# Analytically simple dressing of bound-state wave functions

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An exact solution for the dressing of a bound-state wave function by a plane-wave electromagnetic field is written in terms of the set of bare states. The solution is stated in the radiation gauge as an infinite series involving powers of the field frequency. The leading term of this series gives a simple analytical form for a state dressed by the field when the energy of a single field photon is much less than transition energies in the bound system. The result is closely related to the momentum-translation approximation (MTA) wave function. The MTA result is thereby shown to follow in a fixed gauge from an exact solution of the fully interacting Schrödinger equation.

## I. INTRODUCTION

The primary aim of this work is to establish a simple analytical form for the dressing of a bound quantum state by a low-frequency electromagnetic field. It is hypothesized that the energy of a field photon is very much less than the energy required to make a transition to any other state of the system, and so the presumption is implicit that any real transitions which occur require the presence of some other interaction (the "probe" field) in addition to the background low-frequency field.

The procedure employed is to express the wave function of the bound state in the presence of the background field as a series involving increasing powers of the field frequency. The general term of this series is found in closed form, and so it is possible to verify explicitly that the complete series representation provides an exact solution of the Schrödinger equation. The work is couched in the radiation gauge, i.e., in a gauge where the scalar potential of the field is zero. A long-wavelength approximation is used, with retardation terms neglected. In other words, it is presumed that the vector potential and the electric field vector are both functions only of time: $\mathbf{A}(t)$, $\mathbf{E}(t)$. For a photon energy sufficiently small as compared to a transition energy of the system, the leading term dominates all higher-order terms. The leading term, which is of very simple analytical form, then serves as a convenient approximate wave function for the system dressed by the background field.

The dressed state so determined is nearly identical to the momentum-translation approximation (MTA) wave function previously explored.[1–3] The results thereby clarify some points of contention raised in the past. In the present work, the MTA wave function arises from an explicit construction technique unambiguously carried out in a fixed (radiation) gauge. Furthermore, it is the leading low-frequency term in a solution shown to be an exact solution of the Schrödinger equation in the radiation gauge. This answers the objection of Cohen-Tannoudji *et al.*,[4] who noted an analytical similarity to a gauge transformation, and interpreted the MTA result as if it were nothing more than a gauge transformation. It has been emphasized elsewhere[2,3] that the MTA can be regarded as a unitary transformation within a fixed gauge, but it does not correspond to a gauge transformation. The present results independently confirm that assertion of fixed gauge.

The MTA result is

$$\Psi(\mathbf{r},t)=\exp[ie\mathbf{r}\cdot\mathbf{A}(t)]\Phi(\mathbf{r},t)\ , \tag{1}$$

where $\Phi$ is the bare state, with no field present, and $\Psi$ is this state dressed by the field represented by the vector potential $\mathbf{A}$ in the radiation gauge. Units with $\hbar=c=1$ are used here. However, within the radiation gauge, there remains gauge freedom to within an additive constant vector. This is impermissible within the present approach, and so $\mathbf{A}$ must be constrained by an auxiliary condition which may be expressed as

$$|\mathbf{E}(t)|=\omega|\mathbf{A}(t)| \tag{2a}$$

or as

$$\langle\mathbf{A}(t)\rangle=\mathbf{0}\ , \tag{2b}$$

where $\omega$ is the angular frequency of the background field, and where the angular brackets in Eq. (2b) are meant to imply a time average over a period of the wave. Equivalently, the dressed state may be written expressly in terms of the electric field vector with shifted phase as

$$\Psi(\mathbf{r},t)=\exp\left[ie\mathbf{r}\cdot\frac{\mathbf{E}(t+\pi/2\omega)}{\omega}\right]\Phi(\mathbf{r},t)\ . \tag{3}$$

The MTA wave function in the form of Eq. (1) looks exactly like a wave function gauge transformed from the radiation gauge to the position gauge [also called the $\mathbf{r}$ gauge or the electric field (EF) gauge], and this has caused confusion as noted above. Other authors[5,6] interpret the $\Psi$ of Eq. (1) as a noninteracting wave function in the radiation gauge. The present work shows that this is an impermissible point of view. Because of this interpretational difficulty in the past, special pains are taken to identify the dressed state $\Psi$ unequivocally as a dressed state in the radiation gauge.

