# Phase transitions on strange irrational sets

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Nonanalyticities in the generalized dimensions of fractal sets of physical interest are interpreted as phase transitions. We apply the thermodynamical formalism to the fractal set formed by the irrational winding parameter values of critical circle maps and introduce and investigate in detail several distinct fractal measures on this set. The thermodynamic functions associated with different measures are distinct: We discover that, in all cases that we study, they exhibit phase transitions. The numerical estimates of the Hausdorff dimension from various versions of the thermodynamical formalism and a variety of circle maps yield $D_H = 0.8701 \pm 0.0003$ and are consistent with the conjectured universality of $D_H$.

Nonlinear physics presents us with a perplexing variety of complicated fractal objects and strange sets. Notable examples include strange attractors for chaotic dynamical systems, regions of high vorticity in fully developed turbulence, and fractal growth processes.

The aim of this paper is to point out and discuss in some depth the fact that some sets of physical interest have nonanalytic scaling spectra, a phenomenon which we interpret as "phase transitions" on fractal sets. Moreover, this phenomenon has a direct analogy to the phenomenon of phase transitions in condensed-matter physics. The fact that such a phenomenon can occur in physically important settings, such as the mode-locking problem, was discovered first in Ref. 1, and is discussed in detail in this paper. That the identical kind of phase transition can also appear in problems of purely mathematical interest, such as Julia sets, was subsequently observed by Katzen and Procaccia.[2] Other examples of phase transitions on fractal sets have since been studied in Refs. 3–7. The recent work by Feigenbaum[8] on the scaling theory of strange sets clarifies the nature of these phase transitions.

Our starting point is the conjecture of Jensen *et al.*[9,10] that the Hausdorff dimension of the irrational windings parameter set of the critical circle maps is *universal*, $D_H = 0.87\ldots$ . From the experimental point of view, the extraction of $D_H$ is an efficient way of averaging over all available data, as $D_H$ estimates utilize *all* mode lockings accessible in an experiment, in contradistinction to theoretical predictions such as the golden mean universality.[11–13] However, here we are interested not so much in the calculation of this Hausdorff dimension per se: we use this problem as a convenient testing ground to gain insight into new interesting phenomena implicit in the use of "thermodynamic" formalism.

The straightforward numerical estimate of the Hausdorff dimension of Jensen *et al.* uses no knowledge about the structure of the set; the mode-locking intervals are ordered according to their width and counted. Such calculations yield numerically good estimates of $D_H$, but they offer no clue as to why $D_H$ should be universal. However, the set of rationals $P/Q$ clearly possesses rich number-theoretic structure, which we utilize to introduce and carefully examine several novel thermodynamic formulations of the mode-locking problem. These are based on different partitionings of rationals.

(1) Farey series.
(2) Farey tree levels.
(3) Continued fractions of fixed length.

These are standard number-theoretic concepts; their properties are summarized in Appendix A.

Here we investigate the Farey series and the Farey tree partitionings in some detail. Farey series thermodynamics is introduced here for the first time. The Farey tree partitioning was introduced in Ref. 14 and independently investigated by Feigenbaum.[15] The continued-fractions partitioning will be discussed elsewhere:[16] it has also been utilized by Lanford in his mode-locking renormalization theory.[17]

In Sec. II we establish by a simple argument that $D_H$ is bounded and smaller than 1; this is by no means obvious *a priori*, as we also find that a precise numerical determination of the Hausdorff dimension of the set of irrational winding parameter values is hampered by logarithmically slow convergence. We use the thermodynamical formalism to elucidate the nature of this convergence: the main new theoretical insight is that the thermodynamic sums undergo "phase transitions." We model such sums by analytically tractable number theoretical models. These models play a crucial role in the next step—they gauge the performance of various convergence acceleration algorithms. We apply the most effective algorithms to the evaluation of $D_H$ for a number of different circle maps and obtain a much larger spread in the estimates than the numerical stability of each separate estimate would suggest. We give many examples of the procedures employed, and we hope to encourage experiments to follow these lines.[18]

## I. PARTITION SUMS

Consider a "partition" sum over mode-locking intervals defined by

$$\bar{Z}(\tau)=\sum_{Q=1}^{\infty}\sum_{(P|Q)=1}\Delta_{P/Q}^{-\tau} . \quad (1.1)$$

The sum (1.1) is over all irreducible rationals $P/Q$, $P<Q$, and $\Delta_{P/Q}$ is the width of the parameter interval for which the iterates of a critical circle map lock onto a cycle of length $Q$, with winding number $P/Q$.

For sufficiently negative $\tau$, the sum (1.1) is convergent; for $\tau=-1$, $\bar{Z}(-1)=1$ since for the critical circle maps the mode lockings (are conjectured to) fill the entire $\Omega$ range. However, as $\tau$ increases, the narrow (large-$Q$) mode-locked intervals $\Delta_{P/Q}$ get blown up to $\Delta_{P/Q}^{-\tau}$, and at some critical value of $\tau$ the sum diverges. This occurs for $\tau<0$, as $\bar{Z}(0)\to\infty$.

Although the sum (1.1) is infinite, in practice the experimental or numerical mode-locked intervals are available only for small finite $P$ and $Q$. Hence it is necessary to truncate the sum (1.1), and present the set of irrational windings hierarchically, with $N$ intervals in a given truncation,

$$\bar{Z}_N(\tau)=\sum_{i=1}^{N}\Delta_{N,i}^{-\tau} . \quad (1.2)$$

Alternatively, one may consider the sum

$$Z_N(\tau)=\sum_{i=1}^{N} l_{N,i}^{-\tau} , \quad (1.3)$$

where $l_{N,i}$ is the parameter interval between the $\Delta_{N,i-1}$ and $\Delta_{N,i}$ mode lockings. Since the intervals $l_{N,i}$ cover all parameter values corresponding to irrational winding numbers, we refer to them as the *covering* intervals, and to the critical value of $-\tau$ for which the $N\to\infty$ limit of the sum (1.3) diverges as the *covering dimension* $D_N$. If the choice of covering intervals $l_i$ is optimal (for rigorous definitions, see Ref. 19) $D_\infty$ is the Hausdorff dimension $D_H$ of the fractal set; otherwise it is an upper bound to $D_H$.

Conversely, the mode-locked intervals $\Delta_{N,i}$ create holes between irrational winding parameter values. The $N$th estimate of the value of $-\tau$ for which (1.1) diverges will be referred to as the *hole dimension* $\bar{D}_N$.

The choice of holes in (1.2) is largely arbitrary. Ideally one chooses $\Delta_{N,i}$ such that for asymptotic $N$, the covering intervals in (1.3) are of comparable magnitude.

The two sums (1.2) and (1.3) behave quite differently for sufficiently negative $\tau$. For example, as holes and covers are complementary sets, $Z_N(-1)+\bar{Z}_N(-1)=1$. If $D_H<1$, $Z_N(-1)\to 0$, but $\bar{Z}_N(-1)\to 1$. However, as the cover in (1.3) is refined by punching holes $\Delta_{N',i}$ in covers $l_{N,i}$ (where the initial cover has $N$ intervals, and the refined cover has $N'>N$ intervals), the narrowest holes scale in size as the covering intervals. As the divergence of (1.1) is caused by the narrowest intervals we expect the hole dimension $\bar{D}_\infty$ to coincide with the cover dimension $D_\infty$. For simple fractal sets, such as the original Cantor set, or the sets with two scales, this is indeed easily verified by explicit calculation. We wish to determine the limit of the sums (1.2) and (1.3) for $N\to\infty$: in order to do this effectively, we need to recall a few facts specific to circle maps.

## II. MODE LOCKING IN CRITICAL CIRCLE MAPS

Dynamical systems possessing a natural frequency $\omega_1$ display very rich behavior when driven by an external frequency $\omega_2$; as the "bare" winding number $\Omega=\omega_1/\omega_2$ is varied, such systems sweep through infinitely many mode-locked states. Both quantitatively and qualitatively this behavior is already present in simple models such as the sine map

$$x_{n+1}=x_n+\Omega-\frac{k}{2\pi}\sin(2\pi x_n) \pmod 1, \quad k=1 \quad (2.1)$$

and the piecewise cubic map

$$x_{n+1}=\begin{cases}\Omega+4x_n^3 & (0\le x_n\le\frac{1}{2})\\ \Omega+1+4(x_n^3-1) & (\frac{1}{2}\le x_n\le 1) .\end{cases} \quad (2.2)$$

As $\Omega$ is varied from 0 to 1, the iterates of a circle map either mode lock, with a winding number given by a rational number $P/Q\in(0,1)$, or do not mode lock, in which case the winding number is irrational. The complement of the set of parameter values $\Omega$ for which the map mode locks is the *set of irrational windings,* whose dimension we wish to determine. Circle maps with zero slope at the inflection point [e.g., $x=0$ in (2.1)] are called critical: they lie on the borderline of chaos. In the numerical calculations presented here we concentrate only on the physically relevant case, the maps with cubic inflection points.

For a given cycle length $Q$, the mode-locked intervals vary considerably in size. The narrowest interval shrinks with a power law[10,14,20]

$$\Delta_{1/Q}\propto Q^{-3} \quad (2.3)$$

and so does the covering interval $l_{1/Q}$ which spans the parameter values between $\Delta_{1/Q}$ and $\Delta_{1/(Q-1)}$. This should be compared to the subcritical circle maps in the number-theoretic limit [$k=0$ in (2.1)], where the interval between $1/Q$ and $1/(Q-1)$ winding number value of the parameter $\Omega$ shrinks as $1/Q^2$. For the critical circle maps the $l_{1/Q}$ interval is narrower than in the $k=0$ case, because it is squeezed by the nearby broad $\Delta_{0/1}$ fixed point interval.

