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Power tail of an intensity distribution function of the bad-cavity laser with field and population fluctuations

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We investigate theoretically the statistical properties of the bad-cavity laser with field and population fluctuations. This system is equivalent to the stochastic Toda oscillator, whose probability distribution obeys the Kramers equation with a position-dependent diffusion coefficient. An approximate probability distribution function is calculated in the stationary state by the method of the orthogonal polynomial expansion. We predict novel statistical features of the laser intensity resulting from the *power tail* of the intensity distribution. Diverging intensity moments and the deviation of the photon-counting statistics from the Poisson type are also given.

Deterministic and statistical dynamics of lasers have been extensively studied in recent years from a viewpoint of the nonequilibrium and nonlinear dissipative statistical physics. The laser is a good example of a *deterministic* dynamical system for the study of chaos, instabilities, and bifurcation sequences.[1] On the other hand, *statistical* fluctuations in laser radiation have increased current interests in the field of wide applications of lasers:[2] The information transmission rates in optical communications, the accuracy of optical computing techniques, and the reliability of spectroscopic data are all crucially dependent upon and limited by the fluctuations present in laser light.

The Langevin equations and/or the Fokker-Planck equations of laser variables have been investigated in detail by many researchers to clarify the statistical features of lasers, especially, dye lasers. Dynamics of lasers are characterized by three time scales: the field decay time K^{-1}, the polarization relaxation time $\gamma_\perp^{-1}$, and the population relaxation time $\gamma_\parallel^{-1}$. Thus far, the lasers with a high-Q cavity (under the good-cavity condition) have received great attention because its condition is more easily accessible experimentally. In the limit of $K \ll \gamma_\perp, \gamma_\parallel$, we can employ the simplest model, i.e., the one-variable Langevin equation to study the statistics of dye laser intensity,[3] while in $K, \gamma_\parallel \ll \gamma_\perp$, both intensity and population difference must be considered by the adiabatic elimination of only the polarization.[4] In addition to dye laser, the CO_2 laser has been recently studied in the limit of $\gamma_\parallel \ll K, \gamma_\perp$.[5] On the other hand, the laser with a low-Q cavity (under the band-cavity condition $K \gg \gamma_\perp, \gamma_\parallel$) is also an interesting system in terms of the *dissipative* nonlinear statistical physics. In this system, we need to consider both the atomic polarization and the population difference to discuss its dynamics by eliminating only the electric field. Experimental studies on this system have recently progressed by use of the far-infrared (FIR) lasers.[6] Deterministic modulation properties of the bad-cavity system were comprehensibly clarified in terms of optical chaos and bifurcations.[7,8] In this Rapid Communication, we pay attention to the stochastic dynamics of the laser with a low-Q cavity described by the coupled Langevin equations and the two-variable Fokker-Planck equation and clarify for the first time the statistical properties of the radiation under the bad-cavity condition.

Here we should mention that the number of operating modes affects drastically the deterministic and the statistical dynamics of laser radiation. In accordance with its number, we must use the proper model to describe the lasers. In the single-mode operation, the laser becomes a low-dimensional system if fluctuations can be neglected, which has been studied in terms of chaos. When several longitudinal modes oscillate, on the other hand, the system has many degrees of freedom which requires the high-dimensional model.[1,9] In particular, infinitely many modes operate simultaneously as in the dye laser; stochastic forces can be employed to describe the effects of the off-resonant modes on the relevant radiation mode. However, the central limit theorem cannot be applied in this multimode system because the modes are not independent of one another.

This Rapid Communication involves the bad-cavity laser with field and population fluctuations. The field fluctuation which is paid particular attention in our model includes both the spontaneous emission noise and the off-resonant mode fluctuations to the relevant resonant mode. No polarization fluctuation is considered and it is shown to be suppressed in, e.g., the strong excitation case. Concerning a low-Q cavity, our Langevin equations contain not only the *additive* noises but also the *multiplicative* noises. We treat the relevant Langevin equations as the stochastic Toda oscillator model and transform it to the two-dimensional Fokker-Planck equation. We seek an approximate analytic solution in the stationary state by the method of the orthogonal function expansion developed by Risken.[10] Under the condition that the population difference has the Gaussian distribution, the intensity distribution function with a *power tail* is approximately obtained. Novel statistical features of the laser intensity which come from its power tail are predicted. The photoelectron counting probability for short observation time is also given to show a difference from the Poisson statistics. The comparison with the good-cavity case is also made.

