RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 39, NUMBER 5 MARCH 1, 1989

Geometry of multifractal systems

Martin Blunt*
Cavendish Laboratory, Madingley Road, Cambridge CB3 OHE, United Kingdom
(Received 8 August 1988)

The behavior of many fractal structures has been interpreted by a scaling hypothesis: The number of sites on a fractal of size R with a probability measure $R^{-\alpha}$ scales as $R^{f(\alpha)}$. This defines a series of exponents $f(\alpha)$. If the measure subdivides in a self-similar manner over the structure, then it is shown that the set of sites with a given value of α do not themselves form a fractal, but have a more complicated geometry. The analysis is applicable to many growth processes including diffusion-limited aggregation, viscous fingering, and the screened-growth model. If the measure is defined on a static fractal, then the same conclusion applies to many other systems such as models of turbulence, random-resistor networks, and fractal aggregates in shear flows.

I. INTRODUCTION

A large amount of experimental and numerical work has been devoted to the behavior of structures whose surface has been characterized as fractal.[1] The fractal can be considered to be composed of N discrete sites or particles of unit size. If the overall extent of the structure is R, then N scales as R^D, where, by definition, D is the fractal or mass dimension. A measure or probability p_i can be assigned to a given site i. If the structure is covered with a grid of boxes or blobs of length b, then the measure of a box $p_i(b)$ is the sum of the measures of the particles inside it. The moments $Z(q)$ of the distribution over the boxes are defined as

$$Z(q)=\sum_{i=1}^{N(b)} p_i(b)^q ,$$

where $N(b)$ is the number of boxes. $Z(q)$ may scale with the system size as $(R/b)^{-\tau(q)}$ where $\tau(q)$ is a spectrum of exponents describing the growth.[2] This distribution can be interpreted in terms of a multifractal formalism:[2] The number N_α of boxes with a measure $(R/b)^{-\alpha}$ for some small range of α from α to $\alpha+\delta\alpha$ may scale as $(R/b)^{f(\alpha)}\delta\alpha$. The scaling function $f(\alpha)$ provides a statistical description of the behavior of many fractal systems. The relationship between the exponents $f(\alpha)$ and $\tau(q)$ is[2]

$$\tau(q)=q\alpha_q-f_q(\alpha) , \quad (1)$$

where the subscripts refer to the value of $df/d\alpha$ at which f and α are evaluated.

Objects with a nontrivial $f(\alpha)$ are termed multifractals. The concept was first introduced by Mandelbrot to describe turbulence.[3] For fractal growth processes the measure may be interpreted as the probability that growth next occurs from a given site. The formalism is also used to describe the behavior of a fractal in an external field, the current distribution in percolating networks, and the energy density in fully developed turbulence. Multifractals also characterize the geometry of chaotic attractors, temporal intermittency in discordered systems, and other processes. The reader is recommended the review of Paladin and Vulpiani.[4]

The geometry of sets of sites with a particular value of α is analyzed. It is shown that for many of the systems mentioned above, the set itself is not a simple fractal with a box-counting dimension equal to $f(\alpha)$, but has a more complicated structure, which is discussed. The case of a large fractal structure with a small cutoff of unit length will be considered. This is the more natural description for growth models. In other cases it is more usual to consider a fractal of fixed size and let the size of the small-scale cutoff tend to zero. The results presented below are valid for either interpretation.

II. THE DIVISION OF PROBABILITY MEASURE INSIDE A BOX

The scale invariance of a fractal structure means that inside a box of size b, there are $b^{f(\alpha)}$ points with measure or probability $b^{-\alpha}$, with the same function $f(\alpha)$ as for the whole pattern, if the sum of the measures of the particles in the box is normalized to unity. Imagine that a site has an overall measure $R^{-\alpha}$ and lies in a box whose measure scales as $(R/b)^{-\alpha_{\rm box}}$. The site is chosen to have a normalized measure $b^{-\alpha_{\rm site}}$ *inside* the box. Then, if growth processes are considered,