The widest interval corresponds to $P/Q=F_{n-1}/F_n$, the $n$th continued fraction approximant to the golden mean. The reason for this is that the golden mean sits as far as possible from any short-cycle mode locking. This interval shrinks with a universal exponent

$$\Delta_{P/Q}\propto l_{P/Q}\propto Q^{2\mu_{gm}} , \quad (2.4)$$

where $P=F_{n-1}$, $Q=F_n$, and $\mu_{gm}$ is related to the universal Shenker number[11] and the golden mean (A10) by

$$\mu_{gm}=\frac{\ln\delta}{2\ln\rho}=1.082\,18\ldots, \quad =2.833\,612\ldots . \tag{2.5}$$

The closeness of $\mu_{gm}$ to 1 indicates that the golden-mean approximants barely feel the fact that the map is critical (in the $k=0$ limit this exponent is $\mu=1$).

The preceding estimates motivate rewriting the limit of the sum (1.1) as

$$\bar{Z}(\tau)=\sum_{Q=1}^{\infty}\sum_{(P|Q)=1} Q^{2\tau\mu_{P/Q}}, \quad Q^2\approx N_Q=\Phi(Q) , \tag{2.6}$$

with exponents of bounded variation

$$\mu_{gm}\le\mu_{P/Q}\le\tfrac{3}{2} . \tag{2.7}$$

In the crudest approximation, one can replace $\mu_{P/Q}$ by a "mean" value $\hat{\mu}$ and the sum (2.6) can be evaluated in terms of Riemann $\zeta$ functions,

$$Z(\tau)=\sum_{Q=1}^{\infty}\phi(Q)Q^{2\tau\hat{\mu}}=\frac{\zeta(-2\tau\hat{\mu}-1)}{\zeta(-2\tau\hat{\mu})} . \tag{2.8}$$

As the sum diverges as $-\tau$ approaches the Hausdorff dimension, the "mean" scaling exponent $\hat{\mu}$ and $D_H$ are related by

$$D_H\hat{\mu}=1 . \tag{2.9}$$

While this does not enable us to compute $D_H$, it does immediately establish that $D_H$ for critical maps exists and is smaller than 1, as the $\mu$ bounds (2.7) yield

$$\tfrac{2}{3}<D_H<0.9240\ldots . \tag{2.10}$$

(These bounds are of experimental relevance,[21,22] as finite $Q$ estimates of $D_H$ are sensitive to the choice of mode-locked intervals.) To obtain sharper estimates of $D_H$, we need to describe the distribution of $\mu_{P/Q}$ within the bounds (2.7); this we shall do by means of the thermodynamical formalism.

## III. THERMODYNAMICAL FORMALISM

The "thermodynamical formalism"[23–27] arises from the observation that it is advantageous to reorder the sum (1.3) by increasing interval size $l$ and rewrite it as

$$Z_N(\tau)=e^{tq_t(\tau)}=\int_{\mu_{min}}^{\mu_{max}}d\mu\, e^{t[s_t(\mu)+\mu\tau]} , \tag{3.1}$$

where

$$t=\ln N , \tag{3.2}$$

$$\mu=-\frac{1}{t}\ln l , \tag{3.3}$$

$$q_t(\tau)=\frac{1}{t}\ln Z_N(\tau) , \tag{3.4}$$

$$s_t(\mu)=\frac{1}{t}\ln N(\mu), \quad 0\le s_t(\mu)\le 1 , \tag{3.5}$$

and $N(\mu)d\mu$ is the number of intervals whose scaling exponent $\mu$ falls into the range $[\mu,\mu+d\mu]$.

The functions $Z_N(\tau)$, $q(\tau)$, and $s(\mu)$ are interrelated in the same way as the thermodynamic partition sum, free energy, and Gibbs free energy, hence the name "thermodynamic formalism." We follow the notational conventions of Halsey *et al.*,[26] with exception that here we always assume uniform probability $p=1/N$ and so the quantities $\mu$ and $s(\mu)$ are more convenient in the present context than $\alpha$ and $f(\alpha)$ from Halsey *et al.* They are related by[27]

$$\alpha=1/\mu, \quad f(\alpha)=s(\mu)/\mu . \tag{3.6}$$

The scaling spectrum $s(\mu)$ is a highly irregular function of $\mu$ for finite $N$; in practice one computes $q_t(\tau)$, which is a monotonically increasing function of $\tau$, and replaces $s(\mu)$ by its convex envelope $S(\mu)$, evaluated from the $t\to\infty$ saddle estimate of (3.1)

$$e^{tq_t(\tau)}=C(t)\left[-\frac{2\pi t}{S_t''(\tau)}\right]^{1/2} e^{t[S_t(\bar{\mu})+\tau\bar{\mu}(\tau)]} , \tag{3.7}$$

where $\bar{\mu}(\tau)$ is the solution of the extremum condition

$$\frac{dS_t(\mu)}{d\mu}=-\tau . \tag{3.8}$$

As we shall see in Sec. V, the distinction between the scaling spectrum $s(\mu)$ and its convex envelope $S(\mu)$ is crucial: it offers one way of diagnosing phase transitions. In the $N\to\infty$ limit $q_t(\tau)$ and $S_t(\mu)$ approach finite limiting values $q(\tau)$ and $S(\mu)$ which satisfy the usual thermodynamical Legendre transform relations

$$q(\tau)=S(\mu)+\tau\mu , \tag{3.9}$$

$$\mu=\frac{dq(\tau)}{d\tau}, \quad \tau=-\frac{dS(\mu)}{d\mu} . \tag{3.10}$$

$\mu(\tau)$, the effective scaling exponent at $\tau$, grows monotonically from $\tau=-\infty$ (where the sum (3.1) is dominated by the widest intervals) to $\tau=\infty$ (where the sum is dominated by the narrowest intervals). Hence the second derivative of $q(\tau)$ is strictly positive. It is related to $S''(\mu)$ by

$$\frac{d^2q(\tau)}{d\tau^2}\frac{d^2S(\mu)}{d\mu^2}=-1 . \tag{3.11}$$

The Hausdorff-dimension condition can now be restated in terms of quantities finite in the $N\to\infty$ limit,

$$q(-D_H)=0 . \tag{3.12}$$

Equivalently, by (3.9) and (3.10), the Hausdorff dimension is the slope of the tangent to $S(\mu)$ at $\mu=\mu_H$ satisfying

$$D_H=\left.\frac{dS(\mu)}{d\mu}\right|_{\mu=\mu_H}=\frac{S(\mu_H)}{\mu_H} . \tag{3.13}$$

The dimension associated with the most numerous scaling exponent can be interpreted as the information dimension, with

$$D_I=1/\mu_I, \quad S(\mu_I)=1 . \tag{3.14}$$

$\mu_I$ is the average scaling exponent at $\tau=0$;

$$\mu_I=\mu(0)=\left.\frac{dq(\tau)}{d\tau}\right|_{\tau=0}=\frac{1}{N}\sum_{i=1}^{N}\mu_i \ . \tag{3.15}$$

The asymptotic $q(\tau)$ is invariant under rescaling of $l_i$; however, the finite $N$ estimates $q_t(\tau)$ pickup corrections of order $1/t$,

$$\begin{aligned} &l_i \to e^{-\alpha}\Delta_i \ , \\ &Z_N(\tau)\to e^{\alpha\tau}Z_N(\tau) \ , \\ &q_t(\tau)\to q_t(\tau)+\alpha\tau/t \ . \end{aligned} \tag{3.16}$$

The finite-$N$ effect can be eliminated by computing $q(\tau)$ from ratios of different $Z_N(\tau)$,

$$q_{t-t'}(\tau)=\frac{1}{t-t'}\ln\frac{Z_N(\tau)}{Z_{N'}(\tau)} \ , \tag{3.17}$$

rather than from (3.4). In our numerical work we use two choices of $N'$; either we take

$$N=[2N'] \tag{3.18}$$

(brackets stand for the integer part) or we normalize by the overall covering interval[28] $Z_1(\tau)=l^{-\tau}$,

$$N'=1 \ . \tag{3.19}$$

Beyond that, we accelerate convergence by methods described in Sec. V B.

The best method for extracting thermodynamics from the intrinsic scales of a strange set is the transfer-matrix technique[23,24,27] which utilizes scaling factors. As a scaling function for the Farey series partitioning is not available, and the Farey tree scaling function[14,15] suffers from slow logarithmic convergence, the computations here are carried out directly on the sums (1.2) and (1.3).

## IV. FAREY SERIES THERMODYNAMICS

The *Farey series covering set* thermodynamics is obtained by deleting all mode-locked intervals $\Delta_{P'/Q'}$ of cycle lengths $1\le Q'\le Q$, and forming the partition sum (1.3) from the remaining covering intervals

$$l_i=l(Q_{i-1},Q_i)=\Omega^{\text{left}}_{P_i/Q_i}-\Omega^{\text{right}}_{P_{i-1}/Q_{i-1}} \ . \tag{4.1}$$

The number of intervals is $N_Q=\Phi(Q)\propto Q^2$ (see Appendix A), and $P_0/Q_0=0/1$.

As the widths of the mode-locked and covering intervals shrink exponentially with $Q$, partitioning of rationals into sets with bounded $Q$ is appealing both from experimental and theoretical points of view. They are also of a number-theoretical interest, because they provide uniform coverings of the unit interval with rationals,[29] and because they are closely related to the deepest problems in number theory, such as the Riemann hypothesis.[30,31]

Number theorists have carefully examined the thermodynamics of Farey series in the number-theory limit [$k=0$ in (2.1)]. Their analytic results are instructive, so we review them first, before examining the thermodynamics of the critical map.

