Photon-number-state preparation in nondegenerate parametric amplification

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In nondegenerate parametric amplification of light, one pump photon is destroyed and two photons are simultaneously created, one in each of two modes with different frequencies (the signal and idler). If m signal photons are counted in time t, what is the state of the idler conditioned on this result? The projection postulate cannot be used to answer this question as photon-counting measurements are not "first-kind" measurements. We use a theory of continual quantum-counting measurements to show that in general the idler field is left in a classical mixture of number states with *at least* m quanta. In an appropriate limit this mixture reduces to an exact number state with m quanta.

I. INTRODUCTION

In the nondegenerate optical parametric amplifier, one pump photon is destroyed and two photons, one in the signal field and another in the idler field, are simultaneously created. The coupling between the fields arises from a second-order nonlinearity in the polarizability of certain crystals. This pairwise production of photons results in the conservation of the photon-number difference between the signal and idler modes in the absence of any loss. The high correlation between the signal and idler fields is responsible for the generation of a squeezed-vacuum state in the output of the device.[1-3] In this paper we answer the following question: if, in time t, m signal-field photons are counted, what is the state of the idler field at time t conditioned on this result?

The precise context in which this question is answered is as follows. We assume that the signal and idler fields are single-mode intracavity fields. Let a denote the annihilation operator for the idler field, while b denotes the annihilation operator for the signal field. These two cavity fields are coupled to a third field, the pump, assumed strong and treated classically with a complex amplitude $\mathcal{E}(t)$. The interaction Hamiltonian is

$$H_I=\hbar g[\mathcal{E}(t)a^\dagger b^\dagger+\mathcal{E}(t)^* ab] , \tag{1.1}$$

where

$$\mathcal{E}(t)=Ee^{-i(\omega_a+\omega_b)t} ,$$

with ω_a and ω_b the frequencies of the idler and signal fields, respectively, and g is a coupling constant. We assume that only photons from the b field are counted. The system is represented schematically in Fig. 1.

At first sight the answer to the question posed above is immediate and trivial. As photons are produced in pairs, if m photons are counted in the signal field, the idler field must be in a photon-number eigenstate with exactly m photons. This answer is based on a naive application of the von Neumann projection postulate as we show below. However, this postulate was intended to apply only to so-called "first-kind" measurements. These are instantaneous, arbitrarily accurate measurements made on a system which retains its identity and is not destroyed by the measurement; a highly idealized kind of measurement. An example might be a position measurement on a free particle. Photon-counting measurements are not measurements of the first kind. First, they are not instantaneous. The counter operates continuously and records the arrival of photons at random times over some time interval. Secondly, the detection of a photon necessitates its destruction and removal from the system. As far as the system is concerned photon-counting measurements must be regarded as a loss mechanism. Finally, photon-

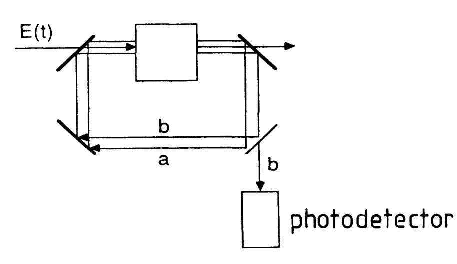

FIG. 1. Schematic plan of the photon-counting scheme discussed in the text.

counting measurements are not arbitrarily accurate. Every photoelectron counter has a quantum efficiency less than unity. The above considerations should be sufficient to cast some doubt on the applicability of the von Neumann projection postulate in this context.

Clearly what is required for a correct description of this model is a quantum theory of continual photon-counting measurements. Fortunately, such a theory is ready at hand and ideally suited to this problem.[4,5] In Sec. II, we will apply this theory to the model under consideration. In the remainder of this section we will apply the state reduction postulate to the model. While, for the reasons discussed above, this is unlikely to describe real measurements, it is relatively simple and enables us to introduce some important concepts and useful techniques in a more familiar setting.

We first state the projection postulate. Let $\hat{\rho}$ be the density operator describing the state of an ensemble of identically prepared systems. Suppose an arbitrarily accurate, instantaneous measurement of some physical quantity, represented by an operator $\hat{A}$, is made on each element of the ensemble. The possible results of the measurement are the eigenvalues $\{a\}$ of the operator $\hat{A}$. We further assume these eigenvalues are nondegenerate. Consider a partition of the ensemble according to whether or not the result of a measurement was some particular eigenvalue a. The density operator describing the state of this subensemble is assumed to be

$$\hat{\rho}^{(a)}=\frac{|a\rangle\langle a|\hat{\rho}|a\rangle\langle a|}{P(a)} = |a\rangle\langle a| \ , \tag{1.2}$$

where $P(a)\equiv \mathrm{tr}(\hat{\rho}|a\rangle\langle a|)$ is the probability to obtain the result a. This is the projection postulate. It enables us to specify the post-measurement state of a system conditioned on the measurement result, at least for measurements of the first kind. We now apply this postulate to the nondegenerate parametric amplifier.

Assume both modes are initially in the vacuum state. In the interaction picture the state of the total system at time t is

$$\hat{\rho}(t)=e^{-is(a^\dagger b^\dagger+ab)}|0\rangle\langle 0|e^{is(a^\dagger b^\dagger+ab)} \ , \tag{1.3}$$

where $|0\rangle\equiv|0\rangle_a\otimes|0\rangle_b$ and $s\equiv gEt$. If m photons are recorded in a first-kind measurement on mode b at time t, the state of mode a conditioned on this result is

$$\hat{\rho}_a^{(m)}(t)=[P(m,t)]^{-1}\mathrm{tr}_b[|m\rangle_b\langle m|\hat{\rho}(t)] \ , \tag{1.4}$$

where $|m\rangle_b\langle m|$ is the projector onto the signal-field number state $|m\rangle_b$, while $P(m,t)$ is the probability to record m quanta and is given by

$$P(m,t)=\mathrm{tr}[|m\rangle_b\langle m|\hat{\rho}(t)] \ . \tag{1.5}$$

The symbol tr_b denotes a partial trace over states of the signal field, while tr denotes a trace over both fields. To be precise it should be noted that Eq. (1.4) is a generalization of the nondegenerate projection postulate in Eq. (1.2). The projector in this case is actually $\hat{1}\otimes|m\rangle_b\langle m|$.

