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Equivalence of the propagator of quasistatical solutions and the quantum harmonic oscillator

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We establish a connection between a two-dimensional field theory using a classical solution which depends only on one of the coordinates, and the quantum-mechanical harmonic oscillator whose frequency is one-coordinate dependent.

The problem of a harmonic oscillator with a time-dependent frequency has been treated in a number of works.[1] Exact solutions and the exact propagator have been found.[2]

On the other hand, the calculation of the contribution to the generating functional from a classical solution and its neighborhood in a two-dimensional field theory is thoroughly investigated.[3]

In this paper we establish a connection between the two problems mentioned above. We show that the case in which the classical solution considered is a static one, or more generally depends on only one of the two independent variables involved, the two-dimensional field theory is formally equivalent to a one-dimensional time-dependent-frequency harmonic oscillator. Moreover, if the classical configuration chosen is an approximate solution of the equation of motion, we will have an equivalent *forced* time-dependent-frequency harmonic oscillator, with the external force being proportional to a quantity that measures the "degree of exactness" of the solution.

Let us consider a two-dimensional scalar-field theory in Euclidean space, with an action of the form

$$S[\Phi]=\int dx\, dt[\tfrac{1}{2}(\partial\Phi)^2+V(\Phi(x,t))-J\Phi(x,t)]\ , \tag{1}$$

where $V(\Phi)$ is the classical potential and J is a constant external current. For definiteness, suppose there exists a classical *static* configuration $\overline{\Phi}(x)$ which is an almost exact solution of the field equation of motion, that is, $\partial^2\overline{\Phi}\approx V'(\overline{\Phi})-J$, where the prime indicates derivation respective to the field Φ, and $\partial^2\equiv\partial_x^2+\partial_t^2$.

In this case the contribution to the generating functional coming from that quasiclassical solution and its neighborhood may be obtained by expanding Φ around $\overline{\Phi}$, $\Phi=\overline{\Phi}+\eta$, and introducing this expansion in Eq. (1). We obtain

$$Z[J]=\frac{\exp(-S[\overline{\Phi}])}{N}\int\mathcal{D}\eta\exp\left[-\int dx\, dt\{\tfrac{1}{2}(\partial\eta)^2+\tfrac{1}{2}V''(\overline{\Phi})\eta^2+[V'(\overline{\Phi})-\partial^2\overline{\Phi}-J]\eta\}\right]\ , \tag{2}$$

where N is the normalization constant,

$$N=\int\mathcal{D}\Phi\exp\left[-\tfrac{1}{2}\int dx\, dt(\partial\Phi)^2\right]\ , \tag{2a}$$

and where we dropped, as usual, higher-order terms in η. We would like to point out that the linear dependence on η in the exponent of Eq. (2) appears only if $\overline{\Phi}(x)$ is not an exact extremum of the action. We note also that the functional integration in Eq. (2) has the same form as the propagator of a theory where the mass depends on only one of the coordinates, subject to an external current.

We are thus led to the calculation of the quantity

$$\frac{\int\mathcal{D}\eta\exp\left[-\tfrac{1}{2}\int dx\, dt\{\eta(x,t)[-\partial^2-f(x)]\eta(x,t)\}+\int dx\, dt\,\epsilon(x)\eta(x,t)\right]}{\int\mathcal{D}\Phi\exp\left[-\tfrac{1}{2}\int(\partial\Phi)^2dx\, dt\right]}\ , \tag{3}$$

where for convenience we have introduced a function $\epsilon(x)$, which measures in some sense the "degree of exactness" of the quasiclassical static solution $\overline{\Phi}(x)$, that is,

$$\epsilon(x)=-\frac{d^2\overline{\Phi}}{dx^2}+V'(\Phi)-J \;, \tag{4}$$

and we have defined the function of x,

$$f(x)=-V''(\overline{\Phi}(x)) \;. \tag{5}$$

To calculate the numerator of Eq. (3) we consider the Schrödinger equation associated to the operator $-\partial^2-f(x)$. In doing this, we enclose the system in a box Λ of sides L and T, the x and t variables varying, respectively, in the domains $-L/2\le x\le +L/2$ and $-T/2\le t\le +T/2$. We take the eigenfunction η to be separable in the variables x and t, $\eta=\phi(x)\chi(t)$, which leads to the equations,[4]

$$-\frac{d^2\chi(t)}{dt^2}=E_1\chi(t) \;, \tag{6a}$$

$$-\frac{d^2\phi(x)}{dx^2}-f(x)\phi(x)=E_2\phi(x) \;. \tag{6b}$$

The eigenfunctions of Eq. (6a) are $\chi_m(t)=(1/\sqrt{L}\,)\exp(ik_m t)$, with continuous eigenvalues of the energy $E_{1m}=k_m^2$, $m\in\mathbb{Z}$; the eigenvalues of Eq. (6b), E_2, belong partly to a discrete set, $E_{2n}=\lambda_n$, $n\in\mathbb{N}$, and partly to a continuous one, $E_{2m}=\omega_m$, $m\in\mathbb{Z}$. The eigenfunctions $\eta_{mn}(x,t)=\phi_n(x)\chi_m(t)$ are chosen to satisfy periodic boundary conditions in the box Λ.

