Improved calculation of total cross section for pair production by relativistic heavy ions

P. B. Eby
Space Science Laboratory, National Aeronautics and Space Administration,
Marshall Space Flight Center, Huntsville, Alabama 35812
(Received 27 September 1988)

A calculation of the total cross section for direct electron-positron pair production by heavy ions is described. It combines the use of the Weizsäcker-Williams method for low-energy transfers and existing calculations for high-energy transfers. Higher-order corrections to the total cross section are calculated based on the Weizsäcker-Williams method and existing results for pair production by photons.

I. INTRODUCTION

Calculation of direct pair production by charged particles was done in the 1930's by Bhabha,[1] Racah,[2] and Landau and Lifshitz,[3] and more recently by Murota, Ueda, and Tanaka[4] (MUT), Ternovskii,[5] Kelner,[6] and Kelner and Kotov[7] (KK). Kokoulin and Petrukhin[8] have given an approximate formula based on Refs. 4–6 and Wright[9] has given a critical review of existing calculations. The Bhabha calculation applied second-order perturbation theory to the Dirac equation with the Coulomb field of the incident particle considered as the perturbation. This calculation is accurate in principle but the results are difficult to evaluate without making approximations. The other calculations are based on the standard lowest-order QED calculation. The MUT calculation separates the contributions from transverse, longitudinal, and scalar photons from the beginning, while the KK calculation uses the usual summing procedure to include all the possible polarizations of the particles involved. The MUT procedure also uses a different coordinate system from the KK calculation to do the integrations and the KK procedure has a more sophisticated treatment of screening. However, both calculations start from the same matrix element and should give equivalent results.

Refs. 1, 4, and 7 show that the approximation known as the Weizsäcker-Williams (WW) method[10–14] is a good one in these calculations and that the results of the calculations agree with what is expected from the WW method. This method is based on a Fourier decomposition of the Coulomb field of the incident particle followed by an integration over impact parameters (see Jackson[10]). The results are accurate in this case provided the incident particle Lorentz factor γ satisfies the condition $\gamma \gg 1$ and the energy transfer e (units of electron rest-mass energy will be used throughout) is such that $e \ll \gamma$. The WW method has an undetermined parameter which corresponds to the minimum impact parameter assumed, however, and this makes it difficult to get specific results from the calculations. The MUT calculation also has an undetermined parameter in the final result as does the Bhabha result. However, the KK calculation has no such undetermined parameter, although it is only good when all the particles involved are relativistic. We therefore adopt the procedure here of comparing the KK and WW results in a range of energy transfer where they both should be valid, i.e., $e \gg 1$ and $e \ll \gamma$, and thereby determine the appropriate value of the undetermined parameter in the WW equivalent photon spectrum.

Use of the WW method allows us to improve existing calculations in two ways. First, we can use known results for the cross section for pair production by photons to more accurately include the contribution from the low-energy transfer region $e \approx 2$. In particular, we can use the Racah formula for the total cross section for pair production by low-energy photons (see Motz, Olsen, and Koch[15] for a review of formulas for pair production by photons). This Racah formula is the Bethe-Heitler cross section integrated over all angles and possible energies of the produced electron and positron. It is not possible to express it in closed form but we will use a numerical approximation due to Maximon which is listed in the Appendix. Secondly, we can include higher-order corrections to photon pair production calculated by Øverbø, Mork, and Olsen[16] and Davies, Bethe, and Maximon.[17] These results are available as tabulated results for low-energy transfer and as approximate formulas for higher energy as listed in the Appendix. These calculations include higher-order terms in absorber atomic number Z_2.

So the plan of the calculation is to use the tabulated results of KK for high-energy transfer, since these are the most accurate results available in this region and also have the best treatment of screening, and to join these to the WW results for low-energy transfer. We will compare the results in the region where both calculations are valid. Higher-order corrections from Refs. 16 and 17 and the results of KK will be interpolated and integrated numerically to obtain the total cross section. An independent calculation using the WW approximation will also be described which terminates the integration at $e = \gamma$. This turns out to give good agreement with the more accurate calculation.

Higher-order terms in projectile atomic number Z_1 will not be calculated here but we believe they are small for the following reasons. Reasoning in analogy to the case of energy loss of heavy ions, there are higher-order corrections in Z_1 from two sources. Low-energy corrections arise from the dipole approximation (see Ahlen[18] for a review) but any similar effects should be small for the higher energies considered here because of the validity of the WW approximation for low-energy transfer. Also, there are higher-order terms for the energy loss calculation for high-energy transfers due to the distortion of the electron wave function by the incident ion (see Eby and Morgan[19] and Eby and Sung[20]). It is possible that an analogous effect exists in this case but it should make little difference to the total cross section since the high-energy transfers contribute little to the total cross section. At any rate, experiments should be able to resolve this issue.

The motivation for this calculation is the recently suggested possibility of measuring energies of very-high-energy heavy cosmic rays by counting the pairs produced in emulsions. The feasibility of doing this is being studied at the present time.

II. THEORY

We first describe the WW theory as given in Ref. 10. For an incident ion with $\gamma \gg 1$ the number of virtual quanta per unit energy interval $N(e)$ is given by (Ref. 10, Eq. 15.59)

$$N(e)=\left[\frac{2Z_1^2\alpha}{\pi e}\right][\ln(1.123\gamma\hbar/eb_{\min})-\tfrac{1}{2}] ,$$

where $b_{\min}$ is the minimum impact parameter and α is the fine-structure constant. We will write this as

$$N(e)=\left[\frac{2Z_1^2\alpha}{\pi e}\right][\ln(\gamma/e)+\epsilon] , \tag{1}$$

where

$$\epsilon=\ln(1.123\lambda/b_{\min})-\tfrac{1}{2}$$

and λ is the Compton wavelength of the electron. This distribution is valid for $e \ll \gamma$ and the more accurate expression involving Bessel functions goes rapidly to zero for $e>\gamma$. Equation (1) should be a good approximation for $e \ll \gamma$ but as already mentioned the parameter ϵ is not determined *a priori*. So we compare the WW method with that due to KK. This is a treatment based on the lowest-order Feynman diagrams listed in Ref. 7. The result is expressed as

$$d\sigma_{\rm KK}/de=(16/\pi)[(Z_1Z_2\alpha r_0)^2][F(E,v)/e] , \tag{2}$$

where E is the incident particle energy, $v=e/E$, and r_0 is the classical electron radius. This calculation was done for muons but should be valid for any heavy particle. The results for $F(E,v)$ are tabulated in KK for $Z_2=11$ and 82. They are assumed to scale as Z_1^2. These results have no undetermined parameter and include screening in a more sophisticated way than the other available calculations.

The WW result for the cross section $d\sigma_{\rm WW}/de$ for pair production is then given by

$$d\sigma_{\rm WW}/de=N(e)\sigma_B(e) ,$$

where $\sigma_B(e)$ is the Racah expression for the total cross section for pair production by photons, valid for any electron and positron energy and neglecting screening. It is computed using the formula in the Appendix. Comparison of this with Eq. (2) allows us to determine ϵ as we will show. We can then use the combination of the WW result for low-energy transfer with the KK result and obtain the total cross section by numerical integration. The corrections to the Born approximation are then computed using the Øverbø *et al.* results to include higher-order corrections to the photon pair production cross section with the same type of numerical integration performed.

According to MUT screening is important if

$$\frac{e}{2e_+e_-} \ll \alpha Z_2^{1/3}/M^2 ,$$

where e_- and e_+ are the electron and positron energies and $M^2=1+e_+e_-/\gamma^2$. Neglecting the second term in M we see that the dividing line between the regions of screening and nonscreening is given by

$$e_{s\pm}=(e/2)(1\pm\sqrt{1-e_z/e}) ,$$

where $e_z=2/(\alpha Z_2^{1/3})$. No roots exist for $e<e_z$ so we can write an expression for the total cross section based only on the WW method as

$$\sigma_T=\int_2^{e_z}N(e)\sigma_B(e)de + \int_{e_z}^{\gamma}N(e)de\left[2\int_1^{e_{s-}}de_+\frac{d\sigma_{\rm NS}(e_+,e_-)}{de_+} + \int_{e_{s-}}^{e_{s+}}de_+\frac{d\sigma_S(e_+,e_-)}{de_+}\right] . \tag{3}$$

The expressions for $d\sigma_{\rm NS}/de_+$ and $d\sigma_S/de_+$ are the high-energy Born approximation formulas given in the Appendix (where the subscripts NS and S refer to nonscreened and screened, respectively). We will see that this formula gives close agreement with the one based on a combination of the WW and KK results as described above, even though the integration is terminated at $e=\gamma$.

The corresponding result from MUT is

$$\sigma_{\rm MUT}=\frac{28}{27\pi}(Z_1Z_2\alpha r_0)^2\ln(e_z/2) \times[3\ln(\alpha_{\rm MUT}\gamma)\ln(2\gamma/e_z)+\ln(e_z/2)^2] , \tag{4}$$

where $\alpha_{\rm MUT}$ is the undetermined parameter in the calculation. Bhabha [Ref. 1, Eq. (47)] gives essentially the same formula except there are additional arbitrary constants in the terms containing e_z. Ternovskii[5] also gives the same formula except e_z is replaced by $(190/Z_2^{1/3})$.

III. RESULTS FOR 200 GeV/NUCLEON

We first describe results for an incident particle energy of 200 GeV/nucleon since beams are now available from the Centre Européen de Recherches Nucléaires (CERN) heavy-ion accelerator at this energy for $Z_1=8$ and 16. Figure 1 shows a comparison of the tabulated results of KK for $\gamma=200$ and $Z_2=11$ as compared to the WW method for various values of the undetermined parameter ϵ occurring in $N(e)$. We see that the value of ϵ that best matches the KK calculation for intermediate energies where both calculations should be accurate is $\epsilon \approx 0$. The

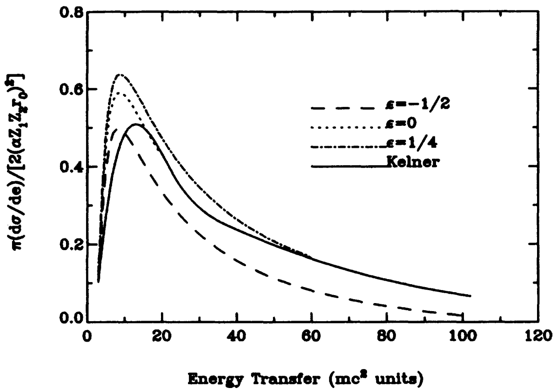

FIG. 1. Differential cross section per unit energy transfer e normalized as shown for the KK tabulated data for $Z_2=11$ compared to the WW calculation for various values of ϵ and $\gamma=200$.

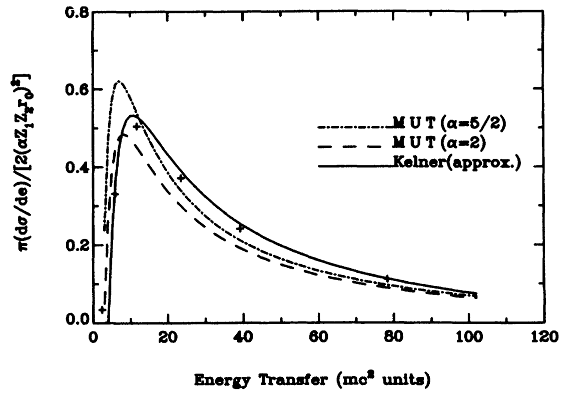

FIG. 2. Differential cross section per unit energy transfer e normalized as shown for $\gamma=200$. The MUT calculation for various values of α_{MUT} is compared to the tabulated values of KK with $Z_2=11$ indicated by + and the approximate formula due to Kelner [Eq. (A1) of the Appendix] given by the solid line.

$\sigma_B(e)$ used in the WW calculation was the complete Racah formula with no screening using the reduced formula due to Maximon as listed in the Appendix. This formula is accurate to one part in 10^4. Neglect of screening is valid for the low-energy transfers where this formula is used. If we join the WW calculation to the KK calculation at $e=20$ where they agree then the combined curve will give the best estimate of the correct cross section. The KK curve is the interpolated values using cubic spline interpolation. The combined curve has been integrated numerically and the results are given in Table I. We see that the use of the KK+WW ($\epsilon=0$) curves increases the KK result for the total cross section by 3.5% for $Z_2=11$ and 6% for $Z_2=82$. (These Z_2 values are the only ones that were tabulated in KK.) The curve for $\epsilon=\frac{1}{4}$ gives a value for the total which is about 7% higher than the $\epsilon=0$ curve for $Z_2=11$ where the curves have been joined as shown in the figure. For $\epsilon=-\frac{1}{2}$ the WW curve does not cross the KK curve and we see that the total cross section is about half the $\epsilon=0$ case if we integrate out to $e_{\max}=120$ where the curve goes to zero. In the other cases the upper limit of integration was taken to be $e_{\max}=1000-1500$ and the results were found to be insensitive to this choice since the KK curve goes quickly to zero for $e>200$. Because the effective Z_2 for emulsions is ≈ 40 the correct results for emulsion should be intermediate between the $Z_2=11$ and 82 so we believe the best estimate for the total cross section normalized as shown is

$$\sigma/[(2/\pi)(\alpha Z_1 Z_2 r_0)^2] \approx 30 \ ,$$

with an accuracy of about 10%.

TABLE I. Total cross sections calculated from the theoretical formulas based on KK, WW, and Øverbø *et al.*

Theoretical formula	$\sigma/[(2/\pi)(\alpha^2 Z_1^2 Z_2^2 r_0^2)]$
Total cross section for $\gamma=200$ and $e_{\max}=1500$	
Kelner($Z_2=11$)	30.0
Kelner($Z_2=82$)	28.4
Kelner($Z_2=11$)+WW($\epsilon=\frac{1}{4}$)	33.2
Kelner($Z_2=11$)+WW($\epsilon=0$)	31.1
Kelner($Z_2=82$)+WW($\epsilon=0$)	30.0
Kelner($Z_2=82$)+WW($\epsilon=0$)+Øverbø($Z_2=44$)	29.4
Kelner($Z_2=82$)+WW($\epsilon=0$)+Øverbø($Z_2=82$)	27.7
WW($\epsilon=-\frac{1}{2}$)	15.8
Total cross section for $\gamma=2\times10^5$ and $e_{\max}=10^6$	
Kelner($Z_2=11$)+WW($\epsilon=0$)	499.0
With Øverbø($Z_2=44$)	486.0
With Øverbø($Z_2=82$)	464.0
Total cross section for $\gamma=10^4$ and $e_{\max}=10^5$	
Kelner($Z_2=11$)+WW($\epsilon=0$)	216.0
With Øverbø($Z_2=44$)	210.0
With Øverbø($Z_2=82$)	199.0

Figure 2 shows the results of a numerical integration of the doubly differential cross sections over the energy distribution of one member of the pair. The cross sections used were the unscreened MUT expression [Ref. 4, Eq. (23)] for $\alpha_{\mathrm{MUT}}=2$ and $\frac{5}{2}$ and we see that these give reasonable agreement with the KK results for $Z_2=11$ as indicated by +. Also, an approximate formula from Kelner [Ref. 6, Eq. (28) listed in the Appendix] was intergrated numerically for the unscreened case and we see that this gives good agreement with the results of KK. This further indicates that screening is not very important at this energy and this is illustrated in Fig. 3 where the KK curves for $Z_2=11$ and 82 are compared. The MUT results are dependent on an undetermined parameter α_{MUT} which is seen to limit the usefulness of this calculation for obtaining precise predictions.

Figures 4 and 5 show the results of Øverbø *et al.*,[16] which we have used to calculate higher-order corrections to the previously described calculations. Figure 4 gives the values taken directly from the tabulated results of Ref. 16. σ_B is the Racah formula calculated as described above. + and × represent the actual data used and the lines the interpolated values used in the calculation. For energy transfers above $e=10$ we used the approximate formula given in Ref. 16 [Eq. (A2) of the Appendix] up to the point where it crossed the Davies-Bethe-Maximon approximate formula [Eq. (A3) of the Appendix] valid for large e. Above this point the latter formula was used as indicated in Fig. 5 by the solid lines. We see that the corrections can be quite large for low-energy transfer but become small for the important energies in the present calculation. For $Z_2=44$, for example, the exact curve is about 5% lower than the Born approximation curve for large energy transfer. In Fig. 6 the cross sections are shown when these corrections are applied. There is very little change in the $Z_2=44$ curve and a slightly larger change for $Z_2=82$. The value of the total cross section calculated using these corrections is decreased by about 2% for $Z_2=44$ and 7.5% for $Z_2=82$ as indicated in Table I. This calculation neglects screening in including the correction but this should not affect the result at this energy. Also, the exact pair production cross section has not been computed above $e=10$ so we have used the approximate formula of Ref. 16 [Eq. (A2) of the Appendix] from $e=10$ to where it joins the Davies-Bethe-Maximon formula and this is probably a good approximation although there are no data in this region to verify this. So for emulsions the higher-order corrections to the Born approximation should be no more than a few percent at this energy.

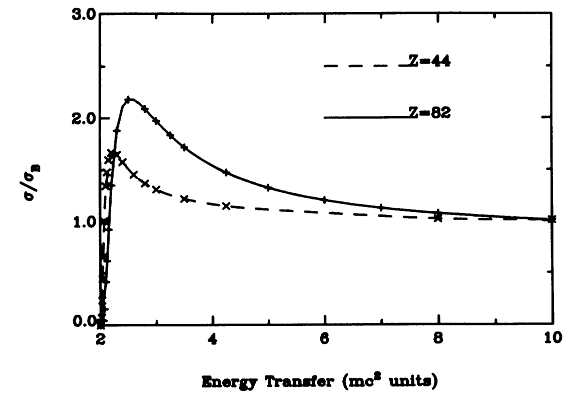

FIG. 4. Ratio of the total cross section for pair production by photons calculated by Øverbø *et al.* to the Born approximation value for $Z_2=44$ and 82. + and × are the data points used and the curve gives the interpolated values used.

Figure 7 gives the average angular distribution of one of the members of the pair with respect to the primary. It was calculated using the Sauter-Gluckstern-Hull formula for $d^2\sigma/de_+d\Omega_+$ integrated over de_+ and over the WW virtual photon spectrum with $\epsilon=0$ This formula is displayed in Ref. 15, Formula 3D-2000, and is valid for a point nucleus with no screening. This integration was terminated at $e=100$ and increasing this upper limit raises the angular distribution only for angles less than a few degrees. We see that there are a few large angle pairs but the bulk of them are less than 10°. This distribution is of interest in experimental methods of counting pairs in emulsions.

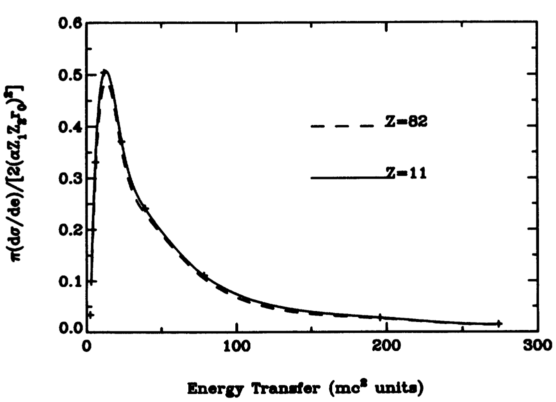

FIG. 3. Comparison of the tabulated data of KK for the differential cross section per unit energy interval e normalized as shown for $Z_2=11$ and 82. $\gamma=200$ and + shows the tabulated data points for $Z_2=11$.

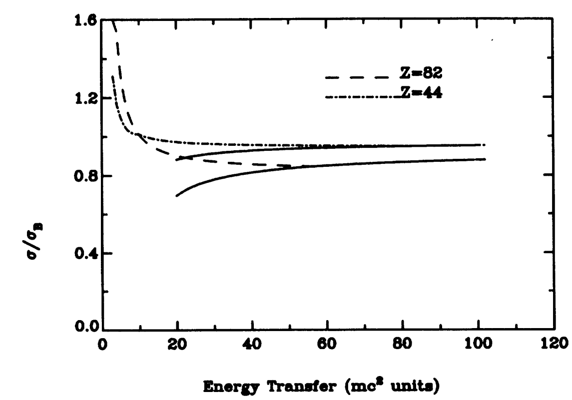

FIG. 5. Ratio of the Øverbø *et al.* approximate calculation of the total cross section for pair production by photons to the Born approximation value for $Z_2=44$ and 82. The solid curves are the approximate Davies-Bethe-Maximon formula.

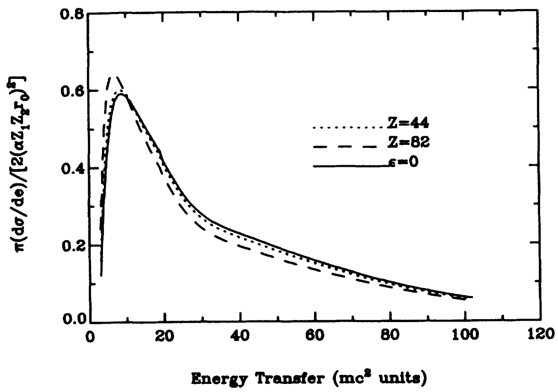

FIG. 6. Differential cross section per unit energy interval e normalized as shown for $\gamma=200$. The solid line is the KK calculation with $Z_2=11$ joined to the WW curve for $\epsilon=0$ as in Fig. 1. The other two curves show the solid line with the Øverbø *et al.* corrections shown in Figs. 4 and 5 applied for $Z_2=44$ and 8.

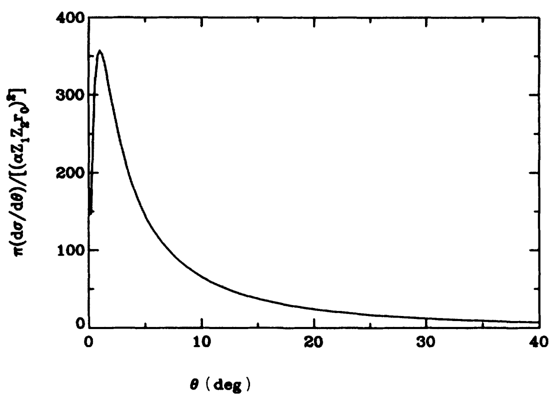

FIG. 7. Angular distribution for either member of the pair vs angle θ with respect to the primary. $\gamma=200$ for this curve and the upper limit on the energy transfer was $e=100$.

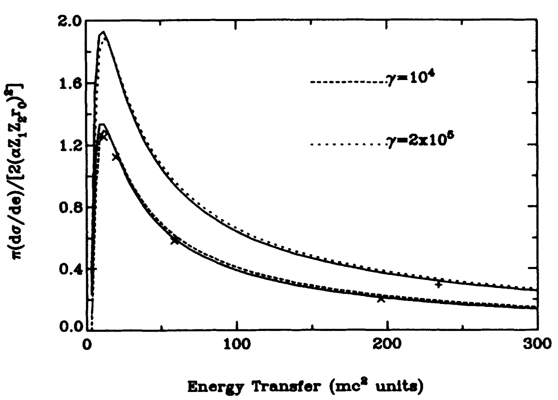

FIG. 8. Differential cross section per unit energy interval e normalized as shown. The solid curves are the WW calculation and the dots and dashes are the approximate Kelner formula $d\sigma_K/de$ listed in the Appendix for the indicated energies. $+$ and $\times$ indicate the tabulated data from KK for $Z_2=11$.

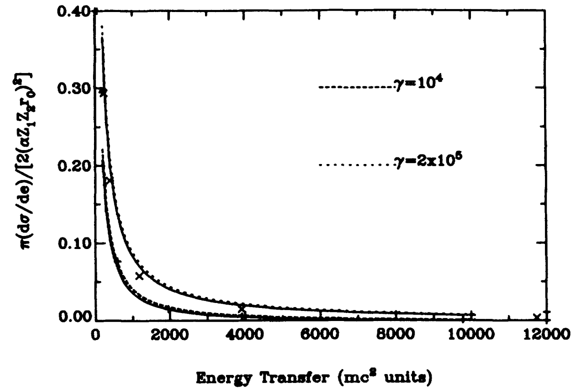

FIG. 9. Same as Fig. 8.

IV. RESULTS FOR LARGE γ

Figures 8 and 9 show a comparison of the results of the WW calculation with the approximate formula $d\sigma_K/de$ due to Kelner and the tabulated results of KK for $Z_2=11$. For the two values of γ chosen we see that the results of all three calculations agree well throughout the entire range of energy transfers although the tabulated values in KK do not cover the lower values of e for $\gamma=2\times10^5$. This indicates that the WW approximation improves in accuracy as γ increases as expected.

Figure 10 shows the total cross section calculated from the curves in Figs. 8 and 9 as the points designated by $\times$. To obtain these values, we used the WW calculation for $e<250$ and the tabulated values of KK with $Z_2=11$ interpolated as before for $e>250$. This was done because the KK calculation is the most accurate for $e\sim\gamma$ and thus handles the high-energy transfers more accurately. The previous $\gamma=200$ result is also included. Table I shows the corresponding calculation using the Øverbø

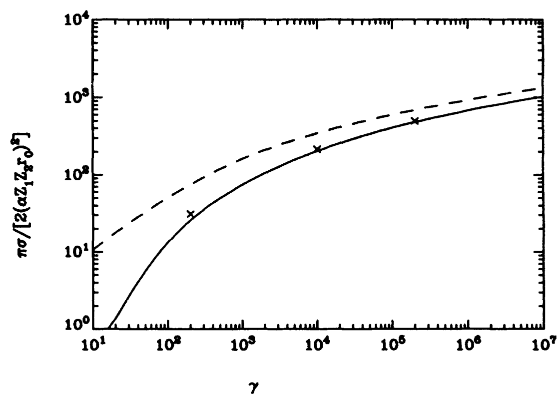

FIG. 10. Total cross section for pair production normalized as shown as a function of incident ion Lorentz factor γ. Solid curve is calculation based on Eq. (3) and dashed line is the approximate formula due to MUT [Eq. (4)]. $\times$ indicates the results of the numerical calculation based on the combination of the WW and KK results as described in the text. Z_2 was taken to be 47 for the solid and dashed curves and $\alpha_{\rm MUT}=1$ was assumed for the dashed curve.

et al. corrections shown in Figs. 4 and 5. The corrections are about the same as for $\gamma=200$. For $Z_2=44$ the results are lowered about 2.5% and for $Z_2=82$ they are lowered 7–8%. This is about half the value of the corrections shown in Fig. 5 over the range of important energy transfers. Thus the large corrections for low-energy transfer (of opposite sign) do not have much effect on the total cross section at the higher energies considered here. The solid curve is the result of the WW calculation [Eq. (3) with $\epsilon=0$] described previously which includes both the screened and nonscreened region and terminates the intergration at $e=\gamma$. Z_2 was taken as 47 for this curve. We see that this curve is in remarkably good agreement with the calculation indicated by $\times$, considering it was a completely independent calculation. The dashed curve is the formula due to MUT [Eq. (4), $Z_2=47$, $\alpha_{\rm MUT}=1$] which is often quoted in the literature. This curve is considerably higher than the other two curves, primarily because the approximations made in the formula which is integrated are not accurate.

V. CONCLUSIONS

We have shown that the total cross section for pair production by heavy ions as calculated here is considerably smaller than existing calculations due to our improved treatment of low-energy transfers and a more accurate integration of the differential cross section. The higher-order corrections to the total cross section reduce it by only a few precent for intermediate absorber Z_2 and by as much as 7–8% for $Z_2\approx 80$ over a large range of energies.

ACKNOWLEDGMENTS

This work was supported by the Marshall Space Flight Center Center Director Discretionary Fund in connection with the CERN EMU04 emulsion experiment.

APPENDIX

For completeness, we list the formulas we have used in the calculation described in the text.

The Racah formula was calculated using the expressions due to Maximon:[16]

$$\sigma_B(e)=\alpha Z_2^2 r_0^2\frac{2\pi}{3}\left[\frac{e-2}{e}\right]^3 \times\left(1+\tfrac{1}{2}x+\tfrac{23}{40}x^2+\tfrac{11}{60}x^3+\tfrac{29}{960}x^4\right),$$

where

$$x=\frac{2e-4}{2+e+2(2e)^{1/2}} .$$

This was used for $e<3.8$. For $e>3.8$ we used

$$\sigma_B(e)=\alpha Z_2^2 r_0^2\left[\tfrac{28}{9}\ln 2e-\tfrac{218}{27}+\left[\frac{2}{e}\right]^2\left[6\ln 2e-\tfrac{7}{2}+\tfrac{2}{3}\ln^3 2e-\ln^2 2e-\frac{\pi^2}{3}\ln 2e+\frac{\pi^2}{6}+2\eta(3)\right]-\left[\frac{2}{e}\right]^4\left(\tfrac{3}{16}\ln 2e+\tfrac{1}{8}\right)-\left[\frac{2}{e}\right]^6\left[\frac{29}{9\times 256}\ln 2e-\frac{77}{27\times 512}\right]\right],$$

with $\eta(3)=1.202$. Øverbø *et al.*[16] state that these formulas are accurate to 1 part in 10^4 in the regions indicated.

The approximate formula due to Kelner [Ref. 6, Eq. (28)] is

$$\frac{d^2\sigma_K}{de_+de_-}=\frac{2Z_1^2Z_2^2\alpha^2r_0^2}{\pi e^4}\left[2\left(e_+^2+e_-^2+\tfrac{2}{3}e_+e_-\right)\left[[\ln(2e_+e_-/e)-\tfrac{1}{2}][\ln(\gamma^2/e_+e_-)-1]-\frac{\pi^2}{12}\right]-\tfrac{2}{3}(e_+-e_-)^2[\ln(2e_+e_-/e)-1]\right],$$

where $e=e_++e_-$. This is valid for $e\ll\gamma$ and $e_+\gg 1$, $e_-\gg 1$ and neglects screening. For reference, we define

$$\frac{d\sigma_K}{de}=\int_1^{e-1}de_+\frac{d^2\sigma_K}{de_+de_-} . \tag{A1}$$

We have calculated this integral numerically for the examples given in the text. The approximate formula due to Øverbø *et al.* for corrections to the Born approximation is

$$\sigma/\sigma_B=1+a+b/(e-2) , \tag{A2}$$

where

$$a=-0.488(\alpha Z_2)^2-0.07(\alpha Z_2)^4 ,$$

$$b=5.06(\alpha Z_2)^2-2.1(\alpha Z_2)^4 .$$

This formula is not derived rigorously but reasons are given in the paper to expect the formula is valid up to $e\approx 20$. The Davies-Bethe-Maximon formula is given by

$$\sigma_{\mathrm{DBM}}(e)=\alpha Z_2^2 r_0^2[\tfrac{28}{9}\ln(2e)-\tfrac{28}{9}f(Z_2)-\tfrac{218}{27}] . \quad \text{(A3)}$$

The values for $f(Z_2)$ used here are $f(44)=0.13$ and $f(82)=0.33$ as given in Ref. 15.

For the calculation based on the WW method we used

$$\frac{d\sigma_{\mathrm{NS}}(e_+,e_-)}{de_+}=4\alpha Z_2^2 r_0^2 \frac{e_+^2+e_-^2+\frac{2}{3}e_+e_-}{e^3}[\ln(2e_+e_-/e)-\tfrac{1}{2}] ,$$

$$\frac{d\sigma_S(e_+,e_-)}{de_+}=4\alpha Z_2^2 r_0^2 \left[\frac{e_+^2+e_-^2+\frac{2}{3}e_+e_-}{e^3}[\ln(183Z_2^{-1/3})]+\tfrac{1}{9}(e_+e_-/e^3)\right] .$$

These are approximate forms of the Bethe-Heitler cross section valid for $e_+ \gg 1$ and $e_- \gg 1$ as given in Ref. 15 for the nonscreened and screened cases.

[1]H. J. Bhabha, Proc. R. Soc. London, Ser. A **152**, 559 (1935).

[2]G. Racah, Nuovo Cimento **14**, 93 (1937).

[3]L. D. Landau and E. M. Lifshitz, Phys. Z. Sowjetunion **6**, 244 (1934).

[4]T. Murota, A. Ueda, and H. Tanaka, Prog. Theor. Phys. **16**, 482 (1956).

[5]F. F. Ternovskii, Zh. Eksp. Teor. Fiz. **37**, 793 (1860) [Sov. Phys.—JETP **10**, 565 (1960)]; **37**, 1010 (1960) [**10**, 718 (1960)].

[6]S. R. Kelner, Yad. Fiz. **5**, 1092 (1966) [Sov. J. Nucl. Phys. **5**, 778 (1967)].

[7]S. R. Kelner and Y. D. Kotov, Yad. Fiz. **7**, 324 (1968) [Sov. J. Nucl. Phys. **7**, 237 (1968)].

[8]R. P. Kokoulin and A. A. Petrukhin, in Proceedings of the 11th International Conference on Cosmic Rays, Budapest, 1970 (unpublished); in *Proceedings of the 12th International Conference on Cosmic Rays, Hobart, Tasmania, 1971* (University of Tasmania Press, Hobart, 1971), p. 2436; Acta Phys. Acad. Sci. Hung. **29**, Suppl. **4**, 73 (1969).

[9]A. G. Wright, J. Phys. A **6**, 79 (1973); **6**, 79 (1973).

[10]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Chap. 15.

[11]A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Wiley, New York, 1965), Chap. 34.

[12]I. Y. Pomeranchuk and I. M. Shushkevich, Nucl. Phys. **23**, 452 (1961).

[13]V. N. Gribov, V. A. Kolkunov, L. B. Okun', and V. M. Shekher, Zh. Eksp. Teor. Fiz. **41**, 1839 (1961) [Sov. Phys.—JETP **14**, 1308 (1962)].

[14]H. A. Olsen, Phys. Rev. D **19**, 100 (1979).

[15]J. W. Motz, H. A. Olsen, and H. W. Koch, Rev. Mod. Phys. **41**, 581 (1969).

[16]I. Øverbø, K. J. Mork, and H. A. Olsen, Phys. Rev. A **8**, 668 (1973).

[17]H. Davies, H. A. Bethe, and L. C. Maximon, Phys. Rev. **93**, 788 (1954).

[18]S. P. Ahlen, Rev. Mod. Phys. **52**, 121 (1980).

[19]P. B. Eby and S. H. Morgan, Phys. Rev. A **5**, 2536 (1972).

[20]P. B. Eby and C. C. Sung, Phys. Rev. A **33**, 3767 (1986).