Dynamics of an excited two-level atom in the presence of $N-1$ unexcited atoms in the free space

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The dynamics of the system of N two-level atoms in the free space is studied, when at the initial moment just one of the atoms is excited. It is shown that though the number of the field modes is sufficiently larger than the number of the atoms and all energy is transferred into radiation field for $t \to \infty$, the radiation suppression (in a broader sense) can be still observed.

I. INTRODUCTION

The QED models describing the two-level atoms interacting with the radiation field[1,2] attract due attention not only for their mathematical simplicity and elegance, but also because they actually describe the physical reality and can be verified in experiment (see recent experiments with Rydberg atoms in high-Q cavities[3–7]). These are the reasons why the Jaynes-Cummings model[8] and its various modifications and generalizations are still in the focus of the interest.

In the present paper we want to reanalyze the dynamics of the system of N identical, but distinguishable, two-level atoms interacting with the radiation field, when at the initial time ($t=0$) only one of the atoms is in the excited state and all others are in the ground state. The field is supposed to be in its vacuum state at $t=0$.

Such a problem was treated earlier by Stroud *et al.*[9] They discussed for the first time the effect of the radiation trapping (in the framework of the semiclassical approximation). This effect consists of the fact that the presence of the $N-1$ unexcited atoms in the cavity prevents the emission of the whole energy of the excited atom. The emitted energy gets shared equally by the field and the $N-1$ initially unexcited atoms.

Later the model was analyzed in a completely quantized fashion by Cummings and Dorri.[10] They showed that the interaction of N atoms, in the equivalent mode position, with the single-mode resonant field leads to the radiation suppression, i.e., the photon never gets a fraction greater than $1/N$ of the energy and the initially excited atom definitely traps $[(N-1)/N]^2$ part of its energy.

Successively Cummings[11] presented the exact solution for the spontaneous emission of a single atom which is initially excited in the presence of the $N-1$ initially unexcited atoms, interacting with the M modes of the field. The model was solved under the condition that the atoms were at random space positions. In this case the radiation suppression was observed when the number of the accessible modes M was less than the number of the atoms N. When M was larger than N the radiation suppression did not persist. Particularly, in the free-space case the radiation-suppression effect is preserved only when the near-continuum limit is accompanied by letting N approach infinity. Recently, the physical origin of the radiation suppression has been given by Benivegna and Messina.[12] By constructing the N collective modes of the atomic sample through which the actual interaction with the field takes place when only one excitation is present in the system, they show that the radiation suppression is a consequence of the interatomic coherence induced via the electromagnetic field.

In the present paper we will study the dynamics of the system of the N two-level atoms in free space. We will show that in spite of the fact that the number of field modes is sufficiently larger than the number of atoms and the fact that all energy is transferred into the radiation field for $t \to \infty$, the radiation suppression in a broader sense (to be detailed below) can be observed.

In Sec. II we will describe the model—actually, we will study the Lehmberg model.[13] Then, in Sec. III we will solve the problem when at $t=0$ one of the atoms is in the excited state and the others are in the ground state. Using the Wigner-Weisskopf approximation[14,15] (WWA) we will derive compact analytical expression for the probability to find any atom of the system in the excited state. Section IV is devoted to discussion and conclusions.

II. MODEL AND EQUATION OF MOTION

We will suppose the system of N identical nonoverlapping two-level atoms, at positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, coupled to a near-continuum electromagnetic field via the electric-dipole interaction. In the rotating-wave approximation the model Hamiltonian is

$$\hat{H} = \sum_{j=1}^{N} \tfrac{1}{2}\hbar\omega_0 \hat{\sigma}_3^{(j)} + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}} + \sum_{j=1}^{N} \sum_{\mathbf{k}} (\hbar\lambda_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_j} \hat{a}_{\mathbf{k}} \hat{\sigma}_{+}^{(j)} + \text{H.c.}) , \tag{1}$$

where $\hat{\sigma}_{+}^{(j)}$, $\hat{\sigma}_{-}^{(j)}$, and $\hat{\sigma}_{3}^{(j)}$ are the Pauli raising, lowering, and inversion operators of the atom at the position $\mathbf{r}_j$, respectively. The two states of the atom are separated by the energy $\hbar\omega_0 = E_+ - E_-$. The coupling constant $\lambda_{\mathbf{k}}$ between the atom and the mode with the wave vector $\mathbf{k}$ is assumed equal for all atoms. Finally, $\hat{a}_{\mathbf{k}}^{\dagger}$ and $\hat{a}_{\mathbf{k}}$ are the creation and annihilation operators for $\mathbf{k}$ mode ($[\hat{a}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{q}}]=\delta_{\mathbf{k},\mathbf{q}}$).