### A. Farey arcs

In the $k=0$ limit the mode-locked intervals have zero width, and the covering intervals (4.1) are the Farey arcs,[32] i.e., the differences of successive rationals in a Farey series of order $Q$ (see Appendix A),

$$l_i=\frac{P_i}{Q_i}-\frac{P_{i-1}}{Q_{i-1}}=\frac{1}{Q_iQ_{i-1}} \ . \tag{4.2}$$

It is plain that

$$Z_N(-1)=1, \quad Z(0)=N=\Phi(Q) \ . \tag{4.3}$$

For the Farey series of order $Q$ the broadest Farey arc is $l(1,Q)=1/Q$. Hence the minimum scaling exponent (3.3) is

$$\mu_{\min}=\mu(1,Q)=\tfrac{1}{2} \ . \tag{4.4}$$

As $Q$ grows, each rational with a small denominator $k$ is bracketed by a pair of broad Farey arcs of size

$$\begin{aligned} &l(k,Q')\simeq\frac{1}{kQ'}, \quad k\ll Q, \quad Q'\approx Q \\ &\mu_k=\tfrac{1}{2}+\frac{1}{t}\ln k \ , \end{aligned} \tag{4.5}$$

where $Q'$ is either the preceding or the next denominator in the Farey series. The number of times $k$ appears as a denominator is given by the Euler function $\phi(k)$ (Appendix A). Hence we can estimate the scaling spectrum $s(\mu)$ for $\mu$ close to $\mu_{\min}$,

$$\sum_{k=1}^{R} l(k,Q')^{-\tau}\simeq 2\sum_{k=1}^{R}\phi(k)(kQ)^{\tau} \quad \text{for } R\ll Q \ .$$

Approximating this sum by the integral

$$\int_{1/2}^{1/2+\epsilon} d\mu\, e^{t(2\mu-1+\tau\mu)} \ , \tag{4.6}$$

we find that close to $\mu_{\min}$ the scaling spectrum is given by

$$s(\mu)=2\mu-1, \quad \mu=\mu_{\min}+\epsilon \tag{4.7}$$

$$\frac{ds}{d\mu}=2 \ . \tag{4.8}$$

The finite slope of $s(\mu)$ at $\mu=\mu_{\min}$ implies that the partition sum (1.3) is dominated by the broad Farey arcs (4.5) for all $\tau\le -2$, and by (4.6) $q(\tau)$ is given by a straight line $q=\tau/2$ over this entire range.

For $\tau$ large and positive, the partition sum (1.3) is dominated by the narrowest arc

$$l(Q,Q-1)=\frac{1}{Q(Q-1)} \ , \tag{4.9}$$

$$\mu_{\max}=1 \ . \tag{4.10}$$

Therefore $q(\tau)$ has slope 1 for $\tau\to\infty$; by (4.7) and (4.3) it must pass through $q(-2)=-1$, $q(-1)=0$, $q(0)=1$; and it must be strictly convex. Hence $q(\tau)$ consists of two straight sections

$$q(\tau)=\begin{cases}\tau/2, & \tau\le -2\\ 1+\tau, & \tau\ge -2 \ .\end{cases} \tag{4.11}$$

We see that $q(\tau)$ for the Farey arc thermodynamics undergoes a first-order phase transition at $\tau=-2$. A more careful number-theory analysis[32-34] (Appendix A) leads to the same result. We conclude that in the $N \rightarrow \infty$ limit the Farey arc thermodynamics is simple.

The question of great practical importance is the size of finite-$N$ effects. In this case the Hausdorff dimension is always equal to 1 by (4.3). The finite-$N$ effects for $\tau \rightarrow \pm\infty$ are easily estimated; for $\tau \rightarrow -\infty$ (4.5) yields correction of order $1/\ln Q$, and for $\tau \rightarrow +\infty$ (4.9) yields corrections of order $1/Q$. At the phase transition point $\tau=-2$, the corrections are of order[32-34] $\ln Q/Q$. The "Euler noisiness" of $N=\Phi(Q)$ introduces further errors of order $\ln Q/Q$. These finite-$N$ error estimates agree with the numerically observed finite-$N$ effects, Figs. 1 and 2.

### B. Farey series for critical maps

For critical maps the $1/Q$ mode-locked intervals lie on a parabolic devil's staircase,[10,14,20] yielding the broadest covering interval $l(1,Q) \simeq kQ^{-2}$, with the minimum scaling exponent (3.3)

$$\mu_{\min}=1 . \tag{4.12}$$

The convergence is in practice very slow.[14]

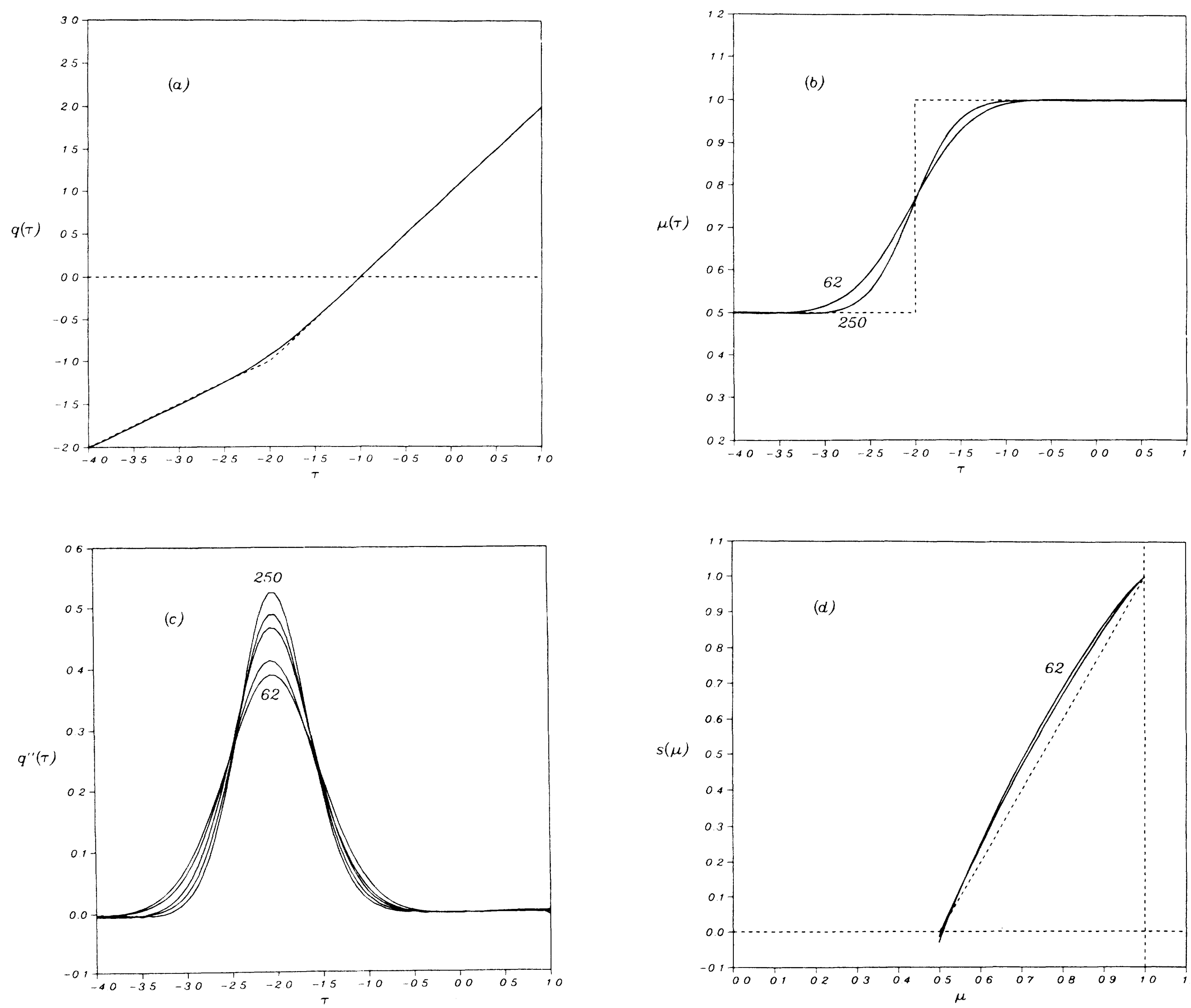


FIG. 1. Numerical estimates of the Farey arcs (4.2) thermodynamic functions, with convergence accelerated by (3.17). (a) $q_t(\tau)$ computed from $Q=250$ and $Q'=177$ ($N_{250}=19\,024$ and $N_{177}=9566$). The asymptotic $q(\tau)$, (4.11), is indicated by the dotted line; there is a first-order phase transition at $\tau=-2$. By (4.3) the Hausdorff dimension equals 1 for all $N$. (b) The scaling exponent $\mu_t(\tau)=q_t'(\tau)$, evaluated for $Q:Q'=250:177$ and 62:44. Slow convergence toward the asymptotic $\mu(\tau)$ (dotted line) is manifest. (c) $q_t''(\tau)$ evaluated for 250:177, 177:125, 125:88, 88:62, and 62:44 gives a finite-$N$ scaling indication of the first-order phase transition at $\tau=-2$. (d) The scaling spectrum $S_t(\mu)$, calculated from $q_t(\tau)$ and $\mu_t(\tau)$ by the Legendre transform (3.9), for 250:177 and 62:44. The asymptotic $S(\mu)$ (dotted line) is concentrated on the points $(\frac{1}{2},0)$ and $(1,1)$.