To explicitly calculate $\hat{\rho}_a^{(m)}(t)$ we will use a c-number formulation. Let $|\alpha\rangle_a$ and $|\beta\rangle_b$ denote coherent states for the idler and signal fields, respectively. The Q function for the total system at time t is then defined as

$$Q(\alpha,\beta,t)\equiv\langle\alpha,\beta|\hat{\rho}(t)|\alpha,\beta\rangle \ , \tag{1.6}$$

where $|\alpha,\beta\rangle\equiv|\alpha\rangle_a\otimes|\beta\rangle_b$. Knowledge of the Q function enables $\hat{\rho}(t)$ to be reconstructed uniquely.[6] We further define the "unnormalized" Q function for the post-measurement state $\hat{\rho}_a^{(m)}(t)$ by

$$\tilde{Q}_a^{(m)}(\alpha,t)\equiv {}_a\langle\alpha|\mathrm{tr}_b[|m\rangle_b\langle m|\hat{\rho}(t)]|\alpha\rangle_a \ . \tag{1.7}$$

Using the completeness relation for coherent states of mode b (Ref. 6) this may be written

$$\tilde{Q}_a^{(m)}(\alpha,t)=\int\frac{d^2\beta_1}{\pi}\int\frac{d^2\beta_2}{\pi}\langle m|\beta_1\rangle\langle\beta_2|m\rangle \times\langle\alpha,\beta_1|\hat{\rho}(t)|\alpha,\beta_2\rangle \ . \tag{1.8}$$

The off-diagonal matrix elements $\langle\alpha,\beta_1|\hat{\rho}(t)|\alpha,\beta_2\rangle$ are found by analytic continuation in β of $Q(\alpha,\beta,t)$.[6] Since the time evolution induces a linear canonical transformation of a and b (Ref. 7) $Q(\alpha,\beta,t)$ directly determines antinormally ordered moments of a and b,[6] the Q function must be a Gaussian and may be constructed as

$$Q(\alpha,\beta,t)=\cosh^{-2}(s)\exp[-|\alpha|^2-|\beta|^2 -i\tanh s(\alpha\beta-\alpha^*\beta^*)] \ . \tag{1.9}$$

Substituting this into Eq. (1.8) with analytic continuation in β the integrals may be performed to give

$$\tilde{Q}_a^{(m)}(\alpha,t)=\frac{\tanh^{2m}s}{\cosh^2 s}\frac{|\alpha|^{2m}}{m!}e^{-|\alpha|^2} \ . \tag{1.10}$$

Thus the normalized Q function for the post-measurement state of mode a is

$$Q_a^{(m)}(\alpha,t)=\frac{|\alpha|^{2m}}{m!}e^{-|\alpha|^2} \ . \tag{1.11}$$

This Q function corresponds to the state

$$\hat{\rho}_a^{(m)}(t)=|m\rangle_a\langle m| \ . \tag{1.12}$$

This result has also been obtained by Yamamato *et al.*[8]

After a first-kind measurement of m quanta in mode b, mode a is found to be in the eigenstate $|m\rangle_a$ of the photon-number operator. This result is independent of the coupling strength g between the two modes (unlike previous measurement models[9,10]) and is a consequence of the perfect correlation in the photon numbers in each mode. If photon-counting measurements are treated as first-kind measurements, the projection postulate does indeed confirm one's intuition for the post-measurement state of mode a.

Before proceeding to a more realistic photon-counting theory an important property of the parametric amplifier should be noted. The reduced state of either mode in the nondegenerate parametric amplifier is a thermal state with mean photon number

$$\bar{n}_a(t)=\bar{n}_b(t)=\sinh^2 s \ .$$

This result has been noted by a number of authors,[11-13] but may easily be deduced from Eqs. (1.7) and (1.10). The reduced state of mode a is defined by

$$\hat{\rho}_a(t)=\mathrm{tr}_b[\hat{\rho}(t)] \ . \tag{1.13}$$

Thus the corresponding Q function is given by

$$Q_a(\alpha,t)=\sum_{m=0}^{\infty}\tilde{Q}_a^{(m)}(\alpha,t) \ . \tag{1.14}$$

This gives a Q function corresponding to the state

$$\hat{\rho}_a(t)=\sum_{m=0}^{\infty}P(m)|m\rangle\langle m| \ ,$$

where

$$P(m)=(1+\bar{n}_a)^{-1}\left[\frac{\bar{n}_a}{1+\bar{n}_a}\right]^m$$

and $\bar{n}_a=\sinh^2 s$.

The above calculation shows that the reduced state of the idler field is a thermal state with a mean number of quanta of $\sinh^2 s$. The same result holds for the reduced state of the signal field. The nondegenerate parametric amplifier directly generates thermal states in either the signal or the idler mode. In what sense then can one speak of this device as producing a photon-number state?

The quantum-mechanical state of a system contains all the information available to an observer concerning the procedure used to prepare the system. If no measurements are made on either the signal or idler field, the state of each field is a thermal state. If, however, measurements are made on the signal field, the state of the idler field must be conditioned on the additional information contained in the results of such measurements. The results of further measurements made on the idler field are likewise conditioned by the result of the photon-counting measurement made on the signal.