Due to the fact that the excitation number operator

$$\hat{R}=\sum_{j=1}^{N}\hat{\sigma}_{+}^{(j)}\hat{\sigma}_{-}^{(j)}+\sum_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}$$

is the integral of motion, the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle=\hat{H}|\psi(t)\rangle \tag{2}$$

for the state vector $|\psi(t)\rangle$ can be solved for the initial condition we are interested in. The solution of the Schrödinger equation strongly depends on the space positions $\mathbf{r}_j$ of the atoms. To make the calculations as easy as possible and the results transparent, we will consider the distances between the atoms equal. This actually can be true just for $N\le 4$ in three-dimensional space. Nevertheless, the results given below can serve generally as a sort of the first approximation for the atomic systems with $N>4$, when $|\mathbf{r}_i-\mathbf{r}_j|=r+\delta_{ij}$. Of course, if $|\mathbf{r}_i-\mathbf{r}_j|\to 0$ for all i,j, then the results are valid for any N.

It is further assumed that the system of the atoms is small enough that the relativistic (time-of-flight) effects may be ignored. In other words, we will impose the restriction that the time required for a light signal to cross the system is small in comparison to the time required for appreciable changes in the atomic levels (for more detailed discussion see Ref. 13).

III. DYNAMICS OF THE N-ATOM SYSTEM

First we briefly describe the dynamics of the two-atom system when at the initial time just one of the atoms is in the excited state. The result (presented here in the WWA) will be given in a closed analytical form for the amplitude of the probability to find the atom at the position $\mathbf{r}_j$ in the excited state.

At $t=0$ let the atom at position $\mathbf{r}_1$ be in the excited state; the atom at $\mathbf{r}_2$ is in the ground state and the field is in its vacuum state,

$$|\psi(t=0)\rangle=|+-;0\rangle\ , \tag{3}$$

where the symbol $+$ $(-)$ in the first (second) place on the right-hand side of (3) denotes the excited (ground) state of the atom at position $\mathbf{r}_1$ ($\mathbf{r}_2$); 0 is for the vacuum state of the radiation field.

The solution of the Schrödinger equation (1) with the initial condition (3) is

$$\begin{aligned}|\psi(t)\rangle=&\exp(-iE_{+}t/\hbar)[C_1(t)|+-;0\rangle+C_2(t)|-+;0\rangle]\\&+\sum_{\mathbf{k}}\exp[-i(E_{-}+\hbar\omega_{\mathbf{k}})t/\hbar]D_{\mathbf{k}}(t)|--;1_{\mathbf{k}}\rangle\ ,\end{aligned} \tag{4}$$

where the amplitude of the probability $C_1(t)$ $[C_2(t)]$ used to find the atom at the position $\mathbf{r}_1$ $[\mathbf{r}_2]$ can be written in the energy representation as

$$C_1(t)=\frac{i}{2\pi}\int_{-\infty}^{\infty}d\varepsilon\, e^{i(\omega_0-\varepsilon)t}\times\frac{\varepsilon-\omega_0-A(\varepsilon,r=0)}{[\varepsilon-\omega_0-A(\varepsilon,r=0)]^2-A^2(\varepsilon,r)}\ ; \tag{5a}$$

$$C_2(t)=\frac{i}{2\pi}\int_{-\infty}^{\infty}d\varepsilon\, e^{i(\omega_0-\varepsilon)t}\times\frac{A(\varepsilon,r)}{[\varepsilon-\omega_0-A(\varepsilon,r=0)]^2-A^2(\varepsilon,r)}\ ; \tag{5b}$$

and

$$A(\varepsilon,r)=\sum_{\mathbf{k}}\frac{|\lambda_{\mathbf{k}}|^2}{\varepsilon-\omega_{\mathbf{k}}+i0}e^{-i\mathbf{k}\cdot\mathbf{r}};\quad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2,\quad r=|\mathbf{r}|\ . \tag{6}$$