The narrowest covering interval is

$$l(Q,Q-1)\approx kQ^{-3}\,,\qquad \mu_{\max}=\tfrac{3}{2}\,. \tag{4.13}$$

The $Q^{-3}$ behavior arises for the same reason as the $Q^{-2}$ scaling leading to (4.12); the successive mode-locked intervals $1/Q$, $1/(Q-1)$ lie on a parabola, so their difference is of order $Q^{-3}$. Numerical estimates of the thermodynamic functions (see Fig. 2) exhibit the same slow convergence and "Euler noise" as the finite-$N$ Farey arc computation, and indicate a first-order phase transition.

However, the finite-$N$ Hausdorff dimension estimates exhibit surprising numerical stability; for example, the sine map $D_N$ estimate (Fig. 3) for $240\leq Q\leq 250$ is $D_N=0.870\,12\pm0.000\,01$. For the cubic map we obtain $D_N=0.870\,02\pm0.000\,01$, and for the sine with 0.075 $\sin^3$ term we obtain $D_N=0.869\,68\pm0.000\,01$. Of course, as it is obvious from Fig. 3, such numerical stability is deceptive, as the computation is not asymptotic. Varying the ratios of $Q$ and $Q'$ in (3.17) has no significant effect on the numerical stability of the preceding estimates.

This completes our numerical investigations of the Farey *series* thermodynamics. We next turn to the other

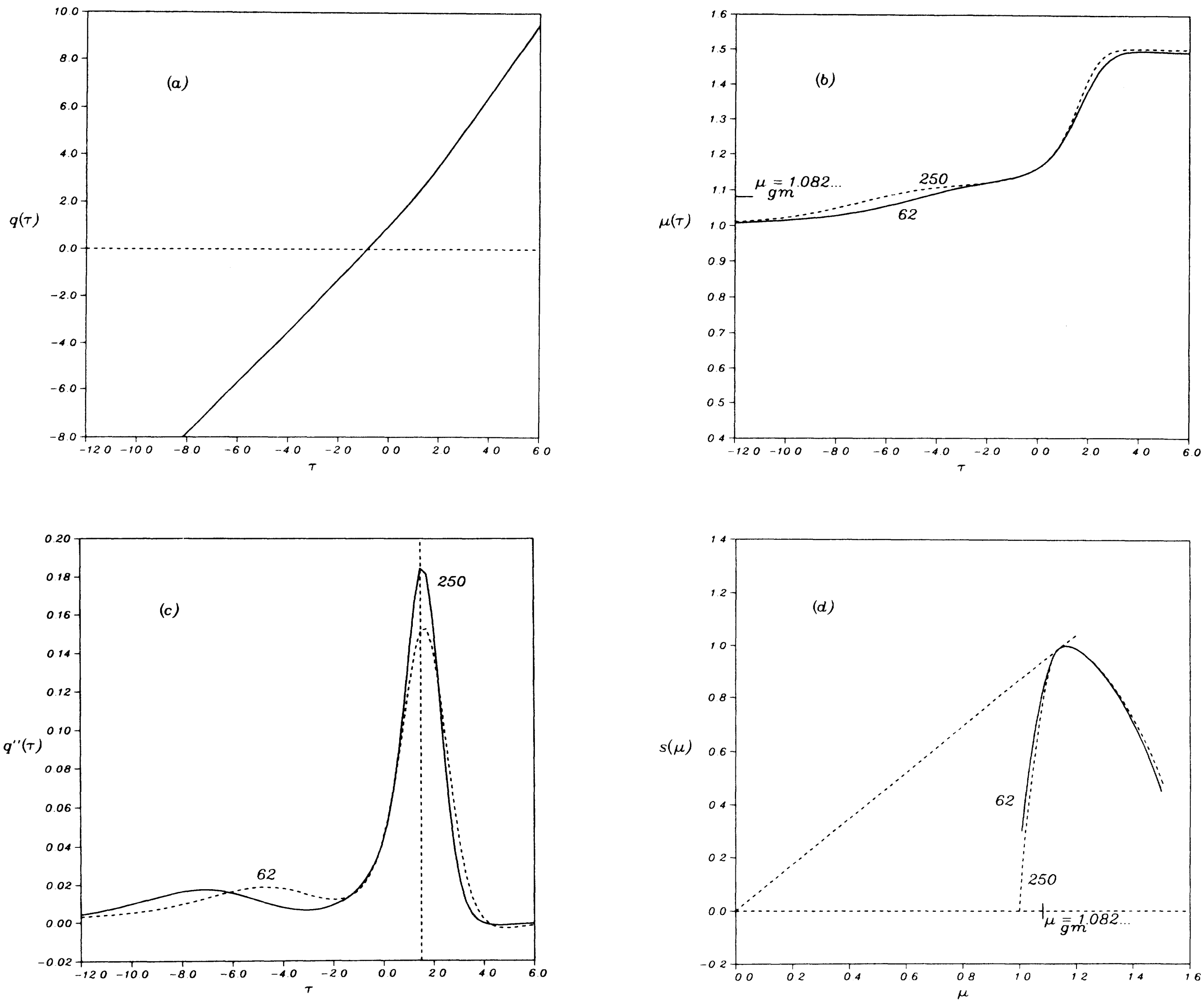


FIG. 2. Numerical estimates of the Farey series covering interval (4.1) thermodynamic functions for the critical sine map (2.1), with convergence accelerated by (3.17). (a) $q_t(\tau)$ computed for $Q{:}Q'=250{:}177$. The Hausdorff dimension is determined by the intersection $q(-D_H)=0$. (b) The scaling exponent $\mu_t(\tau)$ evaluated for 250:177 and 62:44. While the convergence for $\tau\rightarrow\pm\infty$ is slow and "Euler" noisy (as anticipated by the estimates of finite $N$ effects), convergence on the interval $\tau\approx[-1,0]$ is very good. (c) $q_t''(\tau)$ evaluated for 250:177 and 62:44 gives a numerical indication of the first-order phase transition at $\tau_c>1.5$. (d) The scaling spectrum $S_t(\mu)$ calculated from (3.9), for 250:177 and 62:44. $\mu_{\min}\rightarrow 1$ and $\mu_{\max}\rightarrow\frac{3}{2}$, in agreement with (4.12) and (4.13). The slope (3.13) of the dotted line is the Hausdorff dimension. The golden mean (2.5) is the smallest (universal) quadratic irrational scaling exponent. Finite slope $ds/d\mu$ at $\mu=\frac{3}{2}$, $S(\frac{3}{2})\simeq\frac{1}{2}$ gives a numerical indication of a first-order phase transition.

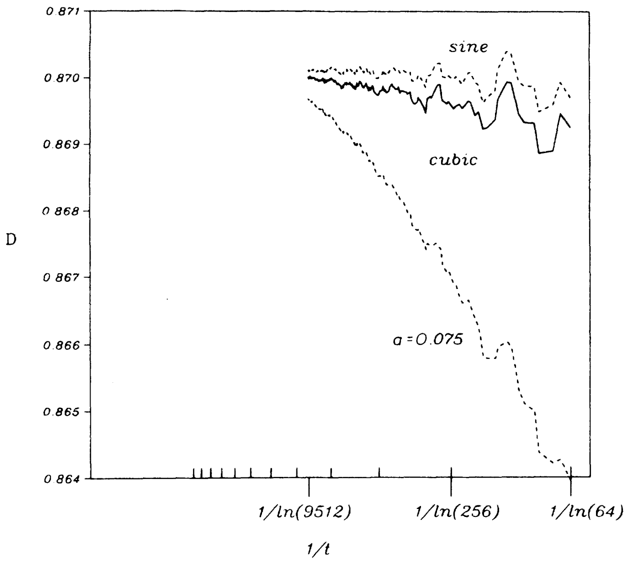


FIG. 3. Finite-$N$ estimates of the Farey series thermodynamics Hausdorff dimension (3.12), with convergence accelerated by use of (3.17) and (3.18), for the sine map (2.1), the cubic map (2.2), and the sine map of Ref. 10, Eq. (4.1), with $a=0.075$. The "noise" is entirely due to the irregular growth of $N=\Phi(Q)$ as a function of $Q$ (Appendix A). $D_N$ is computed for all $Q{:}Q'$, $20 \le Q \le 250$, $Q'=[Q/1.41]$.

partitioning of the rationals employed here, the Farey *level* thermodynamics.

## V. FAREY LEVEL THERMODYNAMICS

The Farey level *hole* thermodynamics is obtained by forming the partition sum (1.2) from the $N=2^n$ mode-locked intervals $\Delta_{P/Q}$ belonging to the $n$th level of the Farey tree (definition 3, Appendix A). The Farey level *covering set* thermodynamics (1.3) is obtained by removing all mode-locked intervals up to the $n$th level of the Farey tree.

At level $n=0$, the sole $l$ interval is obtained by deleting the $\Omega$ values for which the circle map converges to a fixed point. At level $n=1$, $\Omega$ values corresponding to the cycle of length 2 are deleted. By deleting the Farey mediant[35] mode-locked interval $(P+P')/(Q+Q')$ we obtain two $l$ intervals, and so on.

