Multiphoton quantum-statistical theory in driven optical systems without adiabatic elimination

Si Xian Huang and Ren Ming Lin
Department of Physics, Fujian Teacher's University, Fuzhou, Fujian, People's Republic of China
(Received 23 May 1988)

Quantum fluctuations for multiphoton transitions in driven optical systems are treated on the basis of a Fokker-Planck equation for the Wigner function. The equation is considered without adiabatic elimination of the atoms or the field. The linear stability is analyzed and time correlations are derived. A spectrum of transmitted light associated with vacuum Rabi splitting is obtained. The squeezing and antibunching effects in the atomic system and the field are discussed. This theory coincides with the adiabatic elimination theory for the case of both good- and bad-cavity limits.

I. INTRODUCTION

The extensive literature on quantum fluctuations in driven optical systems is restricted mainly to treatments in the good- and bad-cavity limits for one photon.[1-5] Considerable attention has been devoted also to two-photon or multiphoton transitions, for instance, the works of Lugiato and co-workers,[6,7] Walls[8] and Walls and Reid,[9] Loudon,[10] Agarwal and co-workers,[11-13] and Lin and co-workers,[14,15] but these works are restricted exclusively to treatments with adiabatic elimination in the good cavity or bad cavity. Recently, Carmichael treated one-photon transitions in absorptive bistability without adiabatic elimination,[16] but did not discuss the time correlation function and the spectrum of the transmitted light.

In this paper we develop Lugiato's method[1] of the one-photon transition in a driven optical system to treat the multiphoton case without adiabatically eliminating the atoms or the field. So this treatment has very wide application. It includes the multiphoton optical-bistability and multiphoton laser theories. Many new phenomena which have been inadequately treated with adiabatic elimination are carefully analyzed in this paper.

Let μ be the ratio of the cavity linewidth k and atomic linewidth $\gamma_{\parallel},\gamma_{\perp}$:

$$\mu \equiv \frac{k}{\gamma_{\perp}} = \frac{dk}{\gamma_{\parallel}}, \quad d = \frac{\gamma_{\parallel}}{\gamma_{\perp}} . \tag{1.1}$$

When $\mu \ll 1$ ($\mu \gg 1$), this treatment coincides with the adiabatic elimination theory for the good- (bad-) cavity limits.[5-6,14,15,17,18]

In Sec. II we obtain the master equation of the system and the Fokker-Planck equation (FPE). In Sec. III we obtain a linearized FPE and calculate the stationary correlations of different operators. Section IV analyzes the stability. Section V calculates the spectrum of transmitted light and obtains the spectrum of vacuum Rabi splitting.[19,20] Section VI discusses the squeezing and antibunching effects in the field and atomic system.

II. MULTIPHOTON FOKKER-PLANCK EQUATION

We consider a single-cavity mode, which interacts with a collection of identical two-level atoms, and is fed by a coherent field that is injected into the cavity. We call A ($A^{\dagger}$) the annihilation (creation) operator of photons of the mode, r_i^+ (r_i^-) the raising (lowering) operator of the ith atom ($i=1,2,\ldots,N$), and γ_{3i} the corresponding population inversion operator. The commutation rules are

$$[A,A^{\dagger}]=1 ,$$
$$[r_i^+,r_j^-]=2\gamma_{3i}\delta_{ij}, \quad [r_{3i},r_j^{\pm}]=\pm r_i^{\pm}\delta_{ij} . \tag{2.1}$$