$$R^{-\alpha}=\left(\frac{R}{b}\right)^{-\alpha_{\rm box}} b^{-\alpha_{\rm site}} , \quad (2)$$

since the probability of growth is the product of the probabilities of growth anywhere in the box and specifically at a given point inside it. The measure around the fractal is derived from a multiplicative subdivision: The measure at a given site is the product of the measure in an enclosing box times the measure for a subprocess inside the box. This is valid in any space dimension for fractal growth processes, such as diffusion-limited aggregation,[5–7] viscous fingering at finite viscosity ratio,[8] the dielectric breakdown model,[6,7] the screened growth model,[9] Meakin's multifractal Eden model,[10,11] viscous fingering at large viscosity ratio,[12,13] and diffusion-controlled crystallization.[14] The calculations presented here only assume the existence of a multifractal system for which Eq. (2) is valid. Hence, processes in which the "growth probability" can be interpreted as an energy or flux density derived from a hierarchical subdivision are also considered. This is appropriate to current distributions in percolating networks,[15,16] the force distributions on fractal aggregates,[17,18] and β or cascade models of turbulence.[4,19,20]

It is possible to define an exponent f_{eff} such that the total number of the sites mentioned above is given by

$$R^{f_{\text{eff}}}=\left(\frac{R}{b}\right)^{f_{\text{box}}} b^{f_{\text{site}}}, \tag{3}$$

where f_{site} and f_{box} are $f(\alpha_{\text{site}})$ and $f(\alpha_{\text{box}})$, respectively. Then, from Eq. (2):

$$f_{\text{eff}}=\frac{f_{\text{box}}(\alpha-\alpha_{\text{site}})+f_{\text{site}}(\alpha_{\text{box}}-\alpha)}{\alpha_{\text{box}}-\alpha_{\text{site}}}. \tag{4}$$

The exponent f_{eff} is a weighted linear combination of the exponents f_{site} and f_{box}. Since the scaling function is convex,[2] f_{eff} will always lie below $f(\alpha)$. Consequently, the number of sites for which α_{site} and α_{box} are not equal is only an infinitesimal fraction of the total number of points with measure $R^{-\alpha}$, if R, b, and R/b are all large. f_{eff} equals $f(\alpha)$ in three cases only: (i) α equals α_{site} and b is equal to R, (ii) α equals α_{box} and b equals 1, and (iii) α_{box} and α_{site} are both equal to α. In this case f_{eff} is $f(\alpha)$ regardless of b. These sites have the same values of α_{box} and α_{site} independent of the box size.

This analysis of the division of probabilities has also been used to study the screening of a fixed site on a growing fractal as it increases in size.[21]

III. SELECTING SITES OF A GIVEN MEASURE

The sites on the fractal which have a value of α in a small range between α and $\alpha+\delta\alpha$ (measure $R^{-\alpha}$ to $R^{-\alpha-\delta\alpha}$) are selected. It is possible to analyze the geometry of this set and ask whether or not these points themselves describe a fractal, and if they do, whether or not the set has a box-counting dimension equal to $f(\alpha)$. If as above, the fractal is covered with a grid of boxes of length b, then a box is said to be occupied if one or more of the selected sites lie inside it, otherwise, it is empty. The number $n(\alpha,b)$ of occupied boxes is counted as a function of b. If the set were fractal, it would be possible to define a dimension $D(\alpha)$, independent of b from the scaling law

$$n(\alpha,b)\sim(R/b)^{D(\alpha)}\delta\alpha. \tag{5}$$

$n(\alpha,b)$ is determined by the geometry of the sites with a given α. $D(\alpha)$, if it exists, is not necessarily equal to $f(\alpha)$.