Expanding the configuration $\eta(x,t)$ in the functional integrand (3) in terms of the eigenfunctions of the associated Schrödinger operator $-\partial^2-f(x)$, we get

$$\eta(x,t)=\sum_{m,n} a_{nm}\phi_n(x)\chi_m(t) \;, \tag{7}$$

with

$$a_{nm}=\int_{-L/2}^{+L/2}dx\int_{-T/2}^{+T/2}dt\,\eta(x,t)\phi_n(x)\chi_m(t) \;. \tag{7a}$$

The functional integration in the numerator of Eq. (3) can be rewritten in terms of the coefficients a_{nm}. Using that $\int_{-T/2}^{+T/2}\chi_m(t)dt=\sqrt{T}\,\delta_{m0}$, and also that $f(x)$ and $\epsilon(x)$ are functions of x only, the integrations over a_{nm} for $m\neq 0$ are Gaussians, therefore being easily performed. This gives for the numerator of Eq. (3) the result,

$$\prod_{\substack{m,n'\\ m\neq 0}}\left[\frac{2\pi}{E_{1n'}+E_{2m}}\right]^{1/2}\int_{-\infty}^{+\infty}\prod_n da_{n0}\exp\left[-\tfrac{1}{2}\sum_n\left[E_{2n}a_{n0}^2+\int_{-L/2}^{+L/2}dx\sqrt{T}\,a_{n0}\epsilon(x)\phi_n(x)\right]\right] . \tag{8}$$

After some manipulations the exponent in Eq. (8) may be recast in the form

$$-\tfrac{1}{2}\left[\int_{-L/2}^{+L/2}dx\left[\sum_{n'}a_{n'0}\phi_{n'}(x)\right]\left[-\frac{d^2}{dx^2}-f(x)\right]\left[\sum_n a_{n0}\phi_n(x)\right]\right]+\sqrt{T}\int_{-L/2}^{+L/2}dx\,\epsilon(x)\sum_n a_{n0}\phi_n(x) \;. \tag{9}$$

At this point we are in the position to show that the two-dimensional problem that we are dealing with, reduces to a problem in one dimension.

This is done by defining a function $q(x)$,

$$q(x)=\frac{1}{\sqrt{T}}\int_{-T/2}^{+T/2}\eta(x,t)dt \;, \tag{10}$$

which allows us to rewrite Eq. (10) as

$$-\tfrac{1}{2}\int_{-L/2}^{+L/2}dx\,q(x)\left[-\frac{d^2}{dx^2}-f(x)\right]q(x)+\sqrt{T}\int_{-L/2}^{+L/2}\epsilon(x)q(x)dx \;.$$

Therefore, the generating functional Eq. (2) is given by

$$Z[J]=\frac{1}{N}\prod_{\substack{n,m\\ m\neq 0}}\left[\frac{2\pi}{E_{1n}+E_{2m}}\right]^{1/2}\exp(-S[\overline{\Phi}])\int\mathcal{D}q\,\exp\left\{-\tfrac{1}{2}\int_{-L/2}^{+L/2}dx\left[\left[\frac{dq}{dx}\right]^2-f(x)q^2(x)\right]\right.$$

$$\left.+\sqrt{T}\int_{-L/2}^{+L/2}dx\,\epsilon(x)q(x)\right\} , \tag{11}$$

where the normalization constant is now given by

$$N=\prod_{\substack{n,m\in\mathbb{Z}\\ m\neq 0}}\left[\frac{2\pi}{\epsilon_{1m}+\epsilon_{2m}}\right]^{1/2}\times\int\mathcal{D}q\exp\left[-\frac{1}{2}\int_{-L/2}^{+L/2}\left[\frac{dq(x)^2}{dx}\right]\right],$$

where $\epsilon_{1m}=(2\pi n/T)^2$ and $\epsilon_{2n}=(2\pi m/L)^2$, $m,n\in\mathbb{Z}$. We should emphasize that $\epsilon_{1m}=E_{1m}$ but $\epsilon_{2n}\neq E_{2n}$.

Equation (11) is, apart from an overall factor, formally equivalent to the generating functional for a one-dimensional harmonic oscillator $q(x)$ with a "time-dependent" frequency $f(x)$, submitted to an external driving force $\sqrt{T}\,\epsilon(x)$. This problem is the same as the one considered in Ref. 2 where the formal problem has been exactly solved.

From expression (10) we see also that in the particular situation when $\overline{\Phi}(x)$ is an *exact* classical static solution $(\epsilon(x)=0)$ of the field equation, the two-dimensional problem reduces to solve the quantum-mechanical problem of a harmonic oscillator with a time-dependent frequency given by $-V''[\overline{\Phi}(x)]$.

In both cases $[\overline{\Phi}(x)$ is an approximate or exact solution], a memory of the other dimension is kept in the overall factor in front of the functional integration of the quantum harmonic oscillator and in the driving external force. In the overall factor we have the eigenvalues E_{1n} associated to the time dimension t.

The result we found is a general one, valid for any case where we are calculating the contribution coming from the neighborhood of a classical configuration which depends only on one of the two independent coordinates. This includes the calculation of the propagator of a two-dimensional theory with a position or time-dependent mass.

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[2]D. C. Khandekar and S. V. Lawande, J. Math. Phys. **20**, 1870 (1979); **16**, 387 (1975); Bin Kang Cheng, Rev. Bras. Fis. **13**, 220 (1983).

[3]See, for example, R. Rajaraman, *Solitons and Instantons—An Introduction to Solitons and Instantons in Quantum Field Theory* (North-Holland, Amsterdam, 1982), and references therein.

[4]The only restriction to the function $f(x)$ is that $E_2\geq 0$, such that the Gaussian functional integration in Eq. (3) is well defined. The treatment of the zero mode is well known (Ref. 3).