II. QUANTUM THEORY OF CONTINUAL QUANTUM-COUNTING MEASUREMENTS

The theory of continual photon-counting measurements was developed by Davies[4] and Srinivas and Davies.[5] In this section we present a brief summary of as much of this theory as is relevant to our discussion. An application of this theory to other measurement models will be found in Refs. 9 and 10.

Consider a single cavity mode with annihilation operator b. Assume the cavity is empty and that initially the field state is designated $\hat{\rho}(0)$. Further, assume that photons are lost through one end of the cavity and that every photon lost is counted. As far as the description of the cavity mode is concerned the model is equivalent to a damped simple harmonic oscillator for which the state of the system obeys the master equation

$$\frac{d\hat{\rho}}{dt}=-\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}]+\frac{\gamma}{2}(2b\hat{\rho}b^\dagger-b^\dagger b\hat{\rho}-\hat{\rho}b^\dagger b) \ . \tag{2.1}$$

Equation (2.1) essentially defines the class of continual quantum-counting measurements we wish to consider. Real systems would of course only satisfy an equation of this form within various approximations.[14] In optics, however, cavity dynamics of the sort described by Eq. (2.1) is easily achieved.

If m photons are counted in time t the state of the cavity mode "conditioned" on this result is

$$\hat{\rho}^{(m)}=\frac{\mathcal{N}^{(m)}(t)\hat{\rho}(0)}{P(m,t)} \ , \tag{2.2}$$

where

$$\mathcal{N}^{(m)}(t)\hat{\rho}\equiv\int_0^t dt_m\int_0^{t_m}dt_{m-1}\cdots\int_0^{t_2}dt_1\mathcal{S}(t-t_m)\mathcal{J}\cdots\mathcal{J}\mathcal{S}(t_1)\hat{\rho} \ , \tag{2.3}$$

with

$$\mathcal{S}(t)\hat{\rho}=\exp\left[-\frac{i}{h}\hat{H}_0t-\frac{\gamma}{2}b^\dagger bt\right]\hat{\rho}\times\exp\left[\frac{i}{h}\hat{H}_0t-\frac{\gamma}{2}b^\dagger bt\right] \tag{2.4}$$

and

$$\mathcal{J}\hat{\rho}\equiv\gamma b\hat{\rho}b^\dagger \ . \tag{2.5}$$

The state $\hat{\rho}^{(m)}(t)$ is the post-measurement state given that the result of the measurement (m counts) is known. In the theory of measurement $\hat{\rho}^{(m)}(t)$ is referred to as the post-measurement state in the "selective sense."[15] The probability to detect m quanta in time t is $P(m,t)$ where

$$P(m,t)=\mathrm{tr}[\mathcal{N}^{(m)}(t)\hat{\rho}(0)] \ . \tag{2.6}$$

As all photons counted are photons lost from the cavity it is not hard to see that

$$\hat{\rho}(t)=\sum_{m=0}^{\infty}\mathcal{N}^{(m)}(t)\hat{\rho}(0) \tag{2.7}$$

must be a solution of Eq. (2.1), as is easily verified. The state given by Eq. (2.7) is referred to as the post-measurement state in the "nonselective sense."[15] It describes the unpartitioned post-measurement ensemble; that is, it describes the state of the system given that the measurement has occurred but for which the results are unknown.[15]

As an example let $\hat{\rho}(0)$ be the thermal state

$$\hat{\rho}(0)=\sum_{n=0}^{\infty}P(n)|n\rangle\langle n| \ , \tag{2.8}$$

where

$$P(n)=(1+\bar{n})^{-1}\left[\frac{\bar{n}}{1+\bar{n}}\right]^n \ . \tag{2.9}$$

We find

$$\hat{\rho}^{(m)}(t)=[P(m,t)]^{-1}\sum_{n=m}^{\infty}P(n)\begin{bmatrix}n\\m\end{bmatrix}(1-\mu)^{n-m}\mu^{m}\times|n-m\rangle\langle n-m| , \quad (2.10)$$

where

$$P(m,t)=\sum_{n=m}^{\infty}P(n)\begin{bmatrix}n\\m\end{bmatrix}(1-\mu)^{n-m}\mu^{m} , \quad (2.11)$$

and

$$\mu=(1-e^{-\gamma t}) \quad (2.12)$$

is the effective quantum efficiency for detection. Equation (2.11) is the standard result[16] for a detector with quantum efficiency μ. In the limit $\mu\to 1$ $(\gamma t\to\infty)$ $P(m,t)\to P(m)$ and $\hat{\rho}^{(m)}(t)\to|0\rangle\langle 0|$. The limit of unit quantum efficiency is the "perfect measurement" limit for photon counting. We see that the post-measurement state in this limit is certainly not given by the projection postulate (which would have suggested $\hat{\rho}^{(m)}\to|m\rangle\langle m|$).

For the thermal photon-number distribution the sum in Eq. (2.11) may be evaluated to give

$$P(m,t)=(1+\bar{n})^{-1}(\lambda\mu)^{m}[1-\lambda(1-\mu)]^{-(m+1)} , \quad (2.13)$$

where

$$\lambda\equiv\frac{\bar{n}}{1+\bar{n}} . \quad (2.14)$$

This result will be used in Sec. III to interpret the result for state reduction in the parametric amplifier. The mean number of quanta counted in time t is

$$\bar{m}_t=\sum_{m=0}^{\infty}mP(m,t)=\mu\bar{n} . \quad (2.15)$$

For unit quantum efficiency the mean count is just the mean number of quanta in the initial state.

