Effects of pollution on critical population dynamics

R. Kree,* B. Schaub, and B. Schmittmann
Institut für Theoretische Physik III, IV, Universität Düsseldorf, Universitätsstrasse 1, D-4000 Düsseldorf 1, Federal Republic of Germany
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We investigate the effects of pollution on a population that is on the brink of extinction. In the vicinity of the associated critical point, the temporal scales of the population density fluctuations are found to be completely governed by the diffusive behavior of the pollution density fluctuations. Moreover, the mean value of the population density is found to vanish with a larger power-law exponent in the presence of pollution density fluctuations. Results are obtained within a renormalization-group calculation to $O(\epsilon)$ ($\epsilon=4-d$, d being the spatial dimension).

I. INTRODUCTION

The study of ecological problems has become a major research area of all natural sciences within the past decades. Of decisive importance is the understanding of ecological catastrophies, i.e., situations where, for example, a given species becomes extinct.[1] Such events, which in many cases can be described in terms of continuous phase transitions[2-4] in the modern theory of statistical mechanics exhibit a remarkable feature. In the vicinity of such a critical point a system may be described by one single diverging length scale. It is this length scale which characterizes the large scale properties of a macroscopic system. The behavior of physical quantities near a critical point can be successfully explored using renormalization-group (RG) ideas. Among a variety of techniques based upon these ideas, the powerful ϵ expansion provides one of the most reliable methods to determine critical properties. It allows the analytic calculation of universal features by means of a systematic expansion in powers of $\epsilon=d_c-d$ around the upper critical dimension d_c.

It is the purpose of this article to analyze within a RG approach a most important problem in ecology: the effects of pollution on a population which is on the brink of extinction. Before presenting any details of the model to be studied, we stress the following point.

There are, in general, two strategies available for an investigation of the above question: either one starts from a microscopic model which contains very specific assumptions, or one starts from a phenomenological description which contains the more global aspects of a system. In the first case, the resulting model may not be accessible to analytic treatment and therefore may require extensive numerical work, in particular if one is interested in systems with a large number of degrees of freedom. Moreover, the results may depend very specifically on the values and the types of input parameters, and a generalization may not be easy to achieve. Nevertheless this strategy must be followed if one is interested in quantitative results for *stable* ecological systems. On the other hand, the second strategy can be used with great success in the case of systems which are "*unstable*" in the sense that they are close to criticality. The modern theory of critical phenomena allows one to start from a microscopic model and to extract all the features which are important for the critical behavior. In this way, simple phenomenological theories can be constructed for the quantitative determination of critical properties. This enormous simplification results from the fact that in the vicinity of a continuous phase transition, the behavior of the system is dominated by its long-wavelength, large-time-scale fluctuations. Thus it is independent of microscopic details. This is known as "universality" in RG theory. It allows the analysis of macroscopic properties for whole classes of systems. According to universality the relevant characteristics which lead to different macroscopic properties are determined by a few global features like, e.g., the spatial dimension, the range of the interaction (short or long range), and the number of species.

To be more specific, let us consider a model consisting of a population density $n(x,t)$ and a pollution density $c(x,t)$. The population dynamics is assumed to contain the following ingredients.[1] (a) Members of a species are generated and annihilated, with both the production and annihilation rates being functions of the population density. In addition the production and annihilation rates can be affected by the pollution density. (b) Members of a species can diffuse in a d-dimensional space. (c) There is an absorbing stationary state, so that the population density cannot evolve again, once it has become extinct.

The polluting substance is characterized by the properties. (i) Its motion is purely diffusive. (ii) Its total amount is kept constant, thus no poison can be annihilated or produced in its interaction with the species.

The description in terms of densities can be understood in the spirit of the remarks made above. We consider a coarse-graining procedure where a large number of microscopic degrees of freedom are averaged out. These will be modeled as Gaussian noise terms in Langevin equations.

Despite its simplicity the model described here reveals some very interesting features. In the vicinity of the point where the population density becomes extinct, both spatial and temporal scales of its fluctuations become significantly affected by the fluctuations of the pollution density. In particular, the temporal scales are completely governed by the diffusive behavior of the latter.

The present article is organized as follows. In Sec. II we present the Langevin equations for our model and a generating functional which allows to discuss the statistical mechanics most conveniently. In Sec. III we consider the influence of fluctuations close to the transition point. These will cause singularities which can be absorbed by a renormalization of coupling constants. In Sec. IV we determine general scaling properties of some characteristic functions and find the fixed points of the RG transformations. From these we will obtain the critical exponents describing the power-law behavior of observables in the critical region. Section V contains our conclusions.

II. THE MODEL

We will now construct the Langevin equations for a population density $n(x,t)$ and a pollution density $c(x,t)$ in accordance with the assumptions made in Sec. I,

$$\partial_t n(\mathbf{x},t)=\lambda_0\Delta n(\mathbf{x},t)+R_+(n(\mathbf{x},t),c(\mathbf{x},t)) - R_-(n(\mathbf{x},t),c(\mathbf{x},t))+\xi(\mathbf{x},t) , \quad (2.1a)$$

$$\partial_t c(\mathbf{x},t)=\lambda_0'\Delta c(\mathbf{x},t)+f(\mathbf{x},t) . \quad (2.1b)$$

In (2.1a) and (2.1b) the first terms on the right-hand sides model the diffusive motion of the species and the pollution density, and $\xi(x,t), f(x,t)$ are Langevin forces. The quantities R_+, R_- in (2.1a) describe the production and annihilation rates which depend on the pollution density.