The narrowest mode-locked interval at the $n$th level, the golden-mean interval (2.4),

$$\Delta_{F_{n-1}/F_n} \propto \delta^{-n} , \tag{5.1}$$

shrinks exponentially, and for $\tau$ large and positive it dominates $q(\tau)$ and bounds $\mu(\tau)$ [see (3.3)]

$$\mu_{\max} = \frac{\ln\delta}{\ln 2} = 1.502\,642\ldots . \tag{5.2}$$

However, for $\tau$ large and negative, $q(\tau)$ is dominated by the interval (2.3) which shrinks only harmonically, and $\mu(\tau)$ approaches 0 as

$$\mu_{\min} = \frac{3\ln n}{n \ln 2} \to 0 . \tag{5.3}$$

So for finite $n$, $q_n(\tau)$ crosses the $\tau$ axis at $-\tau=D_n$, but in the $n\to\infty$ limit $q(\tau)$ exhibits a phase transition; $q(\tau)=0$ for $\tau<-D_H$, but is a nontrivial function of $\tau$ for $-D_H \le \tau$.

The thermodynamics developed here is radically different from that for the fractal sets which scale everywhere geometrically, such as the period-doubling attractor,[27,36–38] where both small and large scales approach a universal limit, and the $q(\tau)$ zero crossing is smooth. Here the Hausdorff dimension is determined by the transition from geometric scalings (where we know that at least the quadratic irrationals lead to universal scalings) to harmonic scalings, which we know are not universal.[14] More precisely, our problem is that we know that the nonuniversal effects in circle-map scaling functions are numerically small; hence it would be desirable to have (as long as there is no theory for the universality of this Hausdorff dimension) high-precision evaluations of $D_H$. As we shall see, the presence of the phase transitions makes this task difficult.

### A. Order of the phase transition

In order to develop intuition about the nature of this phase transition, we first introduce the "Farey model" and then consider a series of approximations both to the critical circle-map sum (1.2) and to the Farey model. In the Farey model the intervals $l_{P/Q}$ are replaced by $Q^{-2}$,

$$Z_n(\tau) = \sum_{i=1}^{2^n} Q_i^{2\tau} . \tag{5.4}$$

Here $Q_i$ is the denominator of the $i$th Farey rational $P_i/Q_i$. For example,

$$Z_2(\tfrac{1}{2}) = 4+5+5+4 .$$

The Farey model is motivated by the observation that the scaling exponents are of bounded variation, with bounds given by (2.7). With $\tau$ replaced by $\hat{\mu}\tau$, the Farey model can be thought as a "mean" $\hat{\mu}$ approximation (2.8) to the sum (1.3). Alternatively, the Farey model can be thought of as the covering thermodynamics for the trivial ($k=0$) map, as in Sec. IV A. Regardless of the interpretation, the Farey model gives valuable insights into the Farey level thermodynamics: the qualitative behavior is the same in the "trivial" ($k=0$) and the critical ($k=1$) case, but the Farey model is analytically tractable. The sum (5.4) can be evaluated explicitly for positive integer and half-integer values of $\tau$ (Appendix B), or evaluated with high precision numerically for any value of $\tau$. Analogous to the critical case bounds (5.2) and (5.3), the largest and the smallest scaling exponents are given by

$$\mu_{\max} = \frac{2\ln\rho}{\ln 2} = 1.388\,848\ldots , \tag{5.5}$$

$$\mu_{\min} = \frac{2\ln n}{n \ln 2} \to 0 . \tag{5.6}$$

The Farey model also has a phase transition at $\tau=-D_H$.

For the Farey model the Hausdorff dimension equals 1, as the irrational numbers have all the measure on the unit interval. Formally this follows from the divergence of the sum $Z(\tau)=\sum Z_n(\tau)$ at $\tau=-1$ [see (2.8)].

The first rough bound on $D_H$ can be obtained by considering

$$\hat{Z}_n(\tau)=e^{t\tau\mu_{\min}}+Ne^{t\tau\mu_{\max}} . \tag{5.7}$$

In this approximation we have replaced all $l_{P/Q}$, except $l_{1/n}$, the widest interval, by the narrowest interval $l_{F_{n-1}/F_n}$. The crossover from the harmonic dominated to the golden mean dominated behavior occurs at the $\tau$ value for which the two terms in (5.7) contribute equally,

$$D_n=\frac{1}{\mu_{\max}-\mu_{\min}}=\hat{D}+O\left[\frac{\ln n}{n}\right] , \tag{5.8}$$

$$\begin{aligned}\hat{D}&=\frac{1}{\mu_{\max}}=0.7002\ldots \quad \text{(Farey model)}\\ &=0.665\,49\ldots \quad \text{(critical maps)} .\end{aligned} \tag{5.9}$$

As the sum (5.7) is the lower bound to the sum (1.3) for negative $\tau$, $\hat{D}$ is a lower bound on $D_H$. The size of the level-dependent correction in (5.8) is ominous—the finite $n$ estimates converge to the asymptotic value logarithmically,

$$\frac{D_n-D}{D_{n+1}-D}=1+O\left[\frac{1}{n}\right] . \tag{5.10}$$

Such excruciatingly slow convergence cannot be overcome by brute computation, but requires an understanding of the phase transition.

There is an improvement of the preceding bound which exhibits an analytically tractable phase transition. We partition all Farey rationals on the $n$th Farey level into subsets of fixed continued-fraction length,

$$\hat{Z}_n(\tau)=\sum_{k=0}^{n}\binom{n}{k}Q_k^{2\tau} . \tag{5.11}$$

For a fixed continued-fraction length $k$ and fixed Farey level $n$, the denominator of $[a_1,a_2,\ldots,a_k]$ is equal to or less than $Q_k$, the denominator of $[a,a,\ldots,a]$, $a=(n+2)/k$. Asymptotically, $Q_k$ approaches the $k$th power of the larger root of $\sigma^2-\sigma/x-1=0$,

$$\begin{aligned}&Q_k\to\sigma(x)^{nx},\ x=k/n\\ &\sigma(x)=\frac{1+(1+4x^2)^{1/2}}{2x} ,\end{aligned} \tag{5.12}$$

and (5.11) can be approximated by the integral (here we use the Stirling approximation)

$$\begin{aligned}&Z_n(\tau)=\int_0^1 dx\left[\frac{n}{2\pi x(1-x)}\right]^{1/2}e^{t[f(x)+\tau\mu(x)]} ,\\ &f(x)=s(\mu(x))=-\frac{x\ln x+(1-x)\ln(1-x)}{\ln 2} ,\\ &\mu(x)=\frac{2x\ln[\sigma(x)]}{\ln 2} .\end{aligned} \tag{5.13}$$

$q(\tau)$ is given by the saddle point estimate $q(\tau)=f(x)+\tau\mu(x)$, where $x$ is the solution of the extremal condition

$$\begin{aligned}0&=f'(x)+\tau\mu'(x)\\ &=\ln\frac{1-x}{x}+2\tau\left[\ln(\sigma)-\frac{1}{2x\sigma-1}\right] .\end{aligned} \tag{5.14}$$

The saddle point dominates the integral (5.13) as long as $f(x)+\tau\mu(x)>0$. At $\tau=-\hat{D}_H$

$$\begin{aligned}&0=f(x)+\hat{D}_H\mu(x)=f'(x)+\hat{D}_H\mu'(x)\\ &\hat{D}_H=0.8031\ldots,\quad \mu_H=1.2009\ldots ,\end{aligned} \tag{5.15}$$

a *first-order* phase transition takes place, and for $\tau<0$ the integral (5.13) is dominated by the $x\to 0$ (harmonic) end. This phase transition is shown in Fig. 4(b). In Fig. 4(d) we plot $s(\mu)$. As it is easily checked, $s(\mu)$ cannot be convex, as it has slope $\frac{1}{2}$ at $\mu\to 0$. The scaling spectrum $S(\mu)$ (3.9) is the convex envelope of $s(\mu)$ computed from (5.13), and the straight segment from $\mu=0$ to $\mu_H$ implies a first-order phase transition.

$\hat{D}_H$ in (5.15) is an improved lower bound to the Hausdorff dimension. Similarly, from the fact that $s(\mu(x))$ is maximum at $x=\frac{1}{2}$, we obtain a lower bound on the information dimension for the Farey model

$$\begin{aligned}&\mu_I<\mu(\tfrac{1}{2})=\frac{\ln(1+\sqrt{2})}{\ln 2}=1.2715\ldots ,\\ &D_I=\frac{1}{\mu_I}>0.7864\ldots .\end{aligned} \tag{5.16}$$

However, for the Farey model a much better lower bound on $D_I$ is obtained by substituting $\ln Q$ with $\ln\bar{Q}$ in (3.15). $\bar{Q}$ grows as $3^n$ (see Appendix B), so

$$D_I>\frac{\ln 2}{2\ln(\frac{3}{2})}=0.8547\ldots . \tag{5.17}$$

Numerical methods of Sec. V B yield

$$D_I=0.874\,716\,307\ldots \quad \text{(Farey model)} , \tag{5.18a}$$

$$D_I=0.795\,08\ldots \quad \text{(critical maps)} . \tag{5.18b}$$

The bound (5.15) is not a significant improvement of the simple golden-mean bound (5.9), and is still far from $D_H=1$, but we have included the preceding exercise to illustrate how existence of a concave segment in the interval-counting function $s(\mu)$ implies a first-order phase transition. While we are not able to evaluate analytically $s(\mu)$ either for the Farey model or for the critical circle maps, we offer the following argument that $s(\mu)$ starts out concave at the $\mu_{\min}$ end.