We introduce the collective dipole operators of the atomic system:

$$R^{\pm}=\sum_{i=1}^{N} r_i^{\pm}, \quad R_3=\sum_{i=1}^{N} r_{3i} , \tag{2.2}$$

which obey the commutation rules

$$[R^+,R^-]=2R_3, \quad [R_3,R^{\pm}]=\pm R^{\pm} . \tag{2.3}$$

We assume the one-mode quantum statistical model for multiphoton transitions, and that the incident field, the cavity mode, and the atoms are exactly on resonance. In the interaction representation the statistical operator $\rho(t)$ of the system of the atom plus cavity field obeys the master equation[1]

$$\frac{d\rho(t)}{dt}=k\{(A-\alpha),\rho(A-\alpha)^{\dagger}]+\text{H.c.}\}+g[(A^{\dagger n}R^- - A^nR^+),\rho] + \tfrac{1}{2}\sum_{i=1}^{N}\{\gamma_{\uparrow}([r_i^+,\rho r_i^-]+\text{H.c.})+\gamma_{\downarrow}([r_i^-,\rho r_i^+]+\text{H.c.})+\bar{\eta}([r_{3i},\rho r_{3i}]+\text{H.c.})\}+2k\bar{n}[A,[\rho,A^{\dagger}]] , \tag{2.4}$$

where k is the decay rate for the cavity field, α is proportional to a real driving field amplitude, $\gamma_\uparrow$ ($\gamma_\downarrow$) is the upward (downward) transition rate between the lower and upper level, $\bar{\eta}$ is the collision dephasing term, g is the coupling constant in the dipole approximation, and $\bar{n}$ measures the strength of the thermal fluctuations.

The Wigner symmetrical characteristic function is

$$C_W(\xi,\xi^*,\eta,\zeta,\zeta^*,t)=\mathrm{Tr}\{\rho(t)\exp[i(\xi^* R^+ + \xi R^- + \eta R_3 + \zeta^* A^\dagger + \zeta A)]\} , \tag{2.5}$$

The quasiprobability distribution P_W is defined as the Fourier transform of C_W,

$$P_W(\bar{v},\bar{v}^*,\bar{m},\beta,\beta^*,t)=\frac{1}{(2\pi)^5}\int d^2\xi\, d^2\zeta\, d\eta\, e^{-i(\bar{v}^*\xi^*+\bar{v}\xi+\bar{m}\eta+\beta^*\zeta^*+\beta\zeta)}C_W(\xi,\xi^*,\eta,\zeta^*,\zeta,t) . \tag{2.6}$$

The moments of P_W give symmetrized expectation values, for instance,

$$\int d^2\bar{v}\, d\bar{m}\, d^2\beta P_W \beta^*\beta = \langle A^\dagger A\rangle^{(S)} \equiv \tfrac{1}{2}(\langle A^\dagger A\rangle + \langle A A^\dagger\rangle) . \tag{2.7}$$

The longitudinal (transverse) relaxation rate $\gamma_\parallel$ ($\gamma_\perp$) and the parameter of pump σ are defined

$$\gamma_\perp=\tfrac{1}{2}(\gamma_\uparrow+\gamma_\downarrow+\bar{\eta}),\quad \gamma_\parallel=\gamma_\uparrow+\gamma_\downarrow,\quad \sigma=\frac{\gamma_\uparrow-\gamma_\downarrow}{\gamma_\uparrow+\gamma_\downarrow} . \tag{2.8}$$

It is convenient to introduce the normalized variables

$$v=-\left[\frac{N}{2}\sqrt{d}\right]^{-1}\bar{v},\quad m=-\left[\frac{N}{2}\right]^{-1}\bar{m},\quad x=\frac{\beta}{\sqrt{N_{S,n}}},\quad y=\frac{\alpha}{\sqrt{N_{S,n}}} , \tag{2.9}$$

and corresponding normalized operators

$$\hat{v}=-\left[\frac{N}{2}\sqrt{d}\,\right)^{-1}R,\quad \hat{v}^\dagger=-\left[\frac{N}{2}\sqrt{d}\right]^{-1}R^+,\quad \hat{m}=-\left[\frac{N}{2}\right]^{-1}R_3,\quad \hat{x}=\frac{A}{\sqrt{N_{S,n}}},\quad \hat{x}^\dagger=\frac{A^+}{\sqrt{N_{S,n}}} , \tag{2.10}$$

where $N_{S,n}$ is the saturation photon number,

$$N_{S,n}=\left[\frac{\gamma_\perp\gamma_\parallel}{4g^2}\right]^{1/n} . \tag{2.11}$$