Expanding the difference $R_+ - R_-$ in powers of $n(x,t)$ and setting the constant (lowest-order) term equal to zero to ensure the existence of an absorbing state at $n=0$, we find

$$R_+ - R_- = -\lambda_0 r(c(\mathbf{x},t))n(\mathbf{x},t) - \tfrac{1}{2}\lambda_0 g(c(\mathbf{x},t))n^2(\mathbf{x},t)+\cdots \quad (2.2)$$

with couplings λ_0, r, and g. The first term in (2.2) contains the difference between the birth and death rate of the species which is assumed to be affected by the pollution density. Thus

$$r(c(\mathbf{x},t))=r_0+g_0'c_0+g_0'c(x,t)+O(c^2(x,t)) , \quad (2.3a)$$

$$g(c(\mathbf{x},t))=g+\tilde{g}c_0+O(c(\mathbf{x},t)) , \quad (2.3b)$$

where c_0 corresponds to the constant mean-pollution concentration and $c(\mathbf{x},t)$ describes the temporal and spatial fluctuations of the pollution. The constant g_0' may be absorbed in r_0 defining an effective rate $\tau_0=r_0+g_0'c_0$. Similarly, we may define an effective coupling $g_0=g+\tilde{g}c_0$. We then have

$$\partial_t n(\mathbf{x},t)=\lambda_0\Delta n(\mathbf{x},t)-\lambda_0\tau_0 n(\mathbf{x},t)-\lambda_0 g_0' n(\mathbf{x},t)c(\mathbf{x},t) - \tfrac{1}{2}\lambda_0 g_0 n^2(\mathbf{x},t)+\xi(\mathbf{x},t) . \quad (2.4)$$

Here we have only considered the lowest-order terms in the expansion (2.3). Below we will see that the higher-order terms neglected will not modify the critical behavior.

In the absence of the pollution field $c(x,t)$ and the noise term ξ, Eq. (2.4) yields a stable stationary state $n=0$ (the absorbing state) for $\tau_0\geq 0$ and another stable stationary solution $n\neq 0$ for $\tau_0<0$.[1] Thus the parameter τ_0 measures the distance to the critical point.

The Gaussian noise $\xi(x,t)$ must also respect the absorbing state condition, where

$$\langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle=\lambda_0 f(n(\mathbf{x},t))\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') , \quad (2.5a)$$

with

$$f(n(\mathbf{x},t))=\bar{g}_0 n(\mathbf{x},t)+O(n^2(\mathbf{x},t)) . \quad (2.5b)$$

On the other hand, the pollution density is conserved and obeys a purely diffusive relaxational dynamics, where

$$\langle f(\mathbf{x},t)f(\mathbf{x}',t')\rangle=-2\lambda_0''\Delta\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') . \quad (2.6)$$

By a simple rescaling we can ensure $\lambda_0'=\lambda_0''$ which is preserved under the renormalization group. Equations (2.4) and (2.1b) subjected to (2.5) and (2.6) comprise our model.

In the following, we will be interested in the behavior of correlation and response functions in the vicinity of the critical point. We shall extract the critical properties using the powerful apparatus of renormalized field theory in conjunction with an ϵ expansion about the upper critical dimension d_c of the model[3] (which is $d_c=4$ as will be seen below).

Instead of considering all the correlation functions we may use the generating functional

$$Z[h,j]=\exp\int dt\int d^dx[h(\mathbf{x},t)n(\mathbf{x},t)+j(\mathbf{x},t)c(\mathbf{x},t)] . \quad (2.7)$$

After averaging $Z\{h,j\}$ over the Langevin forces $\xi(x,t)$ and $f(x,t)$, we can obtain the correlation functions by functional differentiation of $\langle Z\{h,j\}\rangle_{\xi,f}$ with respect to h and j at the point $h=j=0$. The response to external disturbances may be studied by adding additional sources $\tilde{h}(\mathbf{x},t)$ and $\tilde{j}(\mathbf{x},t)$ on the right-hand side (rhs) of (2.4) and (2.1b), respectively.

A very convenient functional integral representation for the averaged generating functional $\langle Z\{h,j,\tilde{h},\tilde{j}\}\rangle_{\xi,f}$ has been derived in Refs. 5–9. It can be presented in the form

$$\langle Z[h,j,\tilde{h},\tilde{j}]\rangle=\int D(\tilde{n},n)D(\tilde{c},c)\exp\left[-J[\tilde{n},n,\tilde{c},c]+\int d^dx\int dt(hn+jc+\tilde{h}\tilde{n}+\tilde{j}\tilde{c})\right] . \quad (2.8)$$