$\mu_{\min}$ arises from the smallest denominator on the $n$th Farey level, $Q_{[n+2]}=n+2$. Other denominators that are proportional to $n$ in the $n\to\infty$ limit arise from continued fractions of form

$$Q_{[a_1a_2\cdots a_r,n,b_s\cdots b_2b_1]}\to Q_{[a_1a_2\cdots a_r]}nQ_{[b_1b_2\cdots b_s]}+\text{const},\quad a_i,b_i\ll n, \tag{5.19}$$

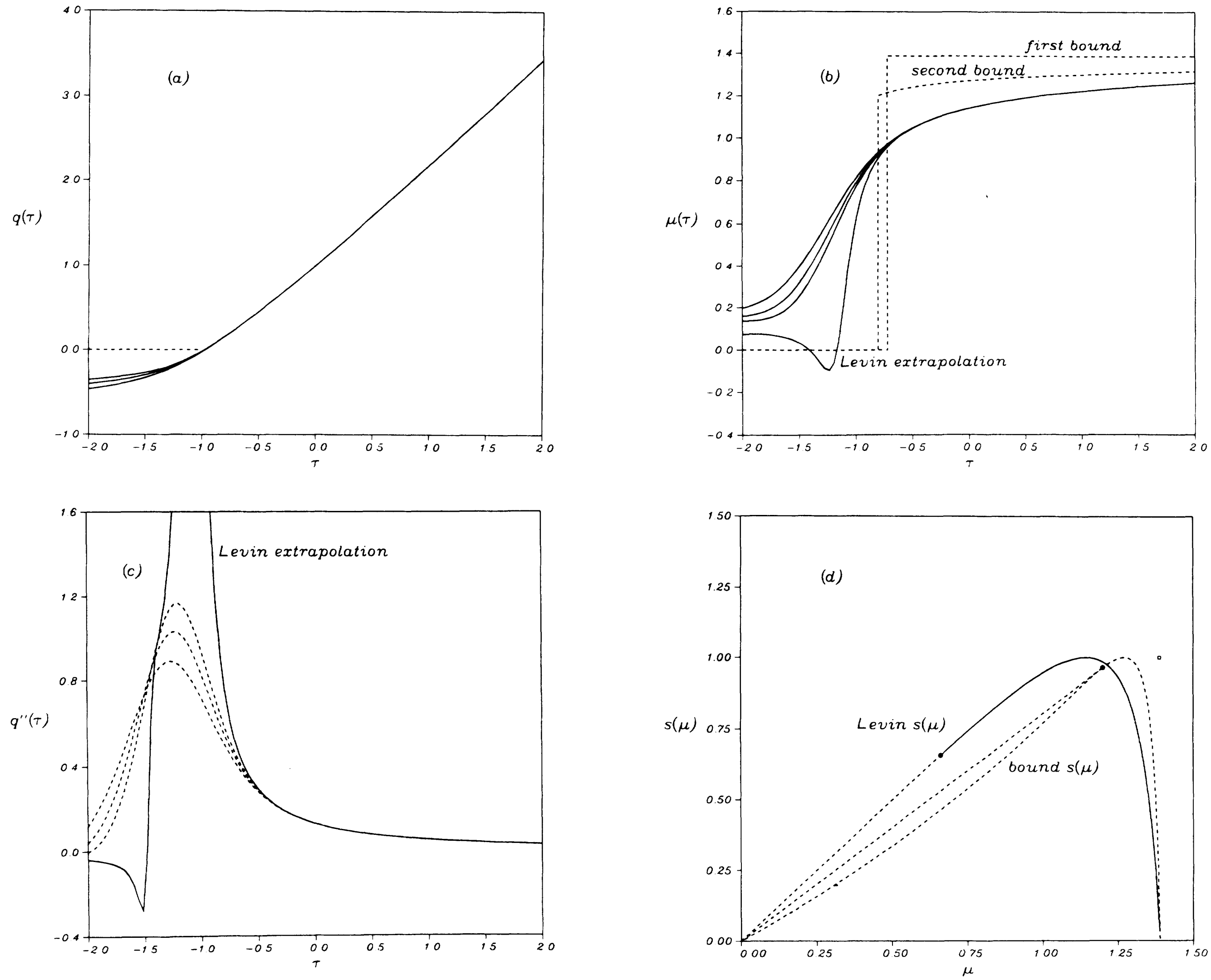


FIG. 4. Farey model (5.4) thermodynamic functions, with convergence accelerated by use of (3.17), computed for Farey levels $n=13$, 15, and 17. For non-negative half-integer $\tau$, $q(\tau)$ is known analytically (Appendix B). The Farey level critical circle maps thermodynamic functions behave qualitatively in the same way and are not plotted here. (a) $q_t(\tau)$. The asymptotically correct $q(\tau)$ equals zero for all $\tau \leq -1$; Hausdorff dimension $\tau=-D_H=-1$ is the phase transition point. (b) The scaling exponent $\mu_t(\tau)=q_t'(\tau)$. For $\tau \rightarrow \infty$, $\mu(\tau)$ tends to $\mu_{\max}=2\ln\rho/\ln 2=1.3888\ldots$ . $\mu_t(0)$ converges geometrically to $\mu_I=1.1432\ldots$ . Slow convergence of $\mu_t(-1)\rightarrow 0$ is manifest. First bound is the golden-mean bound (5.9); second bound is from (5.14). (c) $q_t''(\tau)$ gives a numerical indication of the first-order phase transition at $\tau=-1$. The Levin $u$-transformation estimate from levels $n=9-13$ peaks at $q_t''(-1.05)\simeq 9.6$. (d) The scaling spectrum $S_t(\mu)$, calculated from (3.9), converges geometrically for $\tau>0$, but for $\mu<\mu_I$ the convergence is poor, and $S(\mu)$ is estimated by the Levin $u$ transformation; $D_H$ is the slope of $S(\mu)$ in the $\mu \rightarrow 0$ limit, and the dashed straight line segment indicates a first-order phase transition. The bound (5.9) approximates $S(\mu)$ by two points: the origin and the golden-mean scaling, indicated by a square. The refined bound (5.14) has a concave $s(\mu)$ (lower line), with a first-order phase transition straight segment in $S(\mu)$.

The sums $\sum_i a_i$ and $\sum_i b_i$ are not constrained and the number of denominators with $Q=kn$ is in the $n \rightarrow \infty$ limit given by

$$N_k=\sum_{d \mid k} \phi(d)\phi(k/d), \quad d=1,2,\ldots,k \ . \tag{5.20}$$

Even though we have originally introduced[14] the Farey levels partitioning in order to ensure uniform convergence to the asymptotic limits, the counting of the harmonic intervals is just as "number-theoretically noisy" as the Farey series partitioning. In the $k \rightarrow \infty$, $k \ll n$, limit the number of denominators $Q=kn$ is related to $d(k)$, the number of divisors of $k$,

$$N_k \approx \sum_{d \mid k} \frac{6}{\pi^2} d \frac{6}{\pi^2} \frac{k}{d} \approx k d(k) \ .$$

As $d(k)$ grows slower[35] than any power of $k$, close to $\mu_{\min}$ the scaling exponents and $s(\mu)$ for the Farey model are related by

$$\mu_k = \mu_{\min} + \frac{2\ln k}{n \ln 2} ,$$
$$s(\mu_k) = \mu_{\min} + \tfrac{1}{2}(\mu_k - \mu_{\min}) . \quad (5.21)$$

Hence, in the $k \ll n$, $k \to \infty$, limit the initial slope of $s(\mu)$ is given by

$$\frac{ds(\mu_{\min})}{d\mu} = \tfrac{1}{2} . \quad (5.22)$$

As for the Farey model $D_H = 1$, and $D_H$ is the slope of the tangent of $s(\mu)$ passing through the origin, $s(\mu)$ starts out concave at $\mu_{\min}$. However, as the range of its applicability shrinks toward $\mu = 0$ with increasing $n$, this argument is not sufficient to establish the existence of a finite gap in $\mu(\tau)$ in the $n \to \infty$ limit.

The same counting argument applies to the critical maps as well. The only difference is that the $Q$ dependence is now given by (2.3), and the initial slope at $s(\mu_{\min})$ is $\frac{1}{3}$. This is smaller than $D_H = 0.87\ldots$, and again we expect a first-order phase transition.

Such arguments are further strengthened by numerical computations. Both the numerical convergence acceleration methods of Sec. V B and the finite scaling methods applied to $S''(\mu)$ dependence[18] on the level $n$ consistently yield a finite gap in $\mu$ at the phase transition; for example, they indicate that for the Farey model $\mu(-D_H + \epsilon) \to 0.64 \pm 0.03$. Still, recent studies indicate that this phase transition is not of a first order, but logarithmic of infinite order,[8] and the failure of our numerical and heuristic arguments serves as a warning of how delicate such phase transitions can be.

### B. Numerical results

The convergence of truncated-sum approximations to the arithmetic functions such as a $\zeta$ function is notoriously slow, and no accurate estimate of $D_H$ can be obtained by a mere increase in the number of levels used in the computation (note that computation time grows exponentially fast with the Farey level). As it is unlikely that tractable integral representations for the sums (1.2) can be obtained, the standard Riemann function evaluation methods are inapplicable, and we are forced to resort to the logarithmic convergence acceleration algorithms.[39–42] Their basis is heuristic, and performance uncertain. Fortunately, for the Farey model (5.4) we know the Hausdorff dimension: $D_H = 1$. This enables us to test and develop confidence in the performance of such algorithms.