In the continuum limit[16] the function $A(\varepsilon,r)$ can be calculated explicitly,

$$A(\varepsilon,r)=\Delta(\varepsilon,r)-i\Gamma(\varepsilon,r)\ . \tag{7}$$

For the "Lamb shift" $\Delta(\varepsilon,r)$ and the "linewidth" $\Gamma(\varepsilon,r)$ we have[16]

$$\Delta(\varepsilon,r)=P\int d\omega_{\mathbf{k}}\frac{e^2\omega_{\mathbf{k}}^2|\mathbf{D}_{12}|^2}{6\pi^2\epsilon_0\hbar c^3}\frac{\xi(\varepsilon,r)}{\omega_{\mathbf{k}}-\varepsilon}\ , \tag{8a}$$

$$\Gamma(\varepsilon,r)=\xi(\varepsilon,r)\Gamma(\varepsilon),\quad \Gamma(\varepsilon)=\frac{e^2\varepsilon^3|\mathbf{D}_{12}|^2}{6\pi\epsilon_0\hbar c^3} \tag{8b}$$

where $\xi(\varepsilon,r)=\sin(\varepsilon r)/\varepsilon r$. Here ϵ_0 is the electric permitivity and $\mathbf{D}_{12}$ is the electric dipole matrix.

If the relativistic effects may be ignored, than we can take $\Delta(\varepsilon,r)$ and $\Gamma(\varepsilon,r)$ equal to their constant value at $\varepsilon=\omega_0$ [this corresponds to the WWA (Refs. 14 and 15)], so that

$$\Delta(\varepsilon,r)\to\Delta(\omega_0,r)\cong\xi(\omega_0,r)\Delta(\omega_0)\equiv\xi\Delta\ , \tag{9a}$$

$$\Gamma(\varepsilon,r)\to\xi(\omega_0,r)\Gamma(\omega_0)\equiv\xi\Gamma\ , \tag{9b}$$

where $\Delta(\omega_0)$ and $\Gamma(\omega_0)$ are the Lamb shift and the radiation linewidth in the Wigner-Weisskopf theory[14,15] of the spontaneous decay of the single atom in the free space. Due to the fact that $\Delta(\omega_0)$ is small, we can for a while neglect terms proportional to Δ. Therefore using the WWA the functions $C_i(t)$ [Eqs. (5)] can be found immediately,

$$C_1(t)=\tfrac{1}{2}e^{-\Gamma t}(e^{-\Gamma\xi t}+e^{\Gamma\xi t})\ , \tag{10a}$$

$$C_2(t)=\tfrac{1}{2}e^{-\Gamma t}(e^{-\Gamma\xi t}-e^{\Gamma\xi t})\ , \tag{10b}$$

and for the probability to find at least one of the atoms in the excited state we have

$$I(t)=\sum_{j=1}^{2}|C_j(t)|^2=\tfrac{1}{2}e^{-2\Gamma(1-\xi)t}+\tfrac{1}{2}e^{-2\Gamma(1+\xi)t}\ . \tag{11}$$

The results presented here for two atoms can be generalized for the system of N atoms which are localized at equal distances. The calculations in this case do not in principle differ from that in the two-atom problem (for details and a more general discussion see Ref. 17).

As mentioned in Sec. II, N atoms can be localized at equal distances only for $N \leq 4$. In the opposite case, the solutions given below can be interpreted as an approximation of the exact solutions. Such an approximation makes sense if the linear extensions of the system are less than the wavelength of the resonant mode.