For further details on the functional integral see Ref. 7. In this formulation the dynamic functional J completely describes the statistical mechanics. For our model it has the form

$$J[\tilde{n},n,\tilde{c},c]=J_{\rm pop}[\tilde{n},n]+J_{\rm pol}[\tilde{c},c]+J_{\rm coup}[\tilde{n},n,c] \;, \tag{2.9}$$

where

$$J_{\rm pop}[\tilde{n},n]=\int dt \int d^dx \{\tilde{n}(\mathbf{x},t)[\partial_t+\lambda_0(\tau_0-\Delta)]n(\mathbf{x},t)+\tfrac{1}{2}\lambda_0 g_0\tilde{n}(\mathbf{x},t)n(\mathbf{x},t)n(\mathbf{x},t)-\tfrac{1}{2}\lambda_0\bar{g}_0\tilde{n}(\mathbf{x},t)\tilde{n}(\mathbf{x},t)n(\mathbf{x},t)\} \;, \tag{2.10a}$$

$$J_{\rm pol}[\tilde{c},c]=\int dt \int d^dx [\lambda_0'\tilde{c}(\mathbf{x},t)\Delta\tilde{c}(\mathbf{x},t)+\tilde{c}(\mathbf{x},t)(\partial_t-\lambda_0'\Delta)c(\mathbf{x},t)] \;, \tag{2.10b}$$

$$J_{\rm coup}[\tilde{n},n,c]=\int dt \int d^dx \; \lambda_0 g_0'\tilde{n}(\mathbf{x},t)c(\mathbf{x},t)n(\mathbf{x},t) \;. \tag{2.10c}$$

In Eqs. (2.9) and (2.10) we have introduced a population density response field $\tilde{n}(x,t)$ and a pollution density response field $\tilde{c}(x,t)$ after eliminating the noise terms $\xi(x,t)$ and $f(x,t)$. By a simple rescaling of the fields we can ensure $g_0=\bar{g}_0$. In the absence of the pollution density $c(x,t)$ the functional $J\{\tilde{n},n\}$ is invariant under the transformation

$$n(t,\mathbf{x})\to-\tilde{n}(-t,\mathbf{x}) \;, \tag{2.11a}$$

$$\tilde{n}(t,\mathbf{x})\to-n(-t,\mathbf{x}) \;. \tag{2.11b}$$

This property has two important consequences:[3] For $\tau_0>0$, correlators of the population density vanish. Furthermore, the equality of $\bar{g}_0=g_0$ is preserved under RG transformations. For our model, Eq. (2.9), these properties still hold, as can be seen by integrating out the fields $c(x,t),\tilde{c}(x,t)$ which appear at most quadratically. Then we obtain an effective functional $J_{\rm eff}\{\tilde{n},n\}$ of the form

$$J_{\rm eff}[\tilde{n},n]=J_{\rm pop}[\tilde{n},n]+J_{\rm int}[\tilde{n},n] \;, \tag{2.12a}$$

where

$$J_{\rm int}[\tilde{n},n]=-\tfrac{1}{2}\lambda_0 g_0'^2\int dt\int dt'\int d^dx\int d^dx'\tilde{n}(\mathbf{x},t)n(\mathbf{x},t)C_0(\mathbf{x}-\mathbf{x}',|t-t'|)\tilde{n}(\mathbf{x},t')n(\mathbf{x}',t') \tag{2.12b}$$

with $C_0(x,t)$ being the correlator of the pollution density field (see below). Indeed (2.12) is invariant under (2.11), thus we will put $g_0=\tilde{g}_0$ in the following.

Now we consider the naive scaling dimensions of the fields and couplings in the functional (2.9). Introducing a length scale μ^{-1}, we then find

$$x^{-1}\sim\mu,\quad t^{-1}\sim\lambda_0\mu^2\sim\lambda_0'\mu^2,\quad \tau_0\sim\mu^2 \;,$$
$$\tilde{n}\sim n\sim\mu^{d/2},\quad \tilde{c}\sim c\sim\mu^{d/2} \;, \tag{2.13a}$$

$$g_0\sim\mu^{(4-d)/2},\quad g_0'\sim\mu^{(4-d)/2} \;, \tag{2.13b}$$

signaling $d_c=4$ to be the upper critical dimension. It follows that all higher-order couplings arising from additional terms in the expansion of R_+-R_- [Eqs. (2.2) and (2.3)] or, from the noise correlation function [Eq. (2.5)], are irrelevant in the RG approach as they would carry a negative μ dimension.

For a treatment of (2.9) within renormalized perturbation theory, which is to be set up in Sec. III, we need the free response and correlation propagators. Introducing Fourier transforms for the fields $n,\tilde{n},c,\tilde{c}$ in the quadratic part of (2.10a) and in (2.10b),

$$n(\mathbf{x},t)=\int \exp(i\mathbf{k}\cdot\mathbf{x})n(\mathbf{k},t) \;, \tag{2.14}$$

etc., where $\int=\int d^dk/(2\pi)^{d/2}$, we obtain

$$\tilde{C}_0(\mathbf{k},t-t')=\langle c(\mathbf{k},t)\tilde{c}(\mathbf{k}',t')\rangle_0$$
$$=\Theta(t-t')\delta(\mathbf{k}+\mathbf{k}')e^{-\lambda_0'k^2(t-t')} \;, \tag{2.15a}$$

$$C_0(\mathbf{k},t-t')=\langle c(\mathbf{k},t)c(\mathbf{k}',t')\rangle_0$$
$$=\delta(\mathbf{k}'+\mathbf{k}')e^{-\lambda_0'k^2|t-t'|} \;, \tag{2.15b}$$

and

$$G_0(\mathbf{k},t-t')=\langle n(\mathbf{k},t)\tilde{n}(\mathbf{k}',t')\rangle_0$$
$$=\Theta(t-t')\delta(\mathbf{k}+\mathbf{k}')$$
$$\times e^{-\lambda_0(k^2+\tau_0)(t-t')} \;. \tag{2.15c}$$

For a graphical representation of (2.15b) and (2.15c), see Figs. 1(a) and 1(b). In accordance with the path-integral formulation equations (2.8) and (2.9), we will in the following always use $\Theta(0)=0$,[7,8] a condition which ensures causality.