IV. IS THE SET FRACTAL?

As before, if an occupied box has a measure defined by α_{box} and the site with overall measure $R^{-\alpha}$ has a measure $b^{-\alpha_{\text{site}}}$ within the box, then the number of such boxes scales as $(R/b)^{f_{\text{box}}}$, where α, α_{box}, α_{site}, and b are related by Eq. (2). For large R/b, $n(\alpha,b)$ will be dominated by the largest possible value of f_{box} for which the box will be occupied, i.e., it contains one or more sites of measure $R^{-\alpha}$. The number of such sites in each box scales as $b^{f_{\text{site}}}$.

Hence, for a box to be occupied $f_{\text{site}}\geq 0$. From Eq. (5) an effective exponent $D(\alpha,b)$ is given by

$$D(\alpha,b)=\max(f_{\text{box}}), \tag{6}$$

and if b scales as R^{y}, then from Eq. (2):

$$\alpha_{\text{box}}=\frac{\alpha-y\alpha_{\text{site}}}{1-y}. \tag{7}$$

$\alpha^{\min}$ is defined as the smallest value of α for which $f(\alpha)\geq 0$. For α in the range $\alpha^{\min}\leq\alpha\leq\alpha_0$ (α_0 is the value of α where $df/d\alpha$ is zero: f_0 is the corresponding maximum value of f), then the largest value of f_{box} occurs when $\alpha_{\text{site}}=\alpha^{\min}$. Hence, from Eq. (6):

$$D(\alpha,y)=\begin{cases} f\left(\dfrac{\alpha-y\alpha^{\min}}{1-y}\right), & 0\leq y\leq \dfrac{\alpha_0-\alpha}{\alpha_0-\alpha^{\min}}, \quad (8a)\\[2ex] f_0, & 1\geq y\geq \dfrac{\alpha_0-\alpha}{\alpha_0-\alpha^{\min}}. \quad (8b)\end{cases}$$

Notice that we do *not* have a geometrically fractal set. The exponent $D(\alpha,y)$ is itself a function of box size. The exponent increases from $f(\alpha)$ at $b=1$ until it reaches a maximum f_0 at large b, where f_0 is the dimension of the support of the measure which, for systems which are all active surface, is equal to the mass dimension D. The total number of boxes is dominated by those boxes which contain just a single particle of measure $R^{-\alpha}[f(\alpha^{\min})=0]$.

For $\alpha^{\max}\geq\alpha\geq\alpha_0$ [$\alpha^{\max}$ is the value of α where $f(\alpha)=0$ and $df/d\alpha<0$] Eq. (8) is the same, but with $\alpha^{\max}$ substituted for $\alpha^{\min}$. For $\alpha<\alpha^{\min}$ or $\alpha>\alpha^{\max}$, $f(\alpha)$ is negative. Hence, typically, no such sites are seen on a large cluster. In this case, $n(\alpha,b)$ is independent of b: one box surrounds the single site seen on an atypical fractal.

That the sites with a given value of α might not be fractal was first demonstrated by Cates and Deutsch,[22] who calculated spatial correlations in a curdling model with a hierarchical subdivision of probabilities. In this paper, the correlation functions were again described by exponents which were functions of distance.

It has already been remarked[23] that $f(\alpha)$ cannot be interpreted as the dimension of a set of given α. Here, a much stronger statement is made: an analysis of the geometry yields an effective exponent which is a function of box size. However, a simple description of the physics enables the exact form of the behavior [Eq. (8)] to be determined. From Eq. (8), the only cases where we have a truly fractal set where $D(\alpha)=f(\alpha)$, independent of y, are when (i) $\alpha=\alpha^{\min}$, (ii) $\alpha=\alpha^{\max}$, and (iii) $\alpha=\alpha_0$. For growth models, cases (i) and (ii) represent the most exposed tips and the deepest fjords, respectively.

