin{aligned}|g(K^2)|^2 &= (2\pi)^2\int W_{-\beta}(u,q_u;v,q_v)\\ &\quad\times W_\beta(u,(q_u-K\sqrt{2};v,(q_v-K/\sqrt{2}))\\ &\quad\times du\,dq_u dv\,dq_v \ .\end{aligned} \tag{36}$$

Then, in terms of the J_β function defined in Eq. (26), $g(K^2)$ takes the simpler form

$$g(K^2)=2\pi\int J_{-\beta}(u,q_u)J_\beta(u,(q_u-K/\sqrt{2}))du\,dq_u \ . \tag{37}$$

This overlap integral of the PSD function is illustrated in Fig. 4. The evaluation of this integral is straightforward, and the result is

$$\begin{aligned}g(K^2) &= [2M^2/(2M^2+K^2)]\\ &\quad\times\exp\{-M^2K^2/[2\Omega(2M^2+K^2)]\} \ .\end{aligned} \tag{38}$$

This expression becomes the nonrelativistic form of Eq. (31) for small values of K^2 and becomes 1 for $K^2=0$. It decreases like $1/K^2$ as K^2 becomes large, but does not decrease like $1/(K^2)^2$. How are we going to get an extra $1/K^2$ factor?

Since there are three quarks inside the nucleon, there are two oscillator modes. We can therefore expect that each mode will contribute a $1/(K^2)$ factor to give the net decrease of $1/(K^2)^2$ as K^2 becomes very large. The generalization of the form factor calculation to this three-quark system is straightforward,[10,15,19] for the harmonic-oscillator wave functions. Since the oscillator wave functions are separable, the construction of the PSD function is also straightforward. The result of the calculation is

$$G(K^2)=[2M^2/(2M^2+K^2)]^2 \times\exp\{-M^2K^2/[\Omega(2M^2+K^2)]\} , \quad (39)$$

which is 1 at $K^2=0$, and decreases as $1/(K^2)^2$. The behavior of this function is illustrated in Fig. 2.

The experimental curves are nicely summarized in Ref. 22. For protons, the data for the magnetic and electric form factors are available from $K^2=0$ to 25 and 15 $(\text{GeV}/c)^2$, respectively. For neutrons, the data for the magnetic form factor is available from $K^2=0$ to 7 $(\text{GeV}/c)^2$. All these data are consistent with the form given in Eq. (39) and illustrated in Fig. 2.

As for the charge form factor of neutrons, the coefficient to be multiplied to $G(K^2)$ of Eq. (39) is zero in the harmonic-oscillator model in which only the ground state is taken into account. However, the observed neutron charge form factor is not zero for nonzero values of K^2. This is a clear indication that excited oscillator states should also be taken into account. This point has been discussed by Hussar and Haberman in their recent paper in the conventional harmonic-oscillator formalism.[24]

Throughout this paper, we ignored the effect of spins and assumed that the nucleon form factor can be decomposed into the form of Eq. (13) in the quark model. Indeed, there are models in which the quark spins can be combined for the nucleon to give the form of Eq. (13).[24,25] On the other hand, we still do not know how to treat the spins in the covariant phase-space formalism. This is a challenging future problem.

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