III. RENORMALIZED PERTURBATION THEORY

We will now take into account the influence of fluctuations close to the transition point. These will cause

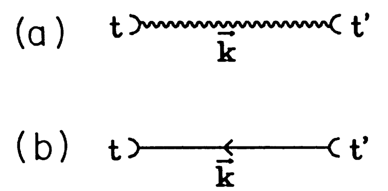

FIG. 1. (a) Free correlation propagator $C_0(k,t-t')$ of the pollution field. (b) Free response propagator $G_0(k,t-t')$ of the population field.

singularities which have to be absorbed by a renormalization of coupling constants.[10] These singularities can be extracted by an evaluation of response functions.

In renormalized perturbation theory, however, it is more convenient to consider the vertex functions. In a graphical representation of the perturbation series they correspond to the one-particle irreducible parts, with extracted external legs, of the diagrams contributing to connected Green's functions. In a RG they can be directly related to the renormalized coupling constants of the theory.[10]

Considering the perturbation expansion in more detail, we find that certain contributions contain divergencies from large internal momenta. Renormalization theory tells us that it suffices to regularize the divergences in the vertex functions corresponding to couplings with nonnegative μ dimension in order to make all individual terms in the perturbation expansion convergent. The regularization subtracts the divergences. These are then absorbed in a multiplicative redefinition of the fields and coupling constants. This simple reparametrization structure of the theory can be exploited to investigate the behavior of response functions on large scales. Technically, this is achieved with the aid of RG equations which will be discussed in Sec. IV.

Consider now the connected response function $G(q_1,\omega_1;q_2,\omega_2)$ which is defined by

$$G(q_1,\omega_1,q_2,\omega_2)=\frac{\delta}{\delta\tilde{h}(q_2,\omega_2)}\langle n(q_1,\omega_1)\rangle$$
$$=\langle \tilde{n}(q_2,\omega_2)n(q_1,\omega_1)\rangle \tag{3.1}$$

[see Eq. (2.8)]. Its corresponding vertex function $\Gamma_{11}(q_1,\omega_1;q_2,\omega_2)$ is the inverse of G. Thus, to lowest order

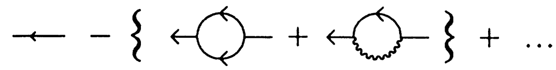

FIG. 2. Graphical representation of the perturbation expansion for $\Gamma_{11}(q,\omega)$ to one-loop order.

$$\Gamma_{11}(q_1,\omega_1,q_2,\omega_2)=\delta(q_1+q_2)\delta(\omega_1+\omega_2)\times[i\omega_1+\lambda_0(\tau_0+q_1^2)] \ . \tag{3.2}$$

To one-loop order we obtain, absorbing a common factor $\delta(q_1+q_2)\delta(\omega_1+\omega_2)$ in the definition of a reduced vertex function $\Gamma_{11}(q,\omega)$,

$$\Gamma_{11}(q,\omega)=i\omega+\lambda_0(\tau_0+q^2)+\tfrac{1}{2}\lambda_0^2 g_0^2\int_p\frac{1}{i\omega+\lambda_0[p^2+(p-q)^2+2\tau_0]}-\lambda_0^2 g_0'^2\int_p\frac{1}{i\omega+\lambda_0(p^2+\tau_0)+\lambda_0'(p-q)^2} \ . \tag{3.3}$$

For a graphical representation of $\Gamma_{11}(q,\omega)$ see Fig. 2. Apart from (3.3) we will need two further vertex functions in order to carry out the renormalization program discussed above. These describe the modifications of the (marginal) couplings g_0,g_0' due to the interactions and their definitions can be read off from (2.10a) and (2.10c). We find up to one-loop order for the reduced three-point vertex function $\Gamma_{\tilde{n}nn}=\Gamma_{12}$ [see Fig. 3(a)]

$$\begin{aligned}\Gamma_{12}(q_A,\omega_A,q_B,\omega_B,q_c,\omega_c)&=\lambda_0 g_0-\tfrac{1}{2}\lambda_0^3 g_0^3\int_p\frac{1}{\{-i\omega_c+\lambda_0[(q_c-p)^2+p^2+2\tau_0]\}\{-i\omega_B-i\omega_c+\lambda_0[(q_B+q_c-p)^2+p^2+2\tau_0]\}}\\&+\lambda_0^3 g_0 g_0'^2\int_p\frac{1}{\{-i\omega_B+\lambda_0[(q_c-p)^2+\tau_0]+\lambda_0'p^2\}}\,\frac{1}{\{-i\omega_B-i\omega_c+\lambda_0[(q_B+q_c-p)^2+(q_c-p)^2+2\tau_0]\}}\\&+2\lambda_0^3 g_0 g_0'^2\int_p\frac{1}{\{-i\omega_c+\lambda_0[(q_c-p)^2+\tau_0]+\lambda_0'p^2\}}\,\frac{1}{\{-i\omega_B-i\omega_c+\lambda_0[(q_B+q_c-p)^2+\tau_0]+\lambda_0'p^2\}}\ ,\end{aligned} \tag{3.4}$$

where we have again absorbed a common factor $\delta(q_A+q_B+q_C)\delta(\omega_A+\omega_B+\omega_C)$. The reduced three-point vertex function $\Gamma_{\tilde{n}nc}=\Gamma_{11,1}$ is given up to one-loop order by [see Fig. 3(b)]

$$\begin{aligned}\Gamma_{11,1}(q_A,\omega_A,q_B,\omega_B,q_c,\omega_c)&=\lambda_0 g_0'-\lambda_0^3 g_0^2 g_0'\\&\times\int_p\frac{1}{\{-i\omega_c+\lambda_0[(q_c-p)^2+p^2+2\tau_0]\}\{-i\omega_B-i\omega_c+\lambda_0[(q_B+q_c-p)^2+p^2+2\tau_0]\}}\\&+\lambda_0^3 g_0'^3\int_p\frac{1}{\{-i\omega_c+\lambda_0[(q_c-p)^2+\tau_0]+\lambda_0'p^2\}\{-i\omega_B-i\omega_C+\lambda_0[(q_B+q_c-p)^2+\tau_0]+\lambda_0'p^2\}}\ .\end{aligned} \tag{3.5}$$

We will now extract the divergences arising from the momentum integrals (for p large) in Eqs. (3.3)–(3.5). To this end we can expand the denominators in $\Gamma_{11}(q,\omega)$ in powers of the external momentum q and frequency ω. We obtain to $O(q^2)$ and $O(\omega)$

$$\Gamma_{11}(q,\omega)=i\omega+\lambda_0(\tau_0+q^2)+\tfrac{1}{4}\lambda_0 g_0^2\int_p\left[\frac{1}{p^2+\tau_0}-\frac{i\omega/2\lambda_0}{(p^2+\tau_0)^2}-\frac{q^2/2}{(p^2+\tau_0)^2}+\frac{(\mathbf{p}\cdot\mathbf{q})^2}{(p^2+\tau_0)^3}\right]$$
$$-\lambda_0 g_0'^2\bar{\rho}_0\int_p\left[\frac{1}{p^2+\tau_0\bar{\rho}_0}-\frac{i\omega\bar{\rho}_0/\lambda_0}{(p^2+\tau_0\bar{\rho}_0)^2}-\frac{\bar{\rho}_0 q^2}{(p^2+\tau_0\bar{\rho}_0)^2}+\frac{4\bar{\rho}_0^2(\mathbf{p}\cdot\mathbf{q})^2}{(p^2+\tau_0\bar{\rho}_0)^3}\right], \tag{3.6}$$

where we have introduced the quantities

$$\rho_0=\frac{\lambda_0'}{\lambda_0},\quad \tilde{\rho}_0=\frac{\rho_0}{1+\rho_0},\quad \bar{\rho}_0=\frac{\tilde{\rho}_0}{\rho_0}\;. \tag{3.7}$$

Performing the momentum integrals using the technique of dimensional regularization,[10] we find

$$\Gamma_{11}(q,\omega)=i\omega+\lambda_0\tau_0+\lambda_0 q^2$$
$$+\tfrac{1}{4}\lambda_0 g_0^2 G_\epsilon\left[-\frac{2}{\epsilon}\frac{\tau_0^{1-\epsilon/2}}{(1-\epsilon/2)}-\frac{i\omega}{\lambda_0}\tau_0^{-\epsilon/2}\frac{1}{\epsilon}-\frac{q^2}{2}\tau^{-\epsilon/2}\frac{1}{\epsilon}\right]$$
$$-\lambda_0 g_0'^2\bar{\rho}_0 G_\epsilon\left[-\frac{2}{\epsilon}\frac{(\tau_0\bar{\rho}_0)^{1-\epsilon/2}}{(1-\epsilon/2)}-\frac{i\omega}{\lambda_0}\bar{\rho}_0(\tau_0\bar{\rho}_0)^{-\epsilon/2}\frac{2}{\epsilon}-q^2\bar{\rho}_0\rho_0(\tau_0\bar{\rho}_0)^{-\epsilon/2}\frac{2}{\epsilon}+q^2\bar{\rho}_0^2\rho_0^2(\tau_0\bar{\rho}_0)^{-\epsilon/2}\frac{2}{\epsilon}\right], \tag{3.8}$$

where $G_\epsilon=\Gamma(1+\epsilon/2)/(4\pi)^{d/2}$ and $\epsilon=4-d$. Expanding $\tau^{-\epsilon}\cong 1-\epsilon\ln\tau$, etc., we see that Eq. (3.8) exhibits singularities for $\epsilon\to 0$ arising from the $O(1)$ terms. These pole terms can be absorbed by introducing multiplicative renormalization factors for the fields and couplings, which we choose in the form

$$n=Z_n^{1/2}n_R,\quad \tilde{n}=Z_n^{1/2}\tilde{n}_R\;, \tag{3.9a}$$

$$c=Z_c^{1/2}c_R,\quad \tilde{c}=Z_{\tilde{c}}^{1/2}\tilde{c}_R\;, \tag{3.9b}$$

$$\lambda_0=Z_n^{-1}Z_\lambda\lambda,\quad \lambda_0'=Z_c^{-1/2}Z_{\tilde{c}}^{-1/2}Z_{\lambda'}\lambda'\;, \tag{3.9c}$$

$$\tau_0=Z_\lambda^{-1}Z_\tau\tau\;, \tag{3.9d}$$

$$g_0^2=G_\epsilon^{-1}Z_\lambda^{-2}Z_n^{-1}Z_u u,\quad u=G_\epsilon g^2\;, \tag{3.9e}$$

$$g_0'^2=G_\epsilon^{-1}Z_\lambda^{-2}Z_v v,\quad v=G_\epsilon g'^2\;. \tag{3.9f}$$

We then obtain

$$\Gamma_{11}^R(q,\omega)=Z_n i\omega+Z_\tau\lambda\tau+Z_\lambda\lambda q^2$$
$$-i\omega\left[\frac{u}{4\epsilon}-\bar{\rho}^2\frac{2v}{\epsilon}\right]-\lambda\tau\left[\frac{u}{2\epsilon}-\bar{\rho}^2\frac{2v}{\epsilon}\right]$$
$$-\lambda q^2\left[\frac{u}{8\epsilon}-\rho\bar{\rho}^3\frac{2v}{\epsilon}\right] \tag{3.10}$$

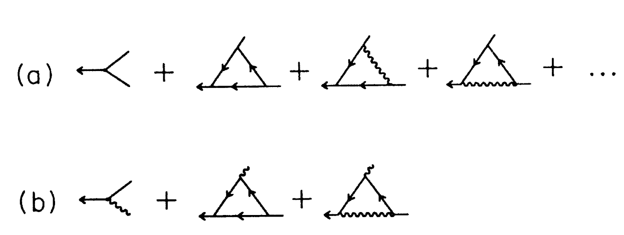

FIG. 3. Graphical representation of the perturbation expansion for (a) the vertex function $\Gamma_{1,2}$ and (b) the vertex function $\Gamma_{11,1}$ to one-loop order.

and thus

$$Z_n=1+\frac{u}{4\epsilon}-\bar{\rho}^2\frac{2v}{\epsilon}\;, \tag{3.11a}$$

$$Z_\tau=1+\frac{u}{2\epsilon}-\bar{\rho}^2\frac{2v}{\epsilon}\;, \tag{3.11b}$$

$$Z_\lambda=1+\frac{u}{8\epsilon}-\rho\bar{\rho}^3\frac{2v}{\epsilon}\;. \tag{3.11c}$$

At this point it is important to realize that neither the fields $c,\tilde{c}$ nor the coupling λ_0' have to be renormalized. Indeed, neither the response function $\langle\tilde{c}(q,\omega)c(-q,-\omega)\rangle$ nor the correlation function $\langle c(q,\omega)c(-q,-\omega)\rangle$ acquire any contribution from the interaction terms in perturbation theory. Thus the response and correlation functions of the pollution field Eqs. (2.15a) and (2.15b) are exact, and therefore

$$Z_c=Z_{\tilde{c}}=Z_{\lambda'}=1\;. \tag{3.12}$$

Next we evaluate the vertex functions $\Gamma_{12},\Gamma_{11,1}$. From the form of the first (zeroth-order) terms in Eqs. (3.4) and (3.5), it follows that we can put the external momenta and frequencies in the denominators in (3.4) and (3.5) to zero. Performing the same steps as before, we finally secure

$$Z_u=1+\frac{2u}{\epsilon}-\bar{\rho}(1+2\bar{\rho})\frac{4v}{\epsilon}\;, \tag{3.13a}$$

$$Z_v=1+\frac{u}{\epsilon}-\bar{\rho}^2\frac{4v}{\epsilon}\;. \tag{3.13b}$$

This concludes our renormalization program.

IV. SCALING PROPERTIES

In this section we will explore the scaling properties of the model under study. The scaling properties describe how some physical quantity will transform under a change of length scales. In fact, we introduced a phenomenological length scale μ which is undetermined [see Eq. (2.13)]. The freedom in the choice of μ, keeping bare couplings $g_0, g'_0, \tau_0, \lambda_0, \lambda'_0$ fixed, can be used to derive a partial differential equation generally denoted as RG equation.

Consider a connected renormalized response function $G_R^{(N)}$ composed of $N(\tilde{n}_R, n_R)$ fields. Following standard arguments[8,10] we obtain a RGE which reads

$$(\mu\partial_\mu+\beta_u\partial_u+\beta_v\partial_v+\beta_\rho\partial_\rho+\kappa\tau\partial_\tau+\tfrac{1}{2}N\gamma_n)G_R^{(N)}=0 \ . \tag{4.1}$$

In (4.1), $\beta_u=\mu\partial_\mu|_0 u$, $\beta_v=\mu\partial_\mu|_0 v$, $\beta_\rho=\mu\partial_\mu|_0\rho$, $\kappa=\mu\partial_\mu|_0\ln\tau$, and $\gamma_n=\mu\partial_\mu|_0\ln Z_n$, where $\partial_\mu|_0$ denotes a derivative at fixed bare parameters. Note that we introduced an effective dimensionless coupling $\rho=\lambda'/\lambda$ in Eq. (4.1) which naturally appears in all Z factors.

Using Eqs. (3.9), (3.11), and (3.13) and taking the μ dimensions into account, we secure, to one-loop order

$$\beta_u=-\epsilon u+u[\tfrac{3}{2}u-v(4\bar{\rho}+2\bar{\rho}^2+4\bar{\rho}^3)] \ , \tag{4.2}$$

$$\beta_v=-\epsilon v+v(\tfrac{3}{4}u-4v\bar{\rho}^3) \ , \tag{4.3}$$

$$\beta_\rho=\tfrac{1}{8}u\rho-2v\rho\bar{\rho}^3 \ , \tag{4.4}$$

$$\kappa=\tfrac{3}{8}u-2v\bar{\rho}^3 \ , \tag{4.5}$$

$$\gamma_n=-\frac{u}{4}+2v\bar{\rho}^2 \ . \tag{4.6}$$

Let us now determine the fixed points u_*, v_*, ρ_* of Eqs. (4.2)–(4.4). These are obtained, if the coupling constants u, v, ρ reach values such that any further change in the scale does not affect them. The fixed points can be identified from

$$\beta_u(u_*,v_*,\rho_*)=0 \ , \tag{4.7a}$$

$$\beta_v(u_*,v_*,\rho_*)=0 \ , \tag{4.7b}$$

$$\beta_\rho(u_*,v_*,\rho_*)=0 \ , \tag{4.7c}$$

and we find

$$(\mathrm{I})\quad u_*=0,\ \ v_*=0,\ \ \rho_*\ \text{arbitrary} \ , \tag{4.8a}$$

$$(\mathrm{II})\quad u_*=0,\ \ v_*=-\frac{\epsilon}{4},\ \ \rho_*=0 \ , \tag{4.8b}$$

$$(\mathrm{III})\quad u_*=\tfrac{2}{3}\epsilon,\ \ v_*=0,\ \ \rho_*=0 \ , \tag{4.8c}$$

$$(\mathrm{IV})\quad u_*=2\epsilon,\ \ v_*=\tfrac{27}{64}\epsilon,\ \ \rho_*=\tfrac{2}{3} \ . \tag{4.8d}$$

As we are interested in extracting the physical behavior on large length scales (or small momenta) we have to investigate whether the fixed points (I)–(IV) are infrared stable. To this end we consider the eigenvalues of the matrix $\partial_{a_i}\beta_{a_i}|_{a_i^*}$ $(a_i=u,v,\rho)$. Whereas (I) and (II) are unstable (i.e., the matrix possesses at least one negative eigenvalue), the fixed point (III) describing critical behavior in the absence of a pollution density[3] is unstable only with respect to the coupling v. It is (IV) which is stable (all eigenvalues are positive) and which governs a new critical behavior in the presence of a pollution density.

Now we turn to the solution of Eq. (4.1) at an infrared stable fixed point u_*, v_*, ρ_*. Using the method of characteristics,[10] one obtains

$$G^{(N)}(\{x,t\},\tau,\mu)=l^{N(\eta+d)/2}G^{(N)}(\{lx,l^z t\},l^{-1/\nu}\tau,\mu) \ , \tag{4.9}$$

where we have introduced a scale variable l by $\mu(l)=\mu l$ and

$$\eta=\gamma_n^*=-u_*/4+2v_*\bar{\rho}_*^2 \ ,$$

$$z=2+\zeta_*=2-u_*/8+2v_*\bar{\rho}_*^3 \quad (\zeta=\mu\partial_\mu|_0\ln\lambda) \ ,$$

and

$$1/\nu=2-\kappa_*=2-3u_*/8=2v_*\bar{\rho}_*^3 \ .$$

From Eq. (4.9) we can determine the general scaling behavior of physical quantities. Putting $\tau l^{-1/\nu}=1$, we find, for example,

$$G^{(1)}(x,t)=|\tau|^\beta f\left[\frac{x}{\xi},\frac{t}{\xi^z}\right] \ , \tag{4.10}$$

where $\beta=\nu(d+\eta)/2$, $\xi=\xi_0|\tau|^{-\nu}$ is the correlation length, z is the dynamical critical exponent, and f is some scaling function. Equation (4.10) has the typical form of a power law and describes the vanishing of the population density $G^{(1)}(x,t)=\langle n(x,t)\rangle$ as the critical point $\tau\to 0$ is approached from below $(\tau<0)$. The critical exponents at the new fixed point (IV) are given by

$$\beta_{\mathrm{new}}=1-\frac{\epsilon}{32},\quad \nu_{\mathrm{new}}=\frac{1}{2}+\frac{\epsilon}{8} \ , \tag{4.11a}$$

$$\eta_{\mathrm{new}}=-\frac{\epsilon}{8},\quad z_{\mathrm{new}}=2 \tag{4.11b}$$

up to order ϵ.

They may be compared with the critical exponents at the fixed point (III) (characterized by a vanishing pollution density),[3]

$$\beta=1-\frac{\epsilon}{6},\quad \nu=\frac{1}{2}+\frac{\epsilon}{16} \ , \tag{4.12a}$$

$$\eta=-\frac{\epsilon}{6},\quad z=2-\frac{\epsilon}{12} \ . \tag{4.12b}$$

Thus we see that the mean value of the population density will vanish with a larger exponent in the presence of pollution density fluctuations $(\beta_{\mathrm{new}}>\beta)$.

Moreover, we find that the temporal scales of the population density fluctuations are completely governed by the diffusive behavior of the pollution density fluctuations. It follows from $z=2+\zeta_*=2-\rho_*^{-1}\beta_{\rho_*}$ that if a fixed point $\rho_*\neq 0$ exists, then $\beta_{\rho_*}=0$ by definition, and therefore $z=2$ in all orders of perturbation theory. Finally, it is remarkable that the exponent ν can also be determined to all orders in perturbation theory. Due to a Ward identify (for details see the Appendix) which implies a relation between Z_v and Z_τ we obtain

$$\nu=\frac{1}{2-\epsilon/2} \tag{4.13}$$

to all orders.

V. CONCLUSIONS

We have investigated the effects of a pollution density on a population which is on the brink of extinction. Two main features are built into the model and determine its universality class. First, the dynamics of the pollution density is purely diffusive, and its mean density is conserved. Second, the population density is subjected to an absorbing state condition. It is now possible to identify those couplings between the pollution density and the population density which are relevant to the behavior of the system close to the (critical) point of extinction, and close to its upper critical dimension. It suffices to include the effect of the pollution on the production and annihilation rates of the species. Beyond that we could, of course, envisage an influence of the population on the diffusion dynamics of the pollution (e.g., a change of the pollution density diffusion constant as a function of the population density). These latter couplings, however, do not alter the critical behavior. Thus, universality in conjunction with an expansion around the upper critical dimension allows for the identification of the relevant model features and hence simplifies the construction of a mathematical model considerably.

Our findings may be summarized as follows. In the vicinity of the point where the population density becomes extinct, both spatial and temporal scales of its fluctuations become significantly affected by the fluctuations of the pollution density. This observation finds its expression in the fact that there is a new stable fixed point which governs the power-law behavior of physical quantities near criticality. In particular, we found that the temporal scales of the population density fluctuations are completely governed by the diffusive behavior of the pollution density fluctuations. This is reflected in the result for the dynamic critical exponent z for the population field: $z=2$ to all orders in perturbation theory.

Another exponent which can be determined to all orders in perturbation theory is the exponent ν which characterizes the correlation length. We obtain $\nu=1/(2-\epsilon/2)$ to all orders. Finally, we find that the mean value of the population density (the order parameter) will vanish with a larger exponent in the presence of pollution density fluctuations.

As a consequence of universality our results can be carried over to phenomena in other fields of science. As important examples let us mention here chemical reactions and cellular automata. In fact, Eq. (2.4) has been proposed in Ref. 2 [with $c(x,t)=0$] for the description of nonequilibrium phase transitions (Schlögl's first model) in chemical reactions. Many related models can be found, e.g., in Ref. 11. Furthermore, the relation of Schlögl's first model[2] to certain cellular automata is well known in the literature.[12,13] Thus our above results may also be tested by numerical simulations of appropriate cellular automata.

For a comparison to numerical and experimental results, the investigation of finite-size effects is of importance. Ecological processes under realistic conditions are naturally restricted to finite environments. In the absence of a pollution field finite-size effects have recently been studied in Ref. 14.

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APPENDIX

In the following, we derive the Ward identity which leads to Eq. (4.13). Consider a diagram contributing to the vertex function Γ_{11} in the bare perturbation series, as a function of external momentum and time. Due to causality, the vertices are ordered from right to left, with increasing time arguments. Thus, a diagram with n vertices consists of $n-1$ time segments. In the time segment (t_i,t_{i-1}), there are bare propagators $G_0(q_\alpha,t_i-t_{i-1})$ and correlators $C_0(q_b,t_i-t_{i-1})$ flowing from right to left, with appropriate internal momenta q_α,q_β. A Fourier transformation from time differences to frequencies yields a dynamical factor

$$F(\{q,\omega\})=\left[i\omega+\sum_\alpha \lambda_0(q_\alpha^2+\tau_0)+\sum_\beta \lambda_0' q_\beta^2\right]^{-1}$$

for each segment.

Let us now insert a vertex $\lambda_0 g_0'$, with zero external momentum and frequency, into a propagator of our time segment (t_i,t_{i-1}). This generates a diagram contributing to $\Gamma_{11,1}$. After Fourier transforming, the dynamical factor is modified, due to the insertion

$$\hat{F}(\{q,\omega\})=\lambda_0 g_0'\left[i\omega+\sum_\alpha \lambda_0(q_\alpha^2+\tau_0)+\sum_\beta \lambda_0' q^2\right]^{-2} .$$

If we sum over all possibilities of inserting the extra vertex into the time segment (t_i,t_{i-1}), we obtain

$$\sum_\alpha \hat{F}(\{q,\omega\})=-g_0'\frac{\partial}{\partial\tau_0}F(\{q,\omega\}) .$$

Summing finally over all ways of inserting the extra vertex into any graph of Γ_{11}, we secure

$$\Gamma_{11,1}(q,\omega,0,0)=-g_0'\frac{\partial}{\partial\tau_0}\Gamma_{11}(q,\omega) , \tag{A1}$$

which is the required Ward identity between vertex functions.

Since the derivative with respect to τ_0 does not generate any new ultraviolet divergences in Γ_{11} we may go over to renormalized quantities, according to (3.9). We find immediately the relation

$$1=Z_v^{1/2}Z_\tau^{-1} . \tag{A2}$$

Using this identity we can rewrite the Wilson function $\kappa=\mu\partial/\partial\mu|_0\ln\tau$, which determines the exponent ν

$$\kappa=\frac{1}{2v}(\beta_v+\epsilon v) ,$$

where we obtain at the fixed point where $\beta_{v_*}=0$ by definition

$$\nu=(2-\kappa^*)^{-1}=1/(2-\epsilon/2) \tag{A3}$$

to all orders in perturbation theory.

The above argument was based on a diagrammatical derivation of (A1). An alternative method exploits the invariance of the theory under a continuous symmetry transformation. The dynamic functional given by (2.10) is invariant under a simultaneous shift of the pollution density $c(x,t)$ and the parameter τ_0 by an arbitrary constant

$$c(\mathbf{x},t)\to c(\mathbf{x},t)+c_0, \quad \tau_0\to\tau_0-g_0'c_0 \ . \tag{A4}$$

One may now follow standard arguments, based on the invariance of the generating functional for the vertex functions (see, e.g., Ref. 10) to rederive (A1).

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*Present address: Universität Essen, Fachbereich Physik, D-4300 Essen, Federal Republic of Germany.

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