III. PHOTON-COUNTING STATE REDUCTION IN NONDEGENERATE PARAMETRIC AMPLIFICATION

We consider a quantum-counting formula of the form discussed in Sec. II for the signal field b alone. The dynamics of the intracavity fields in the interaction picture during the continual measurement of signal photons is given by

$$\frac{d\hat{\rho}}{dt}=-i\chi[a^{\dagger}b^{\dagger}+ab,\hat{\rho}]+\frac{\gamma}{2}(2b\hat{\rho}b^{\dagger}-b^{\dagger}b\hat{\rho}-\hat{\rho}b^{\dagger}b) , \quad (3.1)$$

where $\chi\equiv gE$. If m photons are counted from mode b in time t, the state of mode a at the end of this interval, conditioned on this result is

$$\hat{\rho}_a^{(m)}(t)=\frac{\mathrm{tr}_b[\mathcal{N}^{(m)}(t)\hat{\rho}_a(0)\otimes\hat{\rho}_b(0)]}{P(m,t)} , \quad (3.2)$$

where

$$P(m,t)=\mathrm{tr}[\mathcal{N}^{(m)}(t)\hat{\rho}_a(0)\otimes\hat{\rho}_b(0)] , \quad (3.3)$$

and $\mathcal{N}^{(m)}(t)$ is given by Eqs. (2.3)–(2.5) with $H_0=\hbar\chi(a^{\dagger}b^{\dagger}+ab)$.

To proceed we will calculate the unnormalized density operator defined by

$$\hat{\Lambda}^{(m)}(t)\equiv\mathcal{N}^{(m)}(t)\hat{\rho}_a(0)\otimes\hat{\rho}_b(0) , \quad (3.4)$$

with both modes initially in the coherent states $|\alpha_0\rangle_a$ and $|\beta_0\rangle_b$. An evolution equation for $\hat{\Lambda}^{(m)}(t)$ is derived using the definition of $\mathcal{N}^{(m)}(t)$. We find

$$\frac{d\hat{\Lambda}^{(m)}}{dt}(t)=\mathcal{J}\hat{\Lambda}^{(m-1)}(t)-i\chi[a^{\dagger}b^{\dagger}+ab,\hat{\Lambda}^{(m)}(t)]-\frac{\gamma}{2}[b^{\dagger}b\hat{\Lambda}^{(m)}(t)+\hat{\Lambda}^{(m)}(t)b^{\dagger}b] , \quad (3.5)$$

with the convention $\hat{\Lambda}^{(-1)}(t)=0$.

We will not solve this equation directly but rather use it to obtain a c-number partial differential equation for the matrix elements of $\hat{\Lambda}^{(m)}(t)$ in the coherent-state basis. This is simply the unnormalized Q function

$$\tilde{Q}^{(m)}(\alpha,\beta,t)\equiv\langle\alpha,\beta|\hat{\Lambda}^{(m)}(t)|\alpha,\beta\rangle , \quad (3.6)$$

where $|\alpha,\beta\rangle\equiv|\alpha\rangle_a\otimes|\beta\rangle_b$. Using standard rules[17] we find

$$\frac{\partial\tilde{Q}^{(m)}(\alpha,\beta,t)}{\partial t}=\mathcal{L}\tilde{Q}^{(m)}(\alpha,\beta,t)+\mathcal{K}\tilde{Q}^{(m-1)}(\alpha,\beta,t) , \quad (3.7)$$

where

$$\mathcal{L}=-i\chi\left[\alpha\frac{\partial}{\partial\beta^*}+\beta\frac{\partial}{\partial\alpha^*}+\frac{\partial^2}{\partial\alpha^*\partial\beta^*}\right]+i\chi\left[\alpha^*\frac{\partial}{\partial\beta}+\beta^*\frac{\partial}{\partial\alpha}+\frac{\partial^2}{\partial\alpha\,\partial\beta}\right]-\frac{\gamma}{2}\left[2|\beta|^2+\beta^*\frac{\partial}{\partial\beta^*}+\beta\frac{\partial}{\partial\beta}\right] , \quad (3.8)$$

$$\mathcal{K}=\gamma\left[1+|\beta|^2+\beta\frac{\partial}{\partial\beta}+\beta^*\frac{\partial}{\partial\beta^*}+\frac{\partial^2}{\partial\beta\,\partial\beta^*}\right] . \quad (3.9)$$

The procedure is to first solve for $\tilde{Q}^{(0)}(\alpha,\beta,t)$, which obeys

$$\frac{\partial\tilde{Q}^{(0)}}{\partial t}(\alpha,\beta,t)=\mathcal{L}\tilde{Q}^{(0)}(\alpha,\beta,t) , \quad (3.10)$$

and use this result for successive solutions of Eq. (3.7).

If both modes are initially in the coherent states $|\alpha\rangle_a$ and $|\beta\rangle_b$, we need to solve Eq. (3.10) subject to the initial condition

$$\tilde{Q}^{(0)}(\alpha,\beta,0)=e^{-|\alpha-\alpha_0|^2-|\beta-\beta_0|^2} . \quad (3.11)$$