V. INTERPRETATIONS

This short calculation has important consequences for the interpretation of the multifractal formalism in the many processes where the measure can be considered to be derived from some hierarchical subdivision. If the fractal boundary is decomposed into sets with different values of α, then each set is itself not geometrically fractal, although the number of such points does scale with a constant power of the system size. Equation (4) shows that all but an infinitesimal fraction of sites with a given value of α lie within a box of the same α, regardless of the box size b. Most points will lie within boxes which have α_{site}, α_{box}, and α all equal. Inside each such box will be $b^{f(\alpha)}$ sites with measure $b^{-\alpha}$. These sites lie on a geometrically fractal set of dimension $f(\alpha)$ and can be considered to have a "singularity strength"[2] α, which is independent of box size.

However, the total number of boxes [from which $n(\alpha,b)$ is defined] is dominated by those boxes which typically contain only a single site. Hence, $f(\alpha)$ cannot be measured geometrically from a static cluster, although $f(\alpha)$ is recovered if the probability measure is analyzed on various sized grids or on fractals of different size. Only an infinitesimally small fraction of sites have α_{box} not equal to α_{site}. In such cases, an effective exponent α defined from the probability measure over a length b is dependent on b itself. These sites cannot be considered to possess a unique "singularity strength." The scaling function, $f(\alpha)$, does not itself imply anything about the spatial distribution of the measure, but assumptions about the physics of many multifractal systems allow us to demonstrate that the geometry of sites with different measure is richer and more complicated than originally suggested.

It is likely that the calculation of other properties of fractal growth or the behavior of static self-similar structures may reveal the dominance of other "atypical" subsets, which have different "singularities" α measured on different scales.[21]

ACKNOWLEDGMENTS

The author is grateful for fruitful discussions with M. E. Cates and R. C. Ball. He also acknowledges financial support from British Petroleum plc and the United Kingdom Science and Engineering Research Council.

*Present address: BP Research Centre, Chertsey Road, Sunbury-on-Thames, Middlesex, United Kingdom.

[1]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).

[2]T. C. Halsey *et al.*, Phys. Rev. A **33**, 1141 (1986).

[3]B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).

[4]G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).

[5]T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).

[6]C. Amitrano, A. Coniglio, and F. di Liberto, Phys. Rev. Lett. **57**, 1016 (1986).

[7]Y. Hayakawa, S. Sato, and M. Matsushita, Phys. Rev. A **36**, 1963 (1987).

[8]M. J. Blunt and P. R. King, Phys. Rev. A **37**, 3935 (1988).

[9]P. Meakin, Phys. Rev. A **34**, 790 (1986).

[10]P. Meakin, J. Phys. A **20**, L771 (1987).

[11]P. Meakin, J. Phys. A **20**, L779 (1987).

[12]J. Nittmann, H. E. Stanley, E. Touboul, and G. Daccord, Phys. Rev. Lett. **58**, 619C (1987).

[13]K. J. Måloy, F. Boger, J. Feder, and T. Jossang, in *Time Dependent Effects in Disordered Materials,* NATO Advanced Studies Institute Series B Physics, edited by R. Pynn and T. Riste (Plenum, New York, 1987), pp. 111–138.

[14]S. Ohta and H. Honjo, Phys. Rev. Lett. **60**, 611 (1988).

[15]L. de Arcangelis, S. Redner, and A. Coniglio, Phys. Rev. B **31**, 4725 (1985).

[16]R. Rammal, C. Tannous, P. Breton, and A. M. S. Tremblay, Phys. Rev. Lett. **54**, 1718 (1985).

[17]P. Meakin and J. M. Duetch, J. Chem. Phys. **86**, 4648 (1987).

[18]P. Meakin, Phys. Rev. A **36**, 453 (1985).

[19]C. Meneveau and K. R. Sreenivasan, in *Physics of Chaos and Systems Far From Equilibrium,* edited by M-D. Van and B. Nichols (North-Holland, Amsterdam, 1987).

[20]C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).

[21]R. C. Ball and M. J. Blunt, Phys. Rev. A (to be published).

[22]M. E. Cates and J. M. Deutsch, Phys. Rev. A **35**, 4907 (1987).

[23]P. Grassberger, R. Badii, and A. Politi, J. Stat. Phys. **51**, 135 (1988).