If we suppose the atom localized at $\mathbf{r}_1$ to be excited at $t=0$, then the probability amplitudes $C_j(t)$ are

$$C_1(t)=\frac{N-1}{N}e^{-\Gamma(1-\xi)t}+\frac{1}{N}e^{-\Gamma[1+(N-1)\xi]t}, \tag{12a}$$

$$C_2(t)=\cdots=C_N(t)=\frac{1}{N}e^{-\Gamma[1+(N-1)\xi]t}-\frac{1}{N}e^{-\Gamma(1-\xi)t}, \tag{12b}$$

and

$$I(t)=\sum_{j=1}^{N}|C_j(t)|^2=\frac{N-1}{N}e^{-2\Gamma(1-\xi)t}+\frac{1}{N}e^{-2\Gamma[1+(N-1)\xi]t}. \tag{13}$$

IV. DISCUSSION AND CONCLUSIONS

From the expressions (12) and (13) it follows that in free space for finite N and $r \neq 0$ the radiation suppression really disappears as predicted by Cummings.[11] The radiation suppression disappears in the sense that after some time ($t \gg 1/\Gamma$) all atoms pass to their ground state, i.e.,

$$\lim_{t\to\infty}|C_j(t)|^2=0,$$

and the whole energy is transferred into the radiation field.

Nevertheless, the effect of radiation suppression does

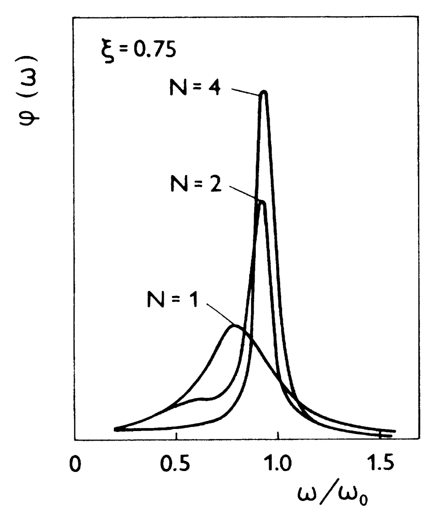

FIG. 1. Spectral line $\phi(\omega)$ for $N=1$, 2, and 4 when $\xi=0.75$ and $\Gamma=\Delta=0.2\omega_0$.

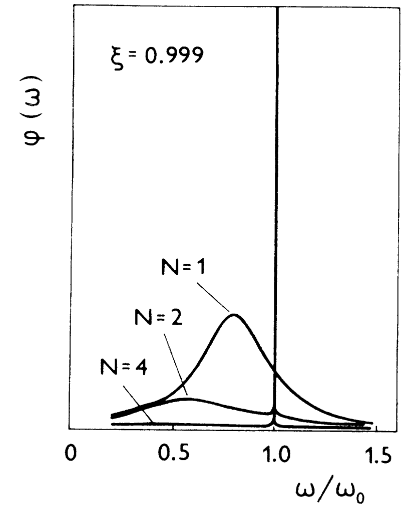

FIG. 2. Spectral line $\phi(\omega)$ for $N=1$, 2, and 4 when $\xi=0.999$ and $\Gamma=\Delta=0.2\omega_0$.

not disappear totally if we understand it in a broader sense: From (12a) it is seen that the initially excited atom does radiate with three different rates and $[(N-1)/N]^2$ part of the energy is radiated with the smallest rate $2\Gamma(1-\xi)$. If the distances between the atoms go to zero, so that $\xi \to 1$, then this fraction of the energy will be "trapped" by the initially excited atom.

It is very instructive to analyze the expression (13) for the probability to find at least one of the atoms in the excited state. This expression gives us information about the global energetical balance in the system and demonstrates that the collection of the N two-level atoms (localized at equal distances) with one excited atom at $t=0$ effectively behaves like a system of two independent, noninteracting even via electromagnetic field, "fictious" two-level atoms in the free space. These two "effective" atoms are characterized by the different damping constants ($\Gamma_1=\Gamma(1-\xi)$ and $\Gamma_2=\Gamma[1+(N-1)\xi]$) and with the different Lamb shifts ($\Delta_1=\Delta(1-\xi)$ and $\Delta_2=\Delta[1+(N-1)\xi]$). Each of these atoms radiates a different fraction of the energy—the "first" $(N-1)/N$ and "second" $1/N$ part of the whole energy. It is seen that the "first" atom is effectively responsible for the radiation suppression.