The method of solution is lengthy and details will be found in Appendix A. The result is

$$\tilde{Q}^{(0)}(\alpha,\beta,t)=\exp\left[-|\alpha|^2-|\beta|^2-|\alpha_0|^2-|\beta_0|^2+if(t)(\alpha^*\beta^*-\alpha\beta)\right.$$
$$+e^{\gamma t/4}\frac{\operatorname{sech}(Kt-c)}{\operatorname{sech}c}(\alpha\alpha_0^*+\beta\beta_0^*e^{-\gamma t/2}+\text{c.c.})+\frac{\gamma t}{2}-2\ln\left[\frac{\cosh(Kt-c)}{\cosh c}\right]$$
$$\left.-if(t)(\alpha_0^*\beta_0^*-\alpha_0\beta_0)\right], \tag{3.12}$$

where

$$f(t)=\frac{\gamma}{4\chi}-\Delta\tanh(Kt-c), \tag{3.13}$$

$$K=\chi\Delta, \tag{3.14}$$

$$\Delta=\left[1+\frac{\gamma^2}{16\chi^2}\right]^{1/2}, \tag{3.15}$$

$$\tanh c=\frac{-\gamma}{4K}. \tag{3.16}$$

The solution to the mth-order equation is

$$\tilde{Q}^{(m)}(\alpha,\beta,t)=\frac{[\mathcal{P}^{(1)}(\alpha,t)]^m}{m!}\tilde{Q}^{(0)}(\alpha,\beta,t), \tag{3.17}$$

where

$$\mathcal{P}^{(1)}(\alpha,t)=|\alpha|^2g(t)+(\alpha\beta_0-\alpha^*\beta_0^*)z(t)+h(t), \tag{3.18}$$

with

$$g(t)=1-f^2(t)-e^{\gamma t/2}\left[1+\frac{\gamma}{2\chi}f(t)-f^2(t)\right], \tag{3.19}$$

$$z(t)=i\Delta(e^{-\gamma t/2}-1)e^{\gamma t/4}\operatorname{sech}(Kt-c)f(t), \tag{3.20}$$

$$h(t)=|\beta_0|^2\left[\Delta^2(1-e^{-\gamma t/2})\operatorname{sech}^2(Kt-c)-\frac{\gamma}{2\chi}f(t)\right]. \tag{3.21}$$

These rather fearsome expressions simplify somewhat for initial vacuum states ($\alpha_0=\beta_0=0$) and henceforward we will only consider this case.

Before proceeding we note that the Q function corresponding to the solution of the master equation (3.1) is given by

$$Q(\alpha,\beta,t)=\sum_{m=0}^{\infty}\tilde{Q}^{(m)}(\alpha,\beta,t), \tag{3.22}$$

which for initial vacuum states is

$$Q(\alpha,\beta,t)=N(t)\exp\{-|\alpha|^2[1-g(t)]-|\beta|^2+if(t)(\alpha^*\beta^*-\alpha\beta)\}, \tag{3.23}$$

where

$$N(t)=1-f^2(t)-g(t)=e^{\gamma t/2}\frac{\cosh^2 c}{\cosh^2(Kt-c)}. \tag{3.24}$$

The Q function for the reduced state of mode a is

$$Q_a(\alpha,t)=\int\frac{d^2\beta}{\pi}Q(\alpha,\beta,t)=[1+\bar{n}_a(t)]^{-1}\exp\left[\frac{-|\alpha|^2}{1+\bar{n}_a(t)}\right], \tag{3.25}$$

where

$$\bar{n}_a(t)=\frac{1}{N(t)}-1 \tag{3.26}$$

is the mean photon number of mode a at time t. The Q function for mode b is calculated in a similar way to give

$$Q_b(\beta,t)=\frac{N(t)}{1-g(t)}\exp\left[-\frac{|\beta|^2N(t)}{1-g(t)}\right]. \tag{3.27}$$

Thus the mean photon number in mode b is

$$\bar{n}_b(t)=\frac{1-N(t)-g(t)}{N(t)}=\bar{n}_a(t)-\frac{g(t)}{N(t)}. \tag{3.28}$$

Note that as $\gamma\to 0$, $g(t)\to 0$ and $\bar{n}_a(t)=\bar{n}_b(t)$, reflecting the conservation law operating in the undamped case. The functions $\bar{n}_a(t)$ and $\bar{n}_b(t)$ determine the functions $f(t)$ and $g(t)$. In fact,

$$f^2(t)=\frac{\bar{n}_b(t)}{\bar{n}_a(t)+1}, \tag{3.29}$$

$$g(t)=\frac{\bar{n}_a(t)-\bar{n}_b(t)}{\bar{n}_a(t)+1}. \tag{3.30}$$

Equations (3.29) and (3.30) are useful in interpreting the results, as will shortly become clear.

The probability $P(m,t)$ to detect m signal photons in the interval $[0,t)$ is [Eq. (3.3)]

$$P(m,t)=\int\frac{d^2\alpha}{\pi}\int\frac{d^2\beta}{\pi}\tilde{Q}^{(m)}(\alpha,\beta,t). \tag{3.31}$$

For initial vacuum states we find

$$P(m,t)=\left[\frac{\cosh c}{\cosh(Kt-c)}\right]^2e^{\gamma t/2}g^m(t)\times[1-f^2(t)]^{-(m+1)}. \tag{3.32}$$

[This is always positive as $f^2(t)\le 1$ for all time, which follows from $\bar{n}_a(t)\ge\bar{n}_b(t)$. See Appendix B.] This may be written in a more easily interpreted form as follows. Define a function

$$\lambda(t)=\frac{\bar{n}_a(t)}{1+\bar{n}_a(t)} , \quad (3.33)$$

where $\bar{n}_a(t)$, is given by Eq. (3.26). Using Eqs. (3.26), (3.29), (3.30), and (3.33), $P(m,t)$ may be written in the form of the probability to detect m quanta in time t from a thermal state with mean photon number $\bar{n}_a(t)$ [Eq. (2.13)], with effective quantum efficiency $\mu_e(t)$, where

$$\mu_e(t)=\frac{\bar{n}_a(t)-\bar{n}_b(t)}{\bar{n}_a(t)} . \quad (3.34)$$

[As $\bar{n}_a(t)\geq\bar{n}_b(t)$ (Appendix B) then $0\leq\mu_e(t)\leq 1$.]