We estimate the asymptotic $D$ by two methods: the Levin's $u$ transformation[40] and the alternating $\epsilon$ algorithm.[41] The $u$ transformation performs remarkably well on all test problems: for example, for the first six Farey levels (5.4), the $u$ transformation yields $D_H = 1$ to five significant digits. The alternating $\epsilon$-algorithm performance is consistently poorer by several significant digits, and more unstable. Acceleration convergence algorithms presume smooth convergence towards the limit. In particular they are inapplicable to the "Euler noisy" Farey series sums of Sec. IV, while they work extremely well applied to the Riemann $\zeta(s)$ function.

We compute the $n$th level Hausdorff dimension from (3.17) taking either successive Farey levels,[43]

$$1 = Z_{n+1}(D_n)/Z_n(D_n) \quad (5.23)$$

or the Hentschel-Procaccia[28] definition,

$$1 = Z_{n+1}(D_n^{\mathrm{HP}})/Z_2(D_n^{\mathrm{HP}}) . \quad (5.24)$$

(The $n = 2$ sum consists just of the single $l$ covering interval between 0/1 and 1/2.) We use these two definitions of $D_n$ and the two acceleration algorithms to estimate $D_H$ for the Farey model (5.4), the critical sine map (2.1), the critical cubic map (2.2), and the family of sine maps[10] with an additional term $-a \sin^3(2\pi x)$ added to the right side of (2.1). The numerical results are summarized in Figs. 5–7 and Table I.

Finite level estimates of $D$ for the cubic map (2.2) do not converge as smoothly as for the sine map, and the accelerated convergence estimates appear unreliable. In particular, it can be shown that in this case Levin's $u$ transformation converges to a local minimum in $D$ at finite $n$, rather than to the asymptotic $n$. The speed and the stability of the Levin $u$ transformation is illustrated by Fig. 7. However, this numerical stability is deceptive, as for the other maps in the sine-family one obtains a variety of similarly stable, though mutually different estimates of Hausdorff dimension.

### VI. CONCLUSIONS

We have investigated the set of irrational windings in two distinct thermodynamic formulations: the Farey

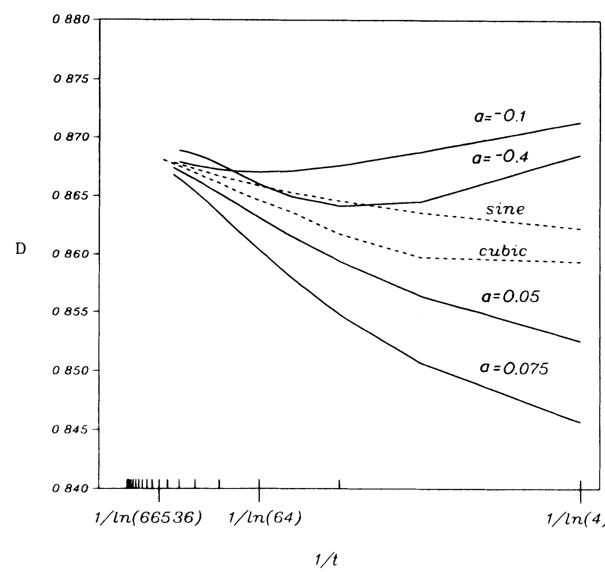


FIG. 5. Finite-$N$ estimates of the Farey tree levels' thermodynamics Hausdorff dimension (5.24), for the sine map (2.1), the cubic map (2.2), and the family of sine maps of Ref. 10, Eq. (4.1), with $a = -0.4$, $-0.1$, 0.05, and 0.075, for Farey levels with $N \leq 32\,768$. While quite stable, these estimates are far from the asymptotic $D_H$ compared to Farey series estimates, Fig. 3.

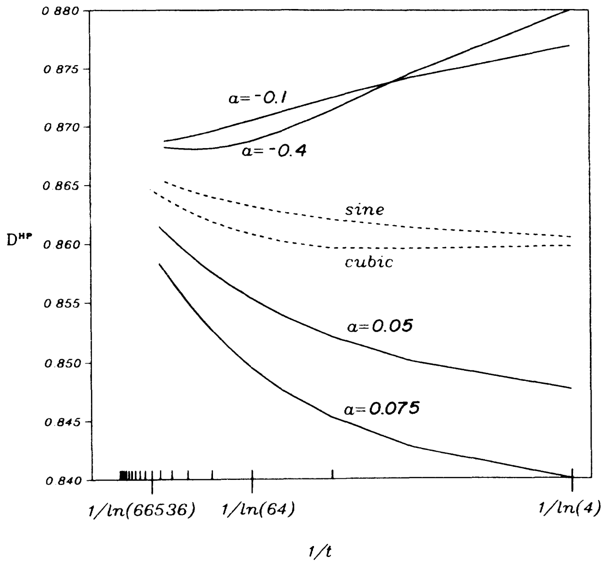


FIG. 6. Same as in Fig. 5, except that $D_N^{\rm HP}$ is computed from (5.25).

*series*[44] (all mode lockings with cycle lengths up to $Q$) and the Farey *levels*[45] ($2^n$ mode lockings on binary Farey tree). The analytic results for a number-theoretic model are helpful in elucidating the "thermodynamics" of the nontrivial critical maps, which we investigate here numerically.

The fractal set discussed here, the set of all parameter values corresponding to irrational windings, has no "natural" measure: for each distinct partitioning of the set of all rationals, we take the probability to be uniform. The

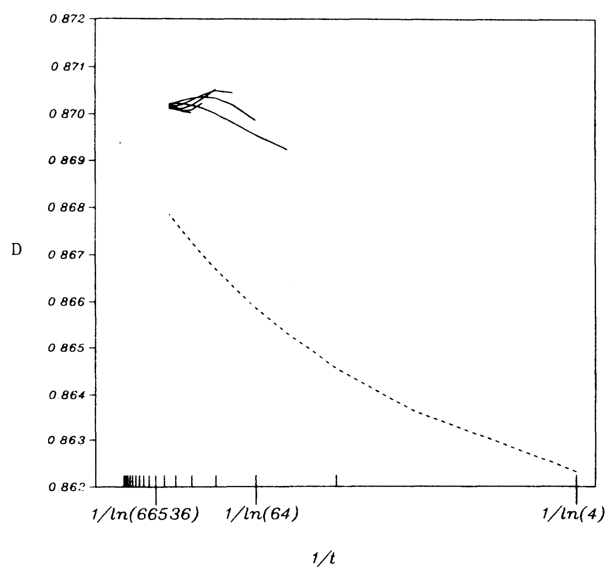


FIG. 7. Finite-$N$ estimates for the sine map, from Fig. 5, compared with 2,3, . . . , 10 term Levin estimates fo the asymptotic $D_H$. The estimates are numerically very stable (see Table I).

TABLE I. Levin $u$-transformation estimates of the Hausdorff dimension for Farey levels thermodynamics computed from finite-$D$ estimates (5.24) and $D_{\rm HP}$ estimates (5.25). Numbers in parentheses indicate the variation in the last digit among all Levin transforms on the last five raw $D_n$ values. Number of distinct mode-locked intervals used in the evaluation of the dimension varies from 4096 to 32 768.

| | Hausdorff dimension estimates | |
|---|---|---|
| map | $D_{\rm HP}$ | $D$ |
| sine, $a=-0.4$ | 0.86 805(4) | 0.8693(4) |
| sine, $a=-0.1$ | 0.8686(1) | 0.8671(8) |
| sine, Eq. (2.1) | 0.8705(2) | 0.8701(1) |
| sine, $a=0.005$ | 0.871(2) | 0.870(1) |
| sine, $a=0.075$ | 0.871(2) | 0.870(1) |
| cubic, Eq. (2.2) | 0.864(5) | 0.868(1) |

thermodynamic functions $q(\tau)$ and $S(\mu)$ are *different* for each distinct partitioning. The only point they have in common is the Hausdorff dimension, which does not depend on the choice of measure.

The Farey *series* thermodynamics turns out to be an excellent method for computing the Hausdorff dimension, but is "number-theoretically" noisy, and does not seem amenable to renormalization treatments. In the Farey *level* partitioning of rationals, $D_H$ plays a double role—it is both the Hausdorff dimension and the location of the phase transition. The second property is the cause of a very slow convergence of $D_n$ toward the asymptotic limit $D_H$, and, despite the claims of Refs. 46 and 21, makes $D_H$ experimentally inaccessible if data is partitioned into Farey levels. Instead, we suggest that from the Farey levels data the experimentalists extract the information dimension (3.14) which, for critical circle maps with cubic inflection, converges quickly (geometrically) to $D_I=0.795\,08\ldots$ .

Our main result is the discovery of phase transitions. They were not anticipated, their identification requires a careful examination of the thermodynamic sums, and while they were overseen in earlier investigations of the same mode-locking problem,[26] we find them in every case that we have studied. To date we are not aware of any other approach that can lead to such a strong conclusion. Not only do the phase transitions enrich our conceptual vocabulary, they should enrich our experimental vocabulary as well. Their measurement should result in new tests of scaling theories of nonlinear systems.

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## APPENDIX A

Here we collect some number-theory results[35] used in this paper.

*Definition 1.* The Euler function $\phi(Q)$ is the number of integers not exceeding and relatively prime to $Q$. For example,

$\phi(1)=1,\phi(2)=1,\phi(3)=2,\ldots,\phi(12)=4,\phi(13)=12,\ldots$ .

*Definition 2.* The *Farey series*[35] of order $Q$ is the monotonically increasing sequence of all irreducible rationals between 0 and 1 whose denominator does not exceed $Q$. Thus $P_i/Q_i$ belongs to $F_Q$ if $0<P_i\le Q_i\le Q$ and $(P_i\mid Q_i)=1$. For example

$$F_5=\{\tfrac{1}{5},\tfrac{1}{4},\tfrac{1}{3},\tfrac{2}{5},\tfrac{1}{2},\tfrac{3}{5},\tfrac{2}{3},\tfrac{3}{4},\tfrac{4}{5},\tfrac{1}{1}\} .$$

If $P_{i-1}/Q_{i-1}$ and $P_i/Q_i$ are consecutive terms of $F_Q$, then

$$P_iQ_{i-1}-P_{i-1}Q_i=1 . \tag{A1}$$

The number of terms in the Farey series $F_Q$ is given by

$$\Phi(Q)=\sum_{n=1}^{Q}\phi(Q)=\frac{3Q^2}{\pi^2}+O(Q\ln Q) . \tag{A2}$$

As $\phi(Q)$ is an irregularly oscillating function of $Q$, the asymptotic limits are not approached smoothly. We refer to this fact as the "Euler noise."

Let $l(Q_{i-1},Q_i)=P_i/Q_i-P_{i-1}/Q_{i-1}=1/(Q_{i-1}Q_i)$ be the $i$th Farey arc. Then the Farey arc partition sum (1.3) is given by

*Theorem 1 (Hall and Tenenbaum[34]).* For $\tau\neq-1$, $\tau>-2$,

$$Z_{Q^2}(\tau)=\left[\frac{6}{\pi^2}+O(Q^{\eta}\ln^3Q)\right]\frac{1-\dfrac{\Gamma^2(2+\tau)}{\Gamma(3+2\tau)}}{(1+\tau)^2}Q^{2(1+\tau)} , \tag{A3}$$

where

$$\eta=\begin{cases}\dfrac{(2+\tau)^2}{6}, & -2<\tau\le-1\\[4pt] \dfrac{2+\tau}{2(2-\tau)}, & -1\le\tau\le0\\[4pt] \dfrac{1}{2-\tau}, & 0\le\tau\le1\\[4pt] 1, & 1\le\tau .\end{cases}$$

*Theorem 2 (Hall[32]).* Hall's estimate gives

$$Z_{Q^2}(-2)=\frac{12}{\pi^2}\left[\ln Q+\gamma+\tfrac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\right]Q^{-2}+O\left[\frac{\ln^2Q}{Q^3}\right] , \tag{A4}$$

where $\gamma$ is the Euler constant (for a sharper estimate, see Ref. 33).

*Theorem 3 (Hall[32]).* For integer $\tau\le-3$

$$Z_{Q^2}(\tau)=\frac{2\zeta(-\tau-1)}{\zeta(-\tau)}Q^{\tau}+O(Q^{\tau-1}\ln^{\theta}Q) , \tag{A5}$$

where $\theta=1$ for $\tau=-3$ and $\theta=0$ for $\tau<-3$.

*Definition 3.* The *n*th *Farey tree level* $T_n$ is the monotonically increasing sequence of those continued fractions $[a_1,a_2,\ldots,a_k]$ whose entries $0<a_i\le n$, $i=1,2,\ldots,k-1$, $1<a_k\le n+2$ add up to[48]

$$\sum_{i=1}^{k}a_i=n+2 . \tag{A6}$$

For example,

$$T_2=\{[4],[2,2],[1,1,2],[1,3]\}=(\tfrac{1}{4},\tfrac{1}{5},\tfrac{3}{5},\tfrac{3}{4}) .$$

The number of terms in $T_n$ is $2^n$. Each rational in $T_{n-1}$ has two "daughters" in $T_n$, given by[8]

$$\begin{array}{ccc} & [a_1,a_2,\ldots,a_k] & \\ \swarrow & & \searrow \\ [a_1,a_2,\ldots,a_k+1] & & [a_1,a_2,\ldots,a_k-1,2]\end{array} . \tag{A7}$$

Iteration of this rule places all rationals on a binary tree, labeling each by a unique binary label.[8] The smallest denominator in $T_n$ is

$$[n-2]=\frac{1}{n-2} , \tag{A8}$$

and the largest denominator is a Fibonacci number

$$[1,1,\ldots,1,2]=\frac{F_{n+1}}{F_{n+2}} ,$$

$$F_0=0,\quad F_1=1,\quad F_{n+1}=F_n+F_{n-1}\propto\rho^{-n} , \tag{A9}$$

where $\rho$ is the golden-mean ratio

$$\rho=\frac{1+\sqrt{5}}{2}=1.618\,03\ldots . \tag{A10}$$

## APPENDIX B

The Farey model sum (5.4) can be evaluated exactly for $\tau=k/2$, $k$ non-negative integer. $Z_n(0)=2^n$, trivially. It is also easy to check that[48] $Z_n(\frac{1}{2})=\sum_iQ_i=2\times3^n$. More surprisingly, $Z_n(\frac{3}{2})=\sum_iQ^3=54\times7^{n-1}$. Such "sum rules" are consequence of the fact that the denominators on a given level are Farey sums of denominators on preceding levels. For example, the denominators on a Farey subtree bracketed by a pair of rationals $P/Q$ and $P'/Q'$ are generated by the Farey mediant rule[35]

$$\begin{array}{ccccc} Q & & & & Q' \\ & & Q+Q' & & \\ & 2Q+Q' & & Q+2Q' & \\ 3Q+Q' & 3Q+2Q' & \cdots & & \end{array} \tag{B1}$$

and the corresponding thermodynamic subsums are given by

$$\tilde{Z}_1(k/2)=(Q+Q')^k \,;$$
$$\tilde{Z}_2(k/2)=(2Q+Q')^k+(Q+2Q')^k \,;$$
$$\vdots \tag{B2}$$
$$\tilde{Z}_j(k/2)=Q^k \sum_{m=0}^{k} c_{km}\chi^m, \quad \chi=Q'/Q \,.$$

For any finite integer $k$, $c_{km}$ are computable combinations of Farey numbers and binomial coefficients, and the dependence of $\tilde{Z}_j(k/2)$ on $\chi^m$ can be eliminated in favor of $\tilde{Z}_{j-1}, \tilde{Z}_{j-2}, \ldots$ . This yields recursion relations in $\tilde{Z}_j$. Such recursion relations[47] for $k=0$ to 7 are summarized in Table II.

The same results have been independently obtained by functional methods in Ref. 8.

TABLE II. Analytic evaluation of the Farey model $Z_n(\tau)$ for $\tau=0,\frac{1}{2},1,\ldots,\frac{7}{2}$.

| $\tau$ | $\lim_{n\to\infty} \frac{Z_{n+1}(\tau)}{Z_n(\tau)}$ | Recursion relation $Z_n(\tau)$ |
|---|---|---|
| 0 | 2 | $2Z_{n-1}$ |
| $\frac{1}{2}$ | 3 | $3Z_{n-1}$ |
| 1 | $(5+\sqrt{17})/2$ | $5Z_{n-1}-2Z_{n-2}$ |
| $\frac{3}{2}$ | 7 | $7Z_{n-1}$ |
| 2 | $(11+\sqrt{113})/2$ | $10Z_{n-1}+9Z_{n-2}-2Z_{n-3}$ |
| $\frac{5}{2}$ | $7+4\sqrt{6}$ | $14Z_{n-1}+47Z_{n-2}$ |
| 3 | 26.202 49. . . | $20Z_{n-1}+161Z_{n-2}+40Z_{n-3}-Z_{n-4}$ |
| $\frac{7}{2}$ | 41.018 3. . . | $29Z_{n-1}+485Z_{n-2}+327Z_{n-3}$ |

## APPENDIX C

A variety of methods for accelerating logarithmic convergence (5.10) is reviewed in Ref. 39. Given a finite truncation of an infinite sum

$$s_n=\sum_{k=1}^{n} a_k=\sum_{k=1}^{\infty} a_k - R_n \,, \tag{C1}$$

where logarithmic convergence means that

$$R_n/R_{n-1}\to 1 \tag{C2}$$

one attempts to make good estimates of $R_n$. We use two methods.

(1) Vanden Broeck and Schwartz transformation:[41] given $s_1,s_2,s_3,\ldots,s_n$ estimates

$$s_n=[n,0]\equiv \sum_{k=1}^{n} a_k, \quad [n,-1]\equiv 0 \tag{C3}$$

the $[N,M+1]$ estimate is given by

$$\frac{1}{[N,M+1]-[N,M]}+\frac{\alpha_m}{[N,M-1]-[N,M]} =\frac{1}{[N+1,M]-[N,M]}+\frac{1}{[N-1,M]-[N,M]} \,. \tag{C4}$$

$\alpha=0$ corresponds to the Shanks transformation, $\alpha=1$ to a Padè transformation, and $\alpha=-1$ accelerates logarithmic convergence and is used here.

(2) Levin[40] transformation: given estimates $s_n, s_{n+1},\ldots,s_{n+k}$, the $s_\infty \simeq u_{nk}$ estimate is given by

$$u_{nk}=\frac{\displaystyle\sum_{j=0}^{k}(-1)^j \binom{k}{j}\left[\frac{n+j}{n+k}\right]^{k-2}\frac{s_{n+j}}{a_{n+j}}}{\displaystyle\sum_{j=0}^{k}(-1)^j \binom{k}{j}\left[\frac{n+j}{n+k}\right]^{k-2}\frac{1}{a_{n+j}}} \,. \tag{C5}$$

In all our tests on analytically known sums, Levin estimates are better than Vanden Broeck and Schwartz by several orders of magnitude. For example, from $\sum_{n=1}^{10} n^{-2}$, $\zeta(2)$ is estimated by Levin method to accuracy $10^{-9}$. A truncated sum would need $10^9$ terms for the same accuracy.

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