The FPE of P_W can be derived by the procedure of Gronchi and Lugiato:[1]

$$\frac{\partial}{\partial t}P_W(v,v^*,m,x,x^*,t)=\Gamma P_W(v,v^*,m,x,x^*,t) ,$$

$$\begin{aligned}\Gamma=&\left[-\frac{\partial}{\partial v}[-\gamma_\perp(v-mx^n)]+\text{c.c.}-\frac{\partial}{\partial x}[-k(x-y+2C_n v x^{*\,n-1})]+\text{c.c.}\right]\\&-\frac{\partial}{\partial m}\{-\gamma_\parallel[m+\sigma+\tfrac{1}{2}(v^*x^n+vx^{*n})]\}+\frac{\gamma_\perp^2 n}{2kC_nN_{S,n}}\left[\frac{\partial^2}{\partial v\partial v^*}+\frac{\sigma d^2}{4}\frac{\partial}{\partial m}\left[\frac{\partial}{\partial v}v+\frac{\partial}{\partial v^*}v^*\right]+\frac{d^2}{4}\frac{\partial^2}{\partial m^2}(1+\sigma m)\right]\\&+\frac{k}{N_{S,n}}(1+2\bar{n})\frac{\partial^2}{\partial x\partial x^*} ,\end{aligned} \tag{2.12}$$

where C_N is cooperation parameter

$$C_N=\frac{nNg^2}{2k\gamma_\perp}N_{S,n}^{(n-1)} . \tag{2.13}$$

In the one-photon case ($n=1$), Eq. (2.12) coincides with Ref. 1, Eq. (176).

From Eq. (2.12) the steady-state expectation values of the field and atomic system quantities $\langle\hat{x}\rangle_S\equiv x_S$, $\langle\hat{v}\rangle_S\equiv v_S$, $\langle\hat{m}\rangle_S\equiv m_S$, can be obtained, where the subscript S denotes the steady state. In the semiclassical approximation the steady-state equations can be obtained,

$$m_S=-\frac{\sigma}{1+x_S^{2n}} , \tag{2.14}$$

$$v_S=m_Sx_S^n=-\frac{\sigma x_S^n}{1+x_S^{2n}} , \tag{2.15}$$

$$y=x_S\left[1-2\sigma C_n\frac{x_S^{2n-2}}{1+x_S^{2n}}\right] . \tag{2.16}$$

III. LINEARIZED FPE AND STEADY-STATE CORRELATIONS

Since $N_{S,n} \gg 1$, these fluctuations will be very small, and the FPE (2.12) can be linearized. Hence we introduce the deviations from steady state,

$$x' = x - x_S, \quad x^{*'} = x^* - x_S, \quad v' = v - v_S, \quad v^{*'} = v^* - v_S, \quad m' = m - m_S , \tag{3.1}$$

corresponding to these operators,

$$\delta\hat{x} = \hat{x} - x_S, \quad \delta\hat{x}^\dagger = \hat{x}^\dagger - x_S, \quad \delta\hat{v} = \hat{v} - v_S, \quad \delta\hat{v}^\dagger = \hat{v}^\dagger - v_S, \quad \delta\hat{m} = \hat{m} - m_S . \tag{3.2}$$