It follows directly from Eq. (2.15) that the mean number of quanta counted in time t is given by

$$\bar{m}_t=\mu_e(t)\bar{n}_a(t) . \quad (3.35)$$

We expect that under conditions for which $\mu_e(t)\sim 1$ the measurement of signal quanta can be used as a good indirect measurement of idler quanta. In Sec. IV we will consider the conditions for which $\mu_e(t)\rightarrow 1$ or equivalently $f^2(t)\rightarrow 0$. We now consider the post-measurement state in the selective sense.

The unnormalized Q function for mode a at time t conditioned on counting m quanta in mode b over the interval $[0,t)$ is

$$\tilde{Q}_a^{(m)}(\alpha,t)=\int\frac{d^2\beta}{\pi}\tilde{Q}^{(m)}(\alpha,\beta,t) . \quad (3.36)$$

For the case of the initial vacuum state the normalized Q function is then found to be

$$Q_a^{(m)}(\alpha,t)=[1-f^2(t)]^{m+1}\frac{|\alpha|^{2m}}{m!}e^{-|\alpha|^2[1-f^2(t)]} . \quad (3.37)$$

This corresponds to the density operator

$$\hat{\rho}_a^{(m)}=\sum_{n=0}^{\infty}P^{(m)}(n,t)|n\rangle\langle n| , \quad (3.38)$$

with

$$P^{(m)}(n,t)=\begin{cases}0, & n<m\\ [1-f^2(t)]^{m+1}\begin{bmatrix}n\\m\end{bmatrix}[f^2(t)]^{n-m}, & n\geq m .\end{cases} \quad (3.39)$$

The result for $n\geq m$ may be equivalently written

$$P^{(m)}(n,t)=\begin{bmatrix}n\\m\end{bmatrix}\{1-\lambda(t)[1-\mu_e(t)]\}^{m+1}\times\{\lambda(t)[1-\mu_e(t)]\}^{n-m} . \quad (3.40)$$

One easily verifies that this is normalized using the result

$$\sum_{n=m}^{\infty}\begin{bmatrix}n\\m\end{bmatrix}x^{n-m}=(1-x)^{-(m+1)} .$$

In the discussion above we identified the limit of $f(t)\rightarrow 0$ as the "ideal" measurement case corresponding to unit quantum efficiency. In this limit indeed we find

$$\lim_{f(t)\rightarrow 0}\hat{\rho}_a^{(m)}(t)=|m\rangle\langle m| , \quad (3.41)$$

that is, the state of mode a is a pure number state. We further discuss this result in Sec. IV. In general, however, the state $\hat{\rho}_a^{(m)}(t)$ will be the classical mixture of number states given by Eq. (3.38). In Fig. 2 we plot $P^{(m)}(n)$ against n for the case $m=5$ and various values for $f(t)$. Only in the case $f(t)\ll 1$ do we obtain a pure number state. In general the distribution is peaked at $n>m$.

IV. DISCUSSION AND CONCLUSION

A realistic theory of continual photon-counting measurements shows that the state of the idler mode, condi-

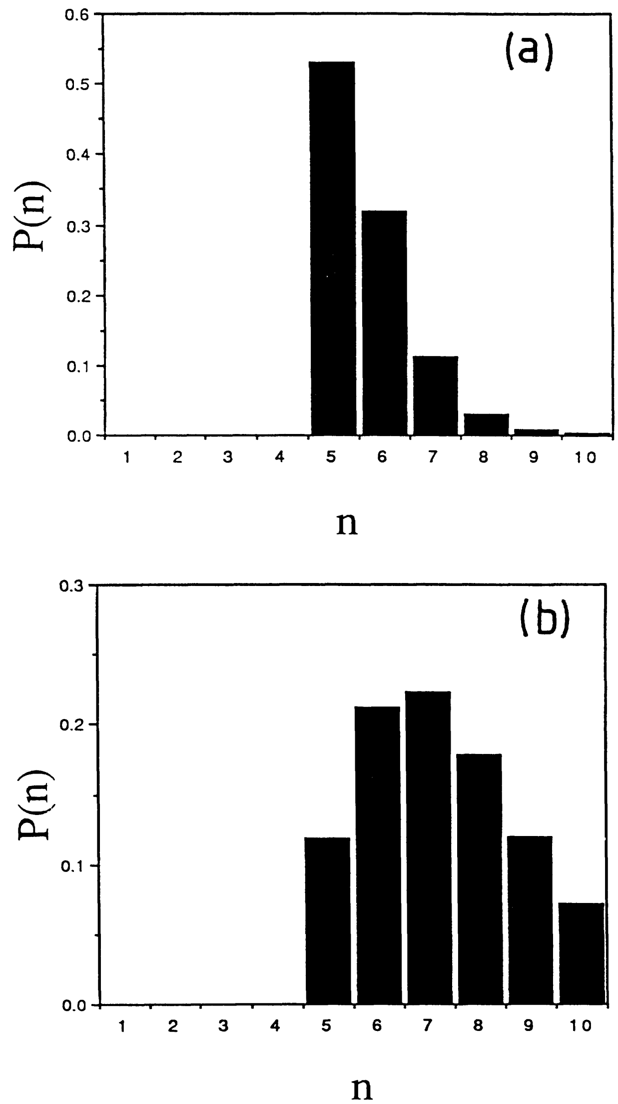

FIG. 2. Plot of the conditional post-measurement photon-number distribution $P^{(m)}(n)$ for $m=5$. (a) $f^2=0.1$. (b) $f^2=0.3$.

tioned on the results of photon counting in the signal mode, is determined by an effective quantum efficiency $\mu_e(t)$ [see Eq. (3.34)]. Only under conditions for which $\mu_e(t)\rightarrow 1$ do we find the conditional state of the idler mode "reduced" to a pure number state at the end of the counting interval. With $\bar{n}_a(t)$ fixed (i.e., the mean photon number of the idler mode fixed) and $\mu_e(t)\rightarrow 1$, the mean number of quanta counted in the interval [0,t) is

$$\bar{m}_t \simeq \bar{n}_a(t) \ .$$

Thus we expect a good measurement of $a^\dagger a$ to correspond to the limit $\mu_e(t)\rightarrow 1$ with $\bar{n}_a(t)$ held fixed. We now consider the physical meaning of these limits.

The limits $\mu_e(t)\rightarrow 1$ is equivalent to $\bar{n}_b(t)\rightarrow 0$ with $\bar{n}_a(t)$ fixed. Consider the case for which $\chi \ll \gamma$ with $\chi t = N$ where N is a constant, and $\chi N/\gamma \ll 1$. To first order in χ/γ we find

$$\begin{aligned} f(t) &\simeq -2\left[\frac{\chi}{\gamma}\right](1-e^{-\gamma t/2}) \ , \\ \mu_e(t) &\simeq 1 \ , \\ \bar{n}_a(t) &\simeq 4\left[\frac{\chi}{\gamma}\right]N \ , \\ \bar{n}_b(t) &\simeq 0 \ . \end{aligned} \tag{4.1}$$

Thus in this limit $\mu_e(t)$ approaches unity while the mean photon number of the signal field at the end of a counting interval is zero. The constant γ determines the rate at which photons are counted in the signal mode, while χ determines the rate at which signal and idler photons are created. Thus $\gamma \gg \chi$ means we are considering systems in which photons are counted, and thus destroyed, in the signal mode at a much greater rate than they are produced. The condition $\chi t = N$, a constant, ensures we are comparing systems for which $\bar{n}_a(t)$ is held fixed as $\mu_e(t)$ varies.

As we saw in Sec. III, in this limit the result for the count probability $P(m,t)$ is equivalent to that for photon counting with unit quantum efficiency from a thermal field in the idler mode. The state of the idler field conditioned on a count of m signal photons in the interval [0,t) is a number state with exactly m photons (for initial vacuum states). In general, however, the conditional state of the idler field is a classical mixture of number states with *at least m* photons, but possibly peaked at greater than m photons.

This result is easily interpreted. In a counting interval not every photon generated in the signal field leaves the cavity to be detected. Only when $\bar{n}_b(t)=0$ can we say that all the signal photons generated over the counting interval have been detected. If m photons are counted over the interval [0,t) and $\bar{n}_b \neq 0$ then *at least m* photons were generated in the signal and idler field, and at time t the conditional state of the idler must contain at least m photons. The condition $\gamma \gg \chi$ ensures that nearly all photons generated in the signal field over a counting interval leaves the cavity to be counted. Note that in the ideal case [$\mu_e(t)\rightarrow 1$] the conditional state of the idler field is a number state, whereas the state of the signal is the vacuum state ($\bar{n}_b=0$). Despite the fact that the von Neumann projection postulate does not apply to this model a theory of quantum counting shows that there is yet a limit in which the idler field can be reduced to a pure number state. In general, however, it is only left in a classical mixture of number states.

In the sense made clear at the end of Sec. I, the nondegenerate parametric amplifier together with signal-field photon-counting measurements may be used to prepare the idler field in a photon-number state. It must be emphasized that this preparation procedure is quite different from that generally used to prepare quantum optical states. Usually states are generated by unitary evolution, the Hamiltonian for which is presumed to be known. Indeed, the nondegenerate parametric amplifier produces thermal states in each mode separately and a squeezed state in a linear combination of the signal and idler modes, through unitary evolution. However, when photon counting on the signal mode is included in the preparation procedure additional information (viz., the results of such measurements) is now available to "condition" the state of the idler field. To verify that a number state has been prepared in the idler field, further experiments could be performed on the idler field; however, one would need to report the results of such measurements together with the result of the signal-field photon-counting measurements used to condition the idler state. Thus one is lead to signal-idler correlation experiments.

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APPENDIX A

The equation for $\hat{Q}^{(m)}(\alpha,\beta,t)$ is

$$\frac{\partial \tilde{Q}^{(0)}}{\partial t}(\alpha,\beta,t)=\mathcal{L}\tilde{Q}^{(0)}(\alpha,\beta,t) \ , \tag{A1}$$

$$\frac{\partial \tilde{Q}^{(m)}}{\partial t}(\alpha,\beta,t)=\mathcal{L}\tilde{Q}^{(m)}(\alpha,\beta,t)+\mathcal{H}\tilde{Q}^{(m-1)}(\alpha,\beta,t) \ , \quad m=1,2,\ldots \ . \tag{A2}$$

The operators $\mathcal{L}$ and $\mathcal{H}$ are given in Eqs. (3.8) and (3.9). The initial conditions are

$$\tilde{Q}^{(0)}(\alpha,\beta,0)=e^{-|\alpha-\alpha_0|^2-|\beta-\beta_0|^2} \ , \tag{A3}$$

$$\tilde{Q}^{(m)}(\alpha,\beta,0)=0, \quad m=1,2,\ldots \ . \tag{A4}$$

To solve for $\tilde{Q}^{(0)}(\alpha,\beta,t)$ let

$$\tilde{Q}^{(0)}(\alpha,\beta,t)=R(\alpha,\beta,t)\exp[-|\alpha|^2-|\beta|^2+i(\alpha^*\beta^*-\alpha\beta)f(t)] \ , \tag{A5}$$

where $f(t)$ satisfies

$$\frac{\partial f}{\partial t}=-\chi[1-f^2(t)]-\frac{\gamma}{2}f(t)\ . \tag{A6}$$