There are three kinds of fluctuations: the field fluctuation $\Gamma_E(t) \equiv \Gamma_1(t) + i\Gamma_2(t)$, the atomic polarization fluctuation $\Gamma_P(t)\equiv\Gamma_3(t)+i\Gamma_4(t)$, and the population fluctuation $\Gamma_D(t)\equiv\Gamma_5(t)$, which are applied respectively to each variable. With a very low-Q cavity ($K\gg\gamma_\perp,\gamma_\parallel$), the electric field can be adiabatically eliminated[11] from the Maxwell-Bloch equations to obtain the Langevin equations:

$$\frac{d}{d\tau}\bar{P}(\tau)=\bar{P}(\bar{D}-1)-i\frac{\bar{D}}{2}\left[\frac{S_E}{\gamma\bar{A}}\right]^{1/2}\Gamma_E(\tau)+\frac{1}{2}\left[\frac{S_P}{\gamma\bar{A}}\right]^{1/2}\Gamma_P(\tau)\,, \tag{1a}$$

$$\frac{d}{d\tau}\bar{D}(\tau)=-\gamma\bar{D}+\gamma(\bar{A}+1)-\gamma\bar{A}|\bar{P}|^2+(\gamma S_D)^{1/2}\Gamma_D(\tau)-\tfrac{1}{2}(\gamma\bar{A}S_E)^{1/2}\,\mathrm{Im}[\bar{P}^*\Gamma_E(\tau)]\,, \tag{1b}$$

where $\tau=\gamma_\perp t$ is a normalized time, $\bar{P}(\tau)$ and $\bar{D}(\tau)$ are the normalized polarization envelope and the normalized population difference, respectively. In the deterministic case ($S_E=S_P=S_D=0$), $|\bar{P}|$ and $\bar{D}$ become unity in the steady state above threshold. Ratio $\gamma=\gamma_\parallel/\gamma_\perp$ is restricted to be less than 2 and the pump parameter $\bar{A}$ is positive above threshold. Noise strengths S_E, S_P, and S_D are assumed to be constant parameters for simplicity.

In the strong excitation case $\bar{A}\gg 1$, fluctuating forces applied on the polarization become weak. Then the phase of polarization is nearly constant in time. Thus, we consider the situation that $\Gamma_E(\tau)$ and $\Gamma_D(\tau)$ play dominant roles as the fluctuations on the population difference $\bar{D}$ of this system. Eliminating $\bar{D}$, Eqs. (1) become the second-order stochastic differential equation of $|\bar{P}(\tau)|$. In addition, using the logarithmic transformation $u(\tau)\equiv 2\ln|\bar{P}(\tau)|$, the stochastic Toda oscillator model with the Toda potential[7] $f(u)=\exp(u)-u-1$ is derived:

$$\frac{d^2u}{d\tau^2}=-\gamma\frac{du}{d\tau}-2\gamma\bar{A}\frac{df}{du}\pm\exp\left[\frac{u}{2}\right](\gamma\bar{A}S_E)^{1/2}\Gamma_1(\tau)+2(\gamma S_D)^{1/2}\Gamma_5(\tau)\,. \tag{2}$$

The third and fourth terms of the right-hand side of Eq. (2) work as a multiplicative and an additive noise on u, respectively. u can also be considered as the position of the Brownian particle in the potential $2\gamma\bar{A}f(u)$ under the position-dependent temperature. In the first approximation, the fluctuations are assumed to be the Gaussian white noises: $\langle\Gamma_i(\tau)\rangle=0$ and $\langle\Gamma_i(\tau)\Gamma_j(\tau')\rangle=2\delta_{ij}\delta(\tau-\tau')$. In this case, this Langevin equation for two variables $u(\tau)$ and $\dot{u}(\tau)\equiv du(\tau)/d\tau$ leads to the two-variable Fokker-Planck equation for the probability distribution $W(u,\dot{u},\tau)$:

$$\frac{\partial W(u,\dot{u},\tau)}{\partial\tau}=\mathcal{L}_{\mathrm{FP}}(u,\dot{u})W(u,\dot{u},\tau)\,, \tag{3}$$

where $\mathcal{L}_{\mathrm{FP}}$ is the Fokker-Planck operator:

$$\mathcal{L}_{\mathrm{FP}}(u,\dot{u})\equiv-\dot{u}\frac{\partial}{\partial u}+2\gamma\bar{A}\left[\frac{df(u)}{du}\right]\frac{\partial}{\partial\dot{u}}+\gamma\frac{\partial}{\partial\dot{u}}\left[\dot{u}+v_d^2(u)\frac{\partial}{\partial\dot{u}}\right]. \tag{4}$$

This Fokker-Planck equation is the same as the Kramers equation[12] with the position-dependent diffusion coefficient $\gamma v_d^2(u)$ which is characterized by the diffusion velocity $v_d(u)\equiv[\bar{A}S_E\exp(u)+4S_D]^{1/2}$.

Applying the natural boundary conditions $W(\pm\infty,\dot{u},\tau)=W(u,\pm\infty,\tau)\equiv 0$, which stem from the shape of the Toda potential $f(u)$, the Fokker-Planck equation can be solved exactly by the Hermite polynomial expansion[10] of the distribution function W with respect to $\dot{u}$:

$$W(u,\dot{u},\tau)=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{c_n(u,\tau)}{\sqrt{n!}}\exp\left[-\frac{(\dot{u})^2}{2v_d^2(u)}\right]\times H_n\left[\frac{\dot{u}}{v_d(u)}\right]\frac{1}{v_d(u)}\,, \tag{5}$$

where $H_n(x)$ is the Hermite polynomials. Using the orthogonal relation of the Hermite functions, we obtain the one-sided recurrence equation of motion for the expansion coefficients $c_n(u,\tau)$, $n=0,1,2,\ldots$, that is,

$$\frac{\partial c_n(u,\tau)}{\partial\tau}=\hat{a}_n^{(-3)}c_{n-3}+\hat{a}_n^{(-1)}c_{n-1}+\hat{a}_n^{(0)}c_n+\hat{a}_n^{(1)}c_{n+1}\,, \tag{6}$$

where $c_n(u,\tau)=0$ for $n<0$ and $\hat{a}_n^{(k)}$'s are the coefficient operators with respect to u:

$$\hat{a}_n^{(-3)}=-[n(n-1)(n-2)]^{1/2}[dv_d(u)/du]\,, \tag{7a}$$

$$\hat{a}_n^{(-1)}=-\sqrt{n}\left[v_d(u)\frac{\partial}{\partial u}+\frac{2\gamma\bar{A}}{v_d(u)}\frac{df(u)}{du}\right]-2n\sqrt{n}[dv_d(u)/du]\,, \tag{7b}$$

$$\hat{a}_n^{(0)}=-n\gamma\,, \tag{7c}$$

$$\hat{a}_n^{(1)}=-\sqrt{n+1}\,v_d(u)\frac{\partial}{\partial u}-(n+1)\sqrt{n+1}[dv_d(u)/du]\,. \tag{7d}$$

These recurrence relations are exactly equivalent to the original Fokker-Planck equation (3).

Now we consider the stationary state ($\partial/\partial\tau=0$) and pay special attention to the case $c_n^{\mathrm{st}}(u)=0$ for $n=1,2,\ldots$ where the probability current in u direction integrated over the "velocity" $\dot{u}$ vanishes, in order to obtain an analytical expression of the solution. This corresponds to the situation in which the population difference D has a Gaussian distribution around its mean $\langle D\rangle$. In this case, the 0th coefficient $c_0^{\mathrm{st}}(u)$ must satisfy the equation $\hat{a}_1^{(-1)}c_0^{\mathrm{st}}(u)=\hat{a}_3^{(-3)}c_0^{\mathrm{st}}(u)=0$, then the probability distribution becomes trivial, i.e., $W^{\mathrm{st}}(u,\dot{u})=0$ in a mathematically rigorous sense. Here we seek an approximate solution to discuss analytically the statistical properties. Integrating the differential equation: $\hat{a}_1^{(-1)}c_0^{\mathrm{st}}(u)=0$, we obtain an approximate expression of the distribution function of u's:

$$W^{\rm st}(u)=\int_{-\infty}^{\infty}W^{\rm st}(u,\dot u)d\dot u=c_0^{\rm st}(u)=N\exp\left[\frac{\gamma\bar A}{2S_D}u\right]\left[1+\frac{\bar A S_E}{4S_D}\exp(u)\right]^{-(1+2\gamma/S_E+\gamma\bar A/2S_D)},\tag{8}$$

where N is the normalization constant independent of u. This result is valid only under the case of $\hat a_3^{(-3)}c_0^{\rm st}(u)\simeq 0$, that is

$$\bar A S_E\exp(u)c_0^{\rm st}(u)/[\bar A S_E\exp(u)+4S_D]^{1/2}\simeq 0\,,\tag{9}$$

which will be discussed later. Here, we note that the solution (8) is exactly correct in the $S_E=0$ case.

The intensity distribution function is easily derived from $W^{\rm st}(u)$ by use of the normalized intensity $\bar I\equiv\bar A\exp(u)\geq 0$ and $W^{\rm st}(\bar I)=c_0^{\rm st}(\ln|\bar I/\bar A|)/\bar I$:

$$W^{\rm st}(\bar I)=\mathcal{N}\bar I^{\gamma\bar A/2S_D-1}\left[1+\frac{S_E}{4S_D}\bar I\right]^{-(1+2\gamma/S_E+\gamma\bar A/2S_D)},\tag{10}$$

where $\mathcal{N}$ is the normalization constant independent of $\bar I$. This resembles the F distribution with a power tail of the exponent $2+2\gamma/S_E$ and is plotted in Fig. 1 for several values of S_E and S_D. In the limiting cases, we have

$$W^{\rm st}(\bar I)=\begin{cases}\mathcal{N}_1\left[\dfrac{1}{\bar I}\right]^{2+2\gamma/S_E}\exp\left[-\dfrac{2\gamma\bar A}{S_E}\dfrac{1}{\bar I}\right] & \text{for } S_E\gg S_D,\ \text{where the approximation is bad}\,, \quad (11\text{a})\\[2ex] \mathcal{N}_2\bar I^{\gamma\bar A/2S_D-1}\exp\left[-\dfrac{\gamma}{2S_D}\bar I\right] & \text{for } S_D\gg S_E,\ \text{where the approximation is good}\,. \quad (11\text{b})\end{cases}$$

Here $\mathcal{N}_1$ and $\mathcal{N}_2$ are the normalization constants. In the $S_E\gg S_D$ case, the exponent of the power tail remains the same as $2+2\gamma/S_E$ of Eq. (10). On the other hand, the laser intensity has the Γ distribution (11b) without a power tail only in the case of $S_D\gg S_E$ where the multiplicative noises are neglected in Eqs. (1).

Stationary moments of the laser intensity are calculated from $W^{\rm st}(\bar I)$ as

$$M_n=\langle\bar I^n\rangle=\left[\frac{4S_D}{S_E}\right]^n B\left[\frac{\gamma\bar A}{2S_D}+n,\frac{2\gamma}{S_E}+1-n\right]\times\left[B\left[\frac{\gamma\bar A}{2S_D},\frac{2\gamma}{S_E}+1\right]\right]^{-1},\tag{12}$$

where $B(x,y)$ is the beta function and integer n is restricted to be less than $1+2\gamma/S_E$ (i.e., M_n does not exist for $n\geq 1+2\gamma/S_E$). Mean intensity $\langle\bar I\rangle$ is $\bar A$, which shows no shift from the deterministic value $\bar A$ in contrast with the good-cavity case in Ref. 3. One novel feature is the fact that the nth intensity moment M_n ($n\geq 2$) diverges as the field noise strength S_E approaches $2\gamma(n-1)$ which is independent of the population noise strength S_D. The stationary cumulants K_n are also calculated from the moments M_n and plotted in Fig. 2 as a function of S_E/γ to show the diverging behaviors. The variance (the second cumulant) K_2 exists when $0\leq S_E/\gamma<2$ as

$$K_2=M_2-M_1^2=\frac{S_E/2\gamma}{1-S_E/2\gamma}\bar A^2+\frac{2S_D/\gamma}{1-S_E/2\gamma}\bar A\,,\tag{13}$$

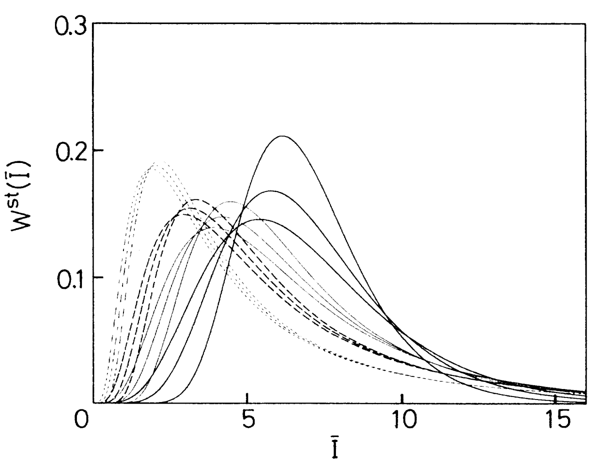

FIG. 1. The probability distribution function of the normalized intensity in the stationary state at $\bar A=7.0$. The tails show power decays. Thick solid lines: $S_E/\gamma=0.09$, thin solid lines: $S_E/\gamma=0.5$, thick broken lines: $S_E/\gamma=1.0$, and thin dotted lines: $S_E/\gamma=2.1$, with three cases of $S_D/\gamma=0.15$, 0.35, and 0.55. The most probable intensities (the maxima of the distribution) becomes small as S_E/γ increases.

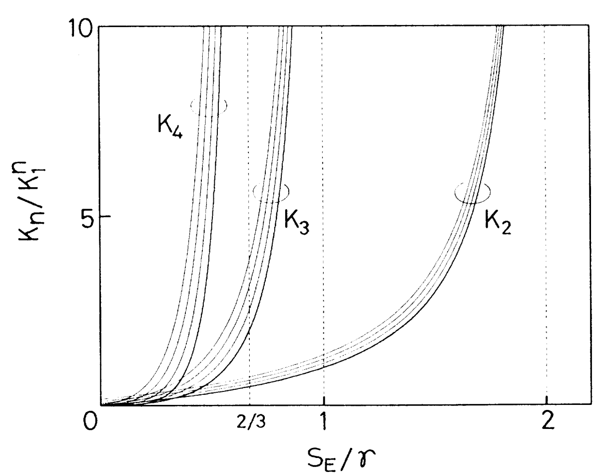

FIG. 2. The stationary cumulants of the intensity normalized by $K_1^n=\langle\bar I\rangle^n=\bar A^n$. The second-, third-, and fourth-order cumulants in the $\bar A=7.0$ case are plotted as a function of the filed noise strength S_E/γ, with a parameter $S_D/\gamma=0.0$, 0.2, 0.4, and 0.6. Thick lines correspond to $S_D/\gamma=0.0$. The nth-order cumulants diverge as $S_E/\gamma\to 2/(n-1)-0$.

which is proportional to the square of the pump parameter $\bar{A}^2$ in the strong excitation regime. This is also different from the good-cavity system[2,3] whose variance is independent of or depends linearly or sublinearly on $\bar{A}$. This intensity variance becomes larger and larger as $S_E/\gamma \to 2$, as shown in Fig. 2.

The statistical properties of the intensity are measured by the photoelectron counting experiments. The relative deviation from the Poisson statistics is described by $\langle (n-\langle n\rangle)^2\rangle = \langle n\rangle + \eta\langle n\rangle^2$, where η is the photon counting coefficient and connected to the intensity moments[13]

$$\eta = \frac{M_2}{M_1^2} - 1 = \frac{S_E/2\gamma}{1-S_E/2\gamma} + \frac{2S_D/\gamma}{1-S_E/2\gamma}\frac{1}{\bar{A}}, \tag{14}$$

which exists when $0 \le S_E/\gamma < 2$. This coefficient η can be defined in $\bar{A} > 0$. Contrary to the good-cavity case,[3,4] η has a finite value even far above threshold ($\bar{A} \gg 1$). The first term of the right-hand side of Eq. (14) $S_E/(2\gamma - S_E)$ is a measure of the deviation from the Poisson statistics[2] in which η becomes zero. Additionally, we consider the stationary photoelectron counting distribution $P(n,T_{\rm obs})$ in which n photoelectrons are registered in an observation time $T_{\rm obs}$. For simplicity, we confine ourselves to the case of the short observation time $T_{\rm obs} \ll \tau_c$, where τ_c is the intensity correlation time and is to be calculated. Then, $P(n,T_{\rm obs})$ is represented as the integral form for $n=0,1,2,\ldots$.[2,14] In a special case of $S_E \gg S_D$, the integration can be carried out and it is expressed by the νth-order modified Bessel function $K_\nu(x)$:

$$P(n,T_{\rm obs}) = \frac{S_E}{n!\gamma\Gamma(2\gamma/S_E)}\left[\frac{2\gamma q T_{\rm obs}\bar{A}}{S_E}\right]^{(n+1)/2+\gamma/S_E} \times K_\nu\left[\left[\frac{8\gamma q T_{\rm obs}\bar{A}}{S_E}\right]^{1/2}\right], \tag{15a}$$

where $\Gamma(x)$ is the gamma function, q is the efficiency of the photon detector, and $\nu = 1 + 2\gamma/S_E - n(> -\frac{1}{2})$. In the another limiting case $S_D \gg S_E$, we have the negative binomial distribution (which is an exact result)

$$P(n,T_{\rm obs}) = \frac{\Gamma(n+\gamma\bar{A}/2S_D)}{n!\Gamma(\gamma\bar{A}/2S_D)}\left[1+\frac{2S_D q T_{\rm obs}}{\gamma}\right]^{-\gamma\bar{A}/2S_D} \times\left[1+\frac{\gamma}{2S_D q T_{\rm obs}}\right]^{-n}. \tag{15b}$$

These show explicitly differences from the Poisson distribution.

The analytical results obtained in this Communication are valid under Eq. (9). In the $S_E = 0$ case, above results are rigorously exact. This approximation (9) is good when (i) $S_D > S_E$ and/or (ii) $\bar{A}$ is not so large. Treating all coefficients $c_n(u,\tau)$ for $n=0,1,2,\ldots$ of Eq. (6), the exact stationary solution can be obtained by the matrix continued fraction method.[10] Although we must discuss the validity of these approximate results by comparing with the exact one, numerical analysis[15] shows that the approximation in this Communication is good. In addition, transient properties are also to be investigated to discuss the correlation function and the correlation time of macrovariables of this system. The relations of the probability distribution with a power tail to the fractal and the Lévy's statistics are an interesting problem. What kind of the stochastic process the laser variables in the bad cavity obey is also to be clarified. Detailed study on this bad-cavity system including the above subjects will be reported in the near future.[15,16]

In conclusion, we investigate the statistical properties of the bad-cavity laser with field and population fluctuations by the stochastic Toda oscillator model. It is shown that the multiplicative field fluctuation results in the power tail of the intensity distribution function and the novel statistics of laser intensity. Comparing with the good-cavity case, the cavity quality is also shown to strongly affect the statistical properties of the laser radiation in addition to the deterministic behaviors.

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