The best way to see this is to study the spectral properties of the radiation from our system of atoms. The spectral line $\phi(\omega)$ defined as

$$\lim_{t\to\infty}\left\langle\psi(t)\left|\sum_{\mathbf{k}}\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}}\right|\psi(t)\right\rangle=1=\frac{1}{\pi}\int d\omega\,\phi(\omega) \tag{14}$$

is the sum of two Lorentzian contours, each of them is characterized by the parameters Δ_i and Γ_i,

$$\phi(\omega)=\frac{N-1}{N}\frac{\Gamma_1}{[(\omega-\omega_0+\Delta_1)^2+\Gamma_1^2]}+\frac{1}{N}\frac{\Gamma_2}{[(\omega-\omega_0+\Delta_2)^2+\Gamma_2^2]}. \tag{15}$$

Here we explicitly write also the Lamb shifts because they indicate the frequency splitting of the radiation from the "effective" atoms. The spectral line $\phi(\omega)$ for $\xi=0.75$ is given in Fig. 1. The narrow peaks (for $N=2,4$) in the figure correspond to the suppressed radiation and can be distinguished from that part of the radiation which is emitted with the high rate Γ_2 proportional to N (wide peak, which is well seen in Fig. 2 for $N=2$). For comparison the spectral line of the radiation from an isolated atom in the free space is plotted in figure also. The radiation suppression is more transparent when $r\rightarrow 0$. In Fig. 2 the spectral line is plotted for $\xi=0.999$ (this value of ξ corresponds, for instance, to interatomic distances in the crystal and for ω_0 proportional to the optical frequencies $\sim 10^{15}\ s^{-1}$). In this case $N=2$ two peaks can easily be distinguished and the narrow peaks become very sharp.

We can conclude that one can find the radiation-suppression effect by studying the spectral properties of the radiation from the system of N atoms prepared at $t=0$ in a fashion defined earlier. We have demonstrated this claim assuming the atoms to be equally separated. In general, the effect persists also if the separations among the atoms are not equal in the sense that the system of N identical atoms with an excitation number equal to 1 radiate more slowly than an isolated atom.[17] However, in the case we have studied the radiation suppression can be seen in the clearest way.

[1]L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).
[2]H.-I. Yoo and J. H. Eberly, Phys. Rep. **118**, 239 (1985).
[3]S. Haroche, in *New Trends in Atomic Physics,* edited by G. Grynberg and R. Stora (North-Holland, Amsterdam, 1984).
[4]S. Haroche and J. M. Raimond, in *Advances in Atomic and Molecular Physics,* edited by D. R. Bates and B. Bederson (Academic, New York, 1985), Vol. 20.
[5]P. Filipowicz, P. Meystre, G. Rempe, and H. Walther, Opt. Acta **32**, 1105 (1985).
[6]D. Meschede, H. Walther, and G. Muller, Phys. Rev. Lett. **54**, 551 (1985).
[7]G. Rempe, H. Walther, and N. Klein, Phys. Rev. Lett. **58**, 353 (1987).
[8]E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).
[9]C. R. Stroud, J. H. Eberly, W. L. Lama, and L. Mandel, Phys. Rev. A **5**, 1094 (1972).
[10]F. W. Cummings and A. Dorri, Phys. Rev. A **28**, 2282 (1983).
[11]F. W. Cummings, Phys. Rev. A **33**, 1683 (1986).
[12]G. Benivegna and A. Messina, Phys. Lett. A **126**, 249 (1988).
[13]R. H. Lehmberg, Phys. Rev. A **2**, 883; **2**, 889 (1970).
[14]V. Weisskopf and E. Wigner, Z. Phys. **63**, 54 (1930); **65**, 18 (1931).
[15]G. S. Agarwal, in *Quantum Statistical Theories of Spontaneous Emission and Their Relation to Other Approaches,* Vol. 70 of *Springer Tracts in Modern Physics,* edited by G. Höhler (Springer-Verlag, Berlin, 1974).
[16]R. Loudon, *The Quantum Theory of Light,* 2nd ed. (Clarendon, Oxford, 1983).
[17]V. Bužek, Czech. J. Phys. B **38**, 1164 (1988).