Thus

$$f(t)=\frac{\gamma}{4\chi}-\Delta\tanh(Kt-c)\ , \tag{A7}$$

with

$$K=\chi\Delta\ , \tag{A8}$$

$$\Delta=\left[1+\frac{\gamma^2}{16\chi^2}\right]^{1/2}\ , \tag{A9}$$

$$\tanh c=-\frac{\gamma}{4K}\ . \tag{A10}$$

The function $R(\alpha,\beta,t)$ then satisfies

$$\frac{\partial R}{\partial t}=\left[2\chi f(t)+\chi f(t)\left[\alpha\frac{\partial}{\partial\alpha}+\beta\frac{\partial}{\partial\beta}+\text{c.c.}\right]-i\chi\left[\frac{\partial^2}{\partial\alpha^*\partial\beta^*}-\text{c.c.}\right]-\frac{\gamma}{2}\left[\beta\frac{\partial}{\partial\beta}+\text{c.c.}\right]\right]R\ , \tag{A11}$$

with

$$R(\alpha,\beta,0)=\exp[-|\alpha_0|^2-|\beta_0|^2+(\alpha\alpha_0^*+\beta\beta_0^*+\text{c.c.})]\ . \tag{A12}$$

R may be found by the method of series. In fact, it has the following rather simple form:

$$R(\alpha,\beta,t)=\exp\left[\exp\left[\chi\int_0^t f(t')dt'\right]\times\left[\alpha\alpha_0^*+e^{-\gamma t/2}\beta\beta_0+\text{c.c.}\right]+\mathcal{F}\right] \tag{A13}$$

(where $\mathcal{F}$ represents a function of α_0, β_0, and time) so that $\tilde{Q}^{(0)}(\alpha,\beta,t)$ is then as given in Eq. (3.12).

To solve for $\tilde{Q}^{(m)}(\alpha,\beta,t)$ we let

$$Q^{(m)}(\alpha,\beta,t)=\mathcal{P}^{(m)}(\alpha,\beta,t)\tilde{Q}^{(0)}(\alpha,\beta,t)\ , \tag{A14}$$

with

$$\mathcal{P}^{(m)}(\alpha,\beta,0)=0,\quad m=1,2,\ldots\ . \tag{A15}$$

Inspection of the resulting equation for $\mathcal{P}^{(1)}$ shows that $\mathcal{P}^{(1)}$ must in fact be independent of β. Similarly, $\mathcal{P}^{(m)}$ is independent of β. Thus

$$\frac{\partial\mathcal{P}^{(m)}(\alpha,t)}{\partial t}=-i\chi\left\{\left[i\alpha f(t)-\beta_0^*\exp\left[-\frac{\gamma t}{2}+\chi\int_0^t f(t')dt'\right]\right]\frac{\partial}{\partial\alpha}-\text{c.c.}\right\}\mathcal{P}^{(m)}+\gamma\left|i\alpha f(t)-\beta_0^*\exp\left[-\frac{\gamma t}{2}+\chi\int_0^t f(t')dt'\right]\right|^2\mathcal{P}^{(m-1)}\ , \tag{A16}$$

where $\mathcal{P}^{(0)}=1$.

Letting

$$\delta=\alpha-\frac{2i\chi\beta_0^*}{\gamma}\exp\left[-\frac{\gamma t}{2}+\chi\int_0^t f(t')dt'\right] \tag{A17}$$

and changing to the new variables

$$s=|\delta|\exp\left[\mathcal{H}\int_0^t f(t')dt'\right]\ , \tag{A18}$$

$$\phi=\arg\delta\ , \tag{A19}$$

the equation becomes

$$\frac{\partial\mathcal{P}^{(m)}}{\partial t}(s,\phi,t)=H(s,\phi,t)\mathcal{P}^{(m-1)}(s,\phi,t)\ . \tag{A20}$$

This has the solution

$$\mathcal{P}^{(m)}(s,\phi,t)=\frac{\left[\int_0^t H(s,\phi,t')dt'\right]^m}{m!}\ , \tag{A21}$$

which may be proved by induction. Evaluating $\mathcal{P}^{(1)}(\alpha,t)$ we find that

$$\mathcal{P}^{(1)}(\alpha,t)=|\alpha|^2 g(t)+(\alpha\beta_0-\alpha^*\beta_0^*)z(t)+h(t)\ , \tag{A22}$$

where $g(t)$, $z(t)$, and $h(t)$ are given in Eq. (3.19)–(3.21). Thus

$$\mathcal{P}^{(m)}(\alpha,t)=\frac{[\mathcal{P}^{(1)}(\alpha,t)]^m}{m!}\ . \tag{A23}$$

APPENDIX B

In this appendix we prove that $f^2(t)<1$ for all time. From Eq. (3.29) it is easy to see that if $\bar{n}_a(t)\geq\bar{n}_b(t)$ then $f^2(t)<1$. To prove that $\bar{n}_a(t)>\bar{n}_b(t)$ we obtain the equation of motion for $[\bar{n}_a(t)-\bar{n}_b(t)]$ from the master equation, Eq. (3.1).

This is

$$\frac{d}{dt}[\bar{n}_a(t)-\bar{n}_b(t)]=\gamma\bar{n}_b(t)\geq 0\ . \tag{B1}$$

The latter inequality follows from the fact that $b^\dagger b$ is a positive operator. Thus

$$\frac{d}{dt}\bar{n}_a(t) \geq \frac{d}{dt}\bar{n}_b(t) , \tag{B2}$$

or

$$\bar{n}_a(t) \geq \bar{n}_b(t) + \text{const} . \tag{B3}$$

At $t=0$, $\bar{n}_a(0)=\bar{n}_b(0)=0$, thus

$$\bar{n}_a(t) \geq \bar{n}_b(t) , \tag{B4}$$

and $f^2(t) \leq 1$.

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