The formulation of the exact solution of the Schrödinger equation is motivated in Sec. II by examining the first-order time-dependent perturbation theory expression of the bound-state wave function in a plane-wave

 

field. An energy factor which occurs is rearranged algebraically into a constant term of order unity and an additional term which is small when the ratio of the field photon energy to the energy difference between bound states is small. This process is extended in Sec. III to the case of second-order time-dependent perturbation theory. In this instance, it is necessary to algebraically combine the outcome of second-order $-e\,\mathbf{A}\cdot\mathbf{p}/m$ terms with $e^2A^2/2m$ terms from a first-order perturbation. It is again found possible to express the resulting energy factor in a form with a leading constant term of order unity, followed by a single energy-denominator term proportional to $\omega$, and a quadratic energy-denominator term proportional to $\omega^2$. At this point, the expression for the general term is apparent. This is written in Sec. IV, and it is shown by direct substitution that the infinite series is an exact solution of the Schrödinger equation in the radiation gauge. The leading term of this series is the dressed-state approximation. The nature of this term, and the limitations upon its use, are examined in Sec. V. It is shown there that this very simple expression for a dressed state contains a full set of energy sidebands to the bare state, and a dressing of the bare state with a set of field-induced angular momentum states.

## II. REARRANGEMENT OF FIRST-ORDER PERTURBATION THEORY

### A. Preliminaries

The total Hamiltonian for the bound system is written as

$$H=H_0+H'+V' , \tag{4}$$

where $H_0$ is the unperturbed Hamiltonian, including the binding potential $V$,

$$H_0=p^2/2m+V , \tag{5}$$

$H'$ is the potential due to the plane-wave dressing field, as expressed in a gauge where the scalar potential is zero, so that

$$H'=-e\,\mathbf{A}\cdot\mathbf{p}/m+e^2A^2/2m , \tag{6}$$

and $V'$ is the potential of the "probe" field, that is, some additional interaction which can supply enough energy to cause a real transition from the initial state. It is presumed that the dressing field is of such low frequency that it cannot of itself cause transitions with any significant probability. The vector potential is taken to be of the form

$$\mathbf{A}=\mathbf{a}e^{-i\omega t} . \tag{7}$$

The so-called rotating-wave form of Eq. (7) is employed because the resulting analytical simplicity then makes possible the detailed construction of the low-frequency series, with all the physical insight gained thereby. When the nature of the low-frequency series associated with Eq. (7) has been established, a full dressed-state theory with both $\exp(-i\omega t)$ and $\exp(+i\omega t)$ parts can be obtained from known series solutions expressed in connection with earlier MTA work.

### B. First-order wave function

In principle, there is no need to use perturbation theory in order to present the exact solution of the Schrödinger equation to be given below. It could simply be stated and its validity as a solution verified directly. Nevertheless, some reference to time-dependent perturbation theory is very useful to motivate and to understand the nature of the solution to be presented.

Let $\Phi$ be the wave function of the bound state in the presence of the noninteracting Hamiltonian $H_0$, and let $\Psi$ be the wave function in the presence of the electromagnetic dressing field as well. That is, $\Psi$ satisfies the Schrödinger equation

$$i\partial_t\Psi=[(1/2m)(-i\partial-e\,\mathbf{A})^2+V]\Psi . \tag{8}$$

Within time-dependent perturbation theory, the first-order solution for $\Psi$ as a perturbation of a particular state $\Phi_0$ is

$$\Psi_0^{(1)}=\Phi_0-i\sum_n\Phi_n\int_{-\infty}^{t}dt_1(\Phi_n,H'^{(1)}\Phi_0) , \tag{9}$$

where $H'^{(1)}$ is the first-order part of Eq. (6), i.e.,

$$H'^{(1)}=-e\,\mathbf{A}\cdot\mathbf{p}/m , \tag{10}$$

and where the $\Phi_n$ are a complete set of unperturbed states. In other words, the $\Phi_n$ satisfy Eq. (8) when $\mathbf{A}$ is set to zero. The standard commutator replacement

$$\mathbf{p}/m=-i[\mathbf{r},H_0] \tag{11}$$

will be used for the momentum operator in Eq. (10), so Eqs. (9)–(11) yield

$$\Psi_0^{(1)}=\Phi_0+\sum_n\Phi_n(\Phi_n,ie\,\mathbf{A}\cdot\mathbf{r}\Phi_0)\times(E_n-E_0)/(E_n-E_0-\omega) , \tag{12}$$

where $E_0$ and $E_n$ are the energies of the unperturbed $\Phi_0$ and $\Phi_n$ states.

The final energy factor in Eq. (12) can be subjected to the decomposition

$$\frac{E_n-E_0}{E_n-E_0-\omega}=1+\frac{\omega}{E_n-E_0-\omega} . \tag{13}$$

The basic hypothesis is now stated that the energy $\omega$ of a single photon of the dressing field is very much less than the energy difference between any pair of levels

$$\omega\ll E_n-E_0 . \tag{14}$$

This means that the first term in Eq. (13) dominates the second term in magnitude. It is thus appropriate to effect the separation represented in Eq. (13) in Eq. (12) as well. When this is done, that part of the summation over the index $n$ which contains the first term of Eq. (13) can be performed exactly, and Eq. (12) becomes

$$\Psi^{(1)}=(1+ie\,\mathbf{A}\cdot\mathbf{r})\Phi_0+\sum_n\Phi_n(\Phi_n,ie\,\mathbf{A}\cdot\mathbf{r}\Phi_0)\omega/(E_n-E_0-\omega) . \tag{15}$$

### C. A condition on the vector potential

An important remark about gauges can be made here. It has been stated that the radiation gauge is to be employed. This is a special case of the Lorentz gauge ($\partial^\mu A_\mu = 0$, in relativistic notation with $\mu = 0,1,2,3$), with the scalar potential $A^0 = 0$. This means that $\nabla \cdot \mathbf{A} = 0$. The question is now posed: How much gauge freedom remains after one specifies the radiation gauge? Consider a gauge transformation $A^\mu \rightarrow A'^\mu = A^\mu + \Gamma^\mu$. The radiation gauge conditions require $\Gamma^0 = 0$ and $\nabla \cdot \Gamma = 0$. Retardation terms have been neglected in the original gauge, so that $\nabla \times \mathbf{A} = \mathbf{0}$. This means, of course, that there is no magnetic field, and this must still be true in the new gauge, so that $\nabla \times \Gamma = \mathbf{0}$. By a basic theorem of vector analysis, $\nabla \cdot \Gamma = 0$ and $\nabla \times \Gamma = \mathbf{0}$ mean that $\Gamma$ is independent of $\mathbf{r}$. The electric field is found from $\mathbf{E} = -\partial_t \mathbf{A} = -\partial_t \mathbf{A}'$, so that $\Gamma$ must be independent of $t$. That is, $\Gamma$ can only be a vector constant in space and time. In principle, then, $\mathbf{A}$ as stated in Eq. (7) could be supplemented by an additive constant vector. However, the results obtained in Eqs. (12)–(15) depend upon $\mathbf{A}$ being free of any such additive constant vector. It is thus necessary to constrain $\mathbf{A}$ by a supplementary condition, which can be stated in any of several ways. One is to require that the magnitude of the electric field is the product of the field frequency and the magnitude of the vector potential, or

$$|\partial_t \mathbf{A}| = \omega |\mathbf{A}| \ . \tag{16}$$

An equivalent way is to specify that the time average of the vector potential over a period of the wave should vanish.

## III. REARRANGEMENT OF SECOND-ORDER PERTURBATION THEORY

Second-order time-dependent perturbation theory presents some new and instructive features not present in the first-order case. It is thus worthwhile to carry through the second-order case despite a considerable increase in complexity.

The extension of Eq. (9) to second order is given by

$$\begin{aligned}\Psi_0^{(2)} = \Phi_0 - i \sum_n \Phi_n \int_{-\infty}^{t} dt_1 (\Phi_n, H' \Phi_0) \\ + (-i)^2 \sum_n \sum_m \Phi_n \int_{-\infty}^{t} dt_1 (\Phi_n, H'^{(1)} \Phi_m) \\ \times \int_{-\infty}^{t_1} dt_2 (\Phi_m, H'^{(1)} \Phi_0) \ ,\end{aligned} \tag{17}$$

where $H'$ contains both terms of the interaction Hamiltonian, as in Eq. (6), and $H'^{(1)}$ is just the first-order term, as in Eq. (10). An important feature of the evaluation of Eq. (17) is that the $e^2 A^2/2m$ term leads to matrix elements of exactly the same form as those arising from $-e\,\mathbf{A}\cdot\mathbf{p}/m$ if the relationship

$$A^2 = -i[\mathbf{A}\cdot\mathbf{r}, \mathbf{A}\cdot\mathbf{p}] \tag{18}$$

is used, together with Eq. (11). Specifically, one obtains

$$\begin{aligned}(\Phi_n, (e^2 A^2/2m)\Phi_0) = \tfrac{1}{2} \sum_m (\Phi_n, \mathcal{A}\Phi_m) \\ \times (\Phi_m, \mathcal{A}\Phi_0)(E_0 - E_m) \\ + \tfrac{1}{2} \sum_m (\Phi_n, \mathcal{A}\Phi_m) \\ \times (\Phi_m, \mathcal{A}\Phi_0)(E_n - E_m) \ ,\end{aligned} \tag{19}$$

where the notation

$$\mathcal{A} \equiv ie\,\mathbf{A}\cdot\mathbf{r} \tag{20}$$

has been introduced. The end result for the explicitly second-order terms in Eq. (17) is

$$\Psi^{(2)} - \Psi^{(1)} = \sum_n \sum_m \Phi_n (\Phi_n, \mathcal{A}\Phi_m)(\Phi_m, \mathcal{A}\Phi_0)\mathcal{E} \ , \tag{21}$$

where $\mathcal{E}$ is an energy-dependent factor given by

$$\begin{aligned}\mathcal{E} = \tfrac{1}{2} \frac{E_m - E_0}{E_n - E_0 - 2\omega} + \tfrac{1}{2} \frac{E_m - E_n}{E_n - E_0 - 2\omega} \\ + \frac{(E_n - E_m)(E_m - E_0)}{(E_n - E_0 - 2\omega)(E_m - E_0 - \omega)} \ .\end{aligned} \tag{22}$$

Equation (22) can be decomposed into fractions with numerators given by increasing powers of $\omega$ to yield

$$\mathcal{E} = \tfrac{1}{2} + \frac{\omega}{E_m - E_0 - \omega} + \frac{\omega^2}{(E_n - E_0 - 2\omega)(E_m - E_0 - \omega)} \ . \tag{23}$$

Again some sums over intermediate states can be performed, and the end result for Eq. (17), with Eqs. (21) and (23) incorporated, is

$$\begin{aligned}\Psi_0^{(2)} = (1 + \mathcal{A} + \mathcal{A}^2/2)\Phi_0 \\ + (1 + \mathcal{A}) \sum_n \Phi_n (\Phi_n, \mathcal{A}\Phi_0) \frac{\omega}{(E_n - E_0 - \omega)} \\ + \sum_n \sum_m \Phi_n (\Phi_n, \mathcal{A}\Phi_m)(\Phi_m, \mathcal{A}\Phi_0) \\ \times \frac{\omega^2}{(E_n - E_0 - 2\omega)(E_m - E_0 - \omega)} \ .\end{aligned} \tag{24}$$

The first term in Eq. (24) is independent of frequency, the single-sum term has a factor of $\omega$, and the double-sum term has a factor of $\omega^2$. This is the desired form.

## IV. EXACT SOLUTION OF THE SCHRÖDINGER EQUATION

Equation (24) suggests a complete solution to the Schrödinger equation, useful in the low-frequency case. This is a series solution, and it will be convenient to write it in a form in which the leading term is separate from the others from the outset. With

$$\Psi_0 = \lim_{N \rightarrow \infty} \Psi_0^{(N)} \ , \tag{25}$$

then Eq. (24) suggests the result

$$\Psi_0^{(N)}=\sum_{j=0}^{N}(\mathcal{A}^j/j!)\Phi_0+\sum_{k=1}^{N}\left[\sum_{j=0}^{N-k}(\mathcal{A}^j/j!)\right]\times\left[\sum_{m_1}\cdots\sum_{m_k}\Phi_{m_1}\mathcal{A}_{m_1m_2}\cdots\mathcal{A}_{m_k0}\frac{\omega^k}{(\omega_{m_10}-k\omega)\cdots(\omega_{m_k0}-\omega)}\right], \tag{26}$$

with $\mathcal{A}$ given by Eq. (20), and the new terminology introduced that

$$\mathcal{A}_{mn}\equiv(\Phi_m,\mathcal{A}\Phi_n),\quad \omega_{m0}\equiv E_m-E_0\ . \tag{27}$$

The demonstration that Eqs. (25) and (26) constitute an exact solution of the Schrödinger equation given in Eq. (8) is tedious, and is relegated to the Appendix. There it is shown that Eq. (26) solves the Schrödinger equation to within terms of order $(\omega\mathcal{A})^{N+1}$. Thus the limit $N\to\infty$ gives an exact solution.

The construction of Eqs. (25) and (26) was motivated by perturbation theory, but since the new solution is an exact solution of the equation of motion, it has an identity entirely independent of the perturbation series. It must be stressed that the vector potential $\mathbf{A}$ in the quantity $\mathcal{A}$ is the radiation gauge vector potential which fully defines the field. Equations (25) and (26) constitute an exact solution of the *radiation gauge* Schrödinger equation given in Eq. (8).

## V. DRESSED-STATE APPROXIMATION

### A. The leading term

The ratio of the magnitude of each term in Eq. (26) as compared to its predecessor behaves approximately as

$$R\cong|\omega\mathcal{A}|/|\Delta E|\ , \tag{28}$$

where the energy difference $\Delta E$ is $\omega_{m0}-k\omega$ for some pair of indices $m,k$. Implicit in each of these energy factors is a contribution from the finite lifetimes of the quantum states involved. That is, a level width contribution $i\Gamma/2$ must also appear in each energy factor, as was recognized long ago by Weisskopf and Wigner. It has been hypothesized that $|\omega_{m0}|\gg\omega$. No matter how large the energy difference $\omega_{m0}$ as compared to $\omega$, eventually the index $k$ will grow so large that $\omega_{m0}\simeq k\omega$ becomes possible. This is where the level width $i\Gamma/2$ plays its role. Even if there should be some index $k$ for which $\omega_{m0}\simeq k\omega$, nevertheless $|i\Gamma/2|\neq 0$. Furthermore, if $k$ is large enough at such an occurrence, there will be many factors in Eq. (26) of order $\omega/|\omega_{m0}|$ combined with the factor of order $\omega/|i\Gamma/2|$. Thus, even if it should be true that $\omega\gg|i\Gamma/2|$, the product of all $k$ factors $\omega/|\Delta E|$ can still be very small without imposing unphysically strong limitations on $\omega$. It will be hypothesized here that $\omega$ is small enough to satisfy this constraint, and so the condition $\omega\ll|\Delta E|$ will be understood to guarantee the smallness of high-order terms in Eq. (26) even in the presence of an eventual near resonance for large $k$. Under these circumstances, the inequality expressed in Eq. (28) justifies the neglect of all terms in the sum over $k$ in Eq. (26) as compared to the leading term.

The dressed-state approximation is thus

$$\Psi_0(\mathbf{r},t)\simeq\lim_{N\to\infty}\sum_{j=0}^{N}(\mathcal{A}^j/j!)\Phi_0(\mathbf{r},t)\ , \tag{29}$$

which is precisely the same as Eq. (1). As indicated in Eq. (16) or Eqs. (2a) or (2b), there is an implied condition on the vector potential in Eq. (1) or (29), and so a more suitable way to write the dressed state expression is the field-dependent form given in Eq. (3). The limitation expressed in Eq. (28) can be restated as

$$e|\mathbf{A}|a_0\omega/|\Delta E|\ll 1\ , \tag{30}$$

where $a_0$ is the Bohr radius for atomic systems, or, more generally, any characteristic radius of the bound system.

It is well known[2,4,7,8] that the MTA gives very poor results when applied uncritically to a problem such as multiphoton ionization of atoms. That is because the conditions discussed here are violated in such a case. There is no probe field to contribute most of the energy, which must all come from the electromagnetic field. The dense level structure of an atom then guarantees that some of the denominators in Eq. (26) will be small for modest values of the index $k$, invalidating the conditions stated above for applicability of the dressed-state approximation.

### B. Validity of the dressed-state approximation

Because Eq. (3) is so close to the MTA expressed in Eq. (1), and because the MTA has been misconstrued[4,5] in the past, the nature of the approximation made here will be examined in more detail.

As noted above, the solution to the Schrödinger equation expressed in Eqs. (25) and (26) is an *exact solution in radiation gauge.* Certainly no change of gauge is incurred in examining the relative magnitudes of the terms in Eq. (26), and so the end result for the dressed-state approximation in Eq. (29) is an approximate solution to the Schrödinger equation in *radiation gauge.* As remarked earlier, the limitation inherent in the low-frequency condition as stated in Eq. (14) is that real transitions cannot occur in a physical system unless a probe field is also present to contribute the major portion of the transition energy, and that intermediate near resonances cannot occur except at very high order.

To assess the accuracy of the dressed-state solution, it can be substituted into the original Schrödinger equation stated in Eq. (8). Equations (25) and (26) satisfy this equation exactly, whereas the substitution of Eq. (1) leaves a residual term

$$\{i\partial_t-[(1/2m)(-i\boldsymbol{\partial}-e\,\mathbf{A})^2+V]\}\Psi^{(D)}=e\mathbf{E}\cdot\mathbf{r}\Psi^{(D)}\ . \tag{31}$$

The notation $\Psi^{(D)}$ has now been introduced for the dressed-state approximation. It is seen from Eq. (31) that a field-dependent term is left over when $\Psi^{(D)}$ is employed in place of $\Psi$, and so the accuracy of the approximation may be judged by comparing the magnitude of this residual term with the magnitude of the leading field-dependent term in the original equation of motion. In terms of matrix elements, this comparison involves the ratio

$$\mathcal{R}=\frac{|\langle e\mathbf{E}\cdot\mathbf{r}\rangle|}{|\langle e\,\mathbf{A}\cdot\mathbf{p}/m\rangle|}=\frac{\omega|\langle e\,\mathbf{A}\cdot\mathbf{r}\rangle|}{|\langle e\,\mathbf{A}\cdot[\mathbf{r},H_0]\rangle|}=\frac{\omega}{|\Delta E|}\ll 1\ , \tag{32}$$

where Eqs. (2a), (11), and (14) have been used. It cannot be stressed too strongly that the residual term $e\mathbf{E}\cdot\mathbf{r}$ is in the radiation gauge, where, for low frequency, its magnitude will be much less than that of the $-e\,\mathbf{A}\cdot\mathbf{p}/m+e^2A^2/2m$ term. This is quite distinct from the situation which arises when a gauge transformation is effected, in which case the $-e\,\mathbf{A}\cdot\mathbf{p}/2m+e^2A^2/2m$ term in the radiation gauge is equivalent to $-e\mathbf{E}\cdot\mathbf{r}$ in the r gauge. This point was discussed in the original MTA papers,[1,2] and has been emphasized anew by Friar and Fallieros.[3] The conclusion is that the dressed state approximation improves in accuracy as the photon energy of the field declines with respect to level spacing.

### C. Physical implications of the dressed-state approximation

To examine the physical implications of the dressed-state approximation, it is convenient to use the form of Eq. (1) for $\Psi^{(D)}$ with the vector potential given by the real part of Eq. (7). This gives immediately

$$\Psi_0^{(D)}=\phi_0(\mathbf{r})\sum_{n=-\infty}^{\infty}i^nJ_n(e\mathbf{a}\cdot\mathbf{r})\exp[-i(E_0-n\omega)t]\ , \tag{33}$$

where the terminology

$$\Phi_0(\mathbf{r},t)=\phi_0(\mathbf{r})\exp(iE_0t) \tag{34}$$

has been used for the field-free stationary state from which $\Psi_0^{(D)}$ is evolved. Clearly, Eq. (33) exhibits the dressed state as a superposition of sideband states with energies $E_0-n\omega$, with the relative amplitude of each sideband state given by the Bessel function of order $n$ and argument $e\mathbf{a}\cdot\mathbf{r}$. As has been already remarked, real transitions from the dressed state require the presence of a probe field, and transitions induced by the probe field can be regarded as starting from one of the energy sidebands.

Equation (33) is familiar in that it has the general character of a Floquet state. It is quite explicit, however, in specifying the form of each of the sideband $\psi_n$ states which are normally indefinite in the Floquet formalism. The presence of the field-dependent Bessel functions in Eq. (33) also introduces a resemblance to a bound-state version of a Kroll-Watson scattering state.[9]

An alternative representation to Eq. (33) is

$$\Psi_0^{(D)}=\Phi_0(\mathbf{r},t)\sum_{l=0}^{\infty}i^l(2l+1)j_l(ear\cos(\omega t))P_l(\cos\theta)\ , \tag{35}$$

where $\theta$ is the angle between $\mathbf{A}$ and $\mathbf{r}$ in spherical polar coordinates based on $\mathbf{A}$ as the polar direction. This partial wave expansion of $\Psi_0^{(D)}$ shows the role of the field in splitting the no-field state $\Phi_0$ into angular momentum substates. These substates occur with amplitudes dependent upon the amplitude of $\mathbf{A}$ in the argument of the spherical Bessel functions $j_l$.

The field-induced energy and angular momentum substates of $\Psi^{(D)}$ make particularly puzzling the interpretation of $\Psi^{(D)}$ as a noninteracting state by Schlicher *et al.*[5] As Friar and Fallieros[3] have pointed out, the interpretation of Schlicher *et al.* has the character of an oxymoron.

### D. Relationship with the MTA

The dressed-state wave function developed here is very closely related to the MTA wave function, but it nevertheless is somewhat different. One important difference is that the new result is subject to restrictions on the vector potential defining it [see Eqs. (2a) and (2b)] which are of such a nature that it is really directly determined by the electric field and not just by the potential. The original derivation[1] of the MTA did not place this restriction on the vector potential. The unitary transformation technique employed in Ref. 1 would account as well for the additive constant vector that is allowable in radiation gauge, as discussed in Sec. II C. Another difference is that the MTA is not limited to the absorption-only vector potential of Eq. (7).

Other differences in the present work as compared to earlier MTA results stem from the method of derivation. The present result arose from separating terms involving the photon energy from other, larger, energy terms, as shown in Eqs. (13) and (23). The unitary transformation leading to the MTA had a different immediate motivation, although the notion of low frequency was present there as well. The two approaches are easily seen to be equivalent, but the philosophy differs. Several papers discussing the MTA have shown[1,2,8,10,11] series expansions of which the MTA is the leading term. These expansions are equivalent. Only the points of view and the specific techniques differ.

An important outcome of the new method of derivation is that it shows the dressed-state result as an approximation to an explicitly specified exact solution. This has the immediate consequence that correction terms are available. It also makes very clear the status of the dressed state as a fixed gauge approximation. Furthermore, the nature of the approximation becomes very explicit, since the neglected terms are in evidence.

The gauge situation is now clear. It is worth remarking on the gauge invariance demonstration of Friar and Fallieros.[3] They obtain the MTA within the context of a manifestly gauge-independent formalism. What this means is that, within any gauge, that part of the electromagnetic field represented by the vector potential $\mathbf{A}$ has its effects on the physical system represented by the

MTA. If a scalar potential exists as well, that portion of the field which it represents remains part of $H_0$. The MTA itself is gauge invariant.

## APPENDIX

It is to be shown here that the $N$th-order wave function in Eq. (26) satisfies the Schrödinger equation of Eq. (8) to within terms of order $(eA)^{N+1}$. First the action of $i\partial_t$ on Eq. (26) will be found. Toward this end, Eqs. (7) and (27) lead to

$$\begin{aligned} &i\partial_t \mathcal{A} = \omega \mathcal{A} \ , \\ &i\partial_t \Phi_n = E_n \Phi_n \ , \\ &i\partial_t \mathcal{A}_{mn} = -(\omega_{mn} - \omega)\mathcal{A}_{mn} \ . \end{aligned} \tag{A1}$$

When $i\partial_t$ is applied to Eq. (26), quantities arising from the first and third elements of Eq. (A1) are found to exactly cancel each other, leaving

$$i\partial_t \Psi_0^{(N)} = \left[\sum_{j=0}^{N} \frac{\mathcal{A}^j}{j!}\right] E_0 \Phi_0 + \sum_{k=1}^{N} \left[\sum_{j=0}^{N-k} \frac{\mathcal{A}^j}{j!}\right] \sum_{n_1} \cdots \sum_{n_k} E_{n_1} \Phi_{n_1} \frac{\mathcal{A}_{n_1 n_2} \cdots \mathcal{A}_{n_k 0} \omega^k}{(\omega_{n_1 0} - k\omega) \cdots (\omega_{n_k 0} - \omega)} \ . \tag{A2}$$

To explore spatial derivatives, Eq. (A1) has to be supplemented with

$$-i\nabla \mathcal{A} = e\,\mathbf{A} \ . \tag{A3}$$

The action of the operator $(-i\nabla - e\,\mathbf{A})$ on $\Psi^{(N)}$ is then

$$\begin{aligned} (-i\nabla - e\,\mathbf{A})\Psi^{(N)} = {} & e\,\mathbf{A} \sum_{j=0}^{N} \left[\frac{j\mathcal{A}^{j-1}}{j!} - \frac{\mathcal{A}^j}{j!}\right] \Phi_0 + \sum_{j=0}^{N} \frac{\mathcal{A}^j}{j!} (-i\nabla \Phi_0) \\ & + \sum_{k=1}^{N} \sum_{j=0}^{N-k} e\,\mathbf{A} \left[\frac{j\mathcal{A}^{j-1}}{j!} - \frac{\mathcal{A}^j}{j!}\right] \sum_{n_1} \cdots \sum_{n_k} \Phi_{n_1} \frac{\mathcal{A}_{n_1 n_2} \cdots \mathcal{A}_{n_k 0} \omega^k}{(\omega_{n_1 0} - k\omega) \cdots (\omega_{n_k 0} - \omega)} \\ & + \sum_{k=1}^{N} \left[\sum_{j=0}^{N-k} \frac{\mathcal{A}^j}{j!}\right] \sum_{n_1} \cdots \sum_{n_k} (-i\nabla \Phi_{n_1}) \frac{\mathcal{A}_{n_1 n_2} \cdots \mathcal{A}_{n_k 0} \omega^k}{(\omega_{n_1 0} - k\omega) \cdots (\omega_{n_k 0} - \omega)} \ . \end{aligned} \tag{A4}$$

Those sums over the index $j$ in Eq. (A4) which contain two terms apiece reduce immediately, since

$$e\,\mathbf{A} \sum_{j=0}^{N-k} \left[\frac{j\mathcal{A}^{j-1}}{j!} - \frac{\mathcal{A}^j}{j!}\right] = -e\,\mathbf{A} \frac{\mathcal{A}^{N-k}}{(N-k)!} \ . \tag{A5}$$

The consequence is that the first and third terms in Eq. (A4) are of order $(eA)^{N+1}$, and so they can be neglected. A second application of the $(-i\nabla - e\,\mathbf{A})$ operator to the remaining terms in Eq. (A4) then gives

$$\begin{aligned} (-i\nabla - e\,\mathbf{A})^2 \Psi^{(N)} = {} & e\,\mathbf{A} \cdot \sum_{j=0}^{N} \left[\frac{j\mathcal{A}^{j-1}}{j!} - \frac{\mathcal{A}^j}{j!}\right] (-i\nabla \Phi_0) + \sum_{j=0}^{N} \frac{\mathcal{A}^j}{j!} (-i\nabla^2 \Phi_0 \\ & + e\,\mathbf{A} \cdot \sum_{k=1}^{N} \sum_{j=0}^{N-k} \left[\frac{j\mathcal{A}^{j-1}}{j!} + \frac{\mathcal{A}^j}{j!}\right] \sum_{n_1} \cdots \sum_{n_k} (-i\nabla \Phi_{n_1}) \frac{\mathcal{A}_{n_1 n_2} \cdots \mathcal{A}_{n_k 0} \omega^k}{(\omega_{n_1 0} - k\omega) \cdots (\omega_{n_k 0} - \omega)} \\ & + \sum_{k=1}^{N} \sum_{j=0}^{N-k} \left[\frac{\mathcal{A}^j}{j!}\right] \sum_{n_1} \cdots \sum_{n_k} [(-i\nabla)^2 \Phi_{n_1}] \frac{\mathcal{A}_{n_1 n_2} \cdots \mathcal{A}_{n_k 0} \omega^k}{(\omega_{n_1 0} - k\omega) \cdots (\omega_{n_k 0} - \omega)} \ . \end{aligned} \tag{A6}$$

With the help of Eq. (A5), one obtains

$$\begin{aligned} \left[\frac{(-i\nabla - e\,\mathbf{A})^2}{2m} + V\right] \Psi^{(N)} = {} & \sum_{j=0}^{N} \left[\frac{\mathcal{A}^j}{j!}\right] \left[\frac{(-i\nabla)^2}{2m} + V\right] \Phi_0 \\ & + \sum_{k=1}^{N} \sum_{j=0}^{N-k} \left[\frac{\mathcal{A}^j}{j!}\right] \sum_{n_1} \cdots \sum_{n_k} \left[\frac{(-i\nabla)^2}{2m} + V\right] \Phi_{n_1} \frac{\mathcal{A}_{n_1 n_2} \cdots \mathcal{A}_{n_k 0} \omega^k}{(\omega_{n_1 0} - k\omega) \cdots (\omega_{n_k 0} - \omega)} \ . \end{aligned} \tag{A7}$$

Since, for any $n$, $\Phi$ satisfies the field-free Schrödinger equation

$$[(-i\nabla)^2/2m + V]\Phi_n = E_n \Phi_n \ , \tag{A8}$$

then the right-hand sides of Eqs. (A2) and (A7) are equal, and so $\Psi^{(N)}$ satisfies Eq. (8) to order $(eA)^N$, *quod erat demonstrandum*.

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