The linearized FPE can be obtained from Eq. (2.12),

$$\frac{\partial}{\partial t} P_W(v', v^{*'}, m', x', x^{*'}, t) = \Gamma' P_W(v', v^{*'}, m', x', x^{*'}, t) , \tag{3.3}$$

$$\begin{aligned}\Gamma' = &\left\{ -\frac{\partial}{\partial v'} \left[-\gamma_\perp \left[v' - x_S^n m' - \frac{nB}{2C_n x_S^{n-1}} x' \right] \right] - \frac{\partial}{\partial x'} \{ -k [x' + (n-1)Bx^{*'} + 2C_n x_S^{n-1} v'] \} + \text{c.c.} \right\} \\ &- \frac{\partial}{\partial m'} \left[-\gamma_\parallel \left[m' + \frac{x_S^n}{2}(v' + v^{*'}) + \frac{nBx_S}{4C_n}(x' + x^{*'}) \right] \right] \\ &+ L_1 \left[\frac{\partial^2}{\partial v' \partial v^{*'}} + L_2 \left[\frac{\partial^2}{\partial m' \partial v'} + \text{c.c.} \right] + L_3 \frac{\partial^2}{\partial m'^2} + L_4 \frac{\partial^2}{\partial x' \partial x^{*'}} \right] ,\end{aligned} \tag{3.4}$$

where

$$B = \frac{y}{x_S} - 1 = -2C_n \sigma \frac{x_S^{2n-2}}{1 + x_S^{2n}} , \tag{3.5a}$$

$$L_1 = \frac{4\gamma_\perp}{Nd} = \frac{\gamma_\perp^2 n}{2C_n k N_{S,n}}, \quad L_2 = \frac{\sigma d^2 B}{8C_n x_S^{n-2}}, \quad L_3 = \frac{d^2}{4} \left[1 + \frac{\sigma B}{2C_n x_S^{2n-2}} \right], \quad L_4 = \frac{2C_n k^2}{n\gamma_\perp}(1 + 2\bar{n}) . \tag{3.5b}$$

Let

$$\{x_i\} = (x_1, x_2, x_3, x_4, x_5) = (v^{*'}, v', m', x^{*'}, x') .$$

Equation (3.3) can be rewritten

$$\frac{\partial}{\partial t} P_W(\{x_i\}, t) = \sum_{ij=1}^{5} \left[-\frac{\partial}{\partial x_i} \gamma_\perp M_{ij} x_j + \frac{\partial^2}{\partial x_i \partial x_j} L_{ij} \right] P_W(\{x_i\}, t) , \tag{3.6}$$

where

$$\underline{M} = \{M_{ij}\} = \begin{bmatrix} -1 & 0 & x_S^n & \frac{nB}{2C_n x_S^{n-1}} & 0 \\ 0 & -1 & x_S^n & 0 & \frac{nB}{2C_n x_S^{n-1}} \\ -dx_S^n/2 & -dx_S^n/2 & -d & -\frac{dnBx_S}{4C_n} & -\frac{dnBx_S}{4C_n} \\ -2\mu C_n x_S^{n-1} & 0 & 0 & -\mu & -\mu(n-1)B \\ 0 & -2\mu C_n x_S^{n-1} & 0 & -\mu(n-1)B & -\mu \end{bmatrix} , \tag{3.7}$$

$$\underline{L} = \{L_{ij}\} = \frac{L_1}{2} \begin{bmatrix} 0 & 1 & L_2 & 0 & 0 \\ 1 & 0 & L_2 & 0 & 0 \\ L_2 & L_2 & 2L_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_4 \\ 0 & 0 & 0 & L_4 & 0 \end{bmatrix} . \tag{3.8}$$

From Eq. (3.6) we have

$$\frac{d}{dt}\langle \hat{x}_i \rangle = \gamma_{\perp} M_{ij} \langle \hat{x}_j \rangle, \quad i,j = 1,2,3,4,5 \ . \tag{3.9}$$

Let the stationary state physical correlation matrix

$$G_{ij}(t) = \langle \hat{x}_i(t) \hat{x}_j(0) \rangle, \quad i,j = 1,2,3,4,5 \ . \tag{3.10}$$

By the regression theorem[4] the time evolution of $G_{ij}(t)$ is

$$\frac{d}{dt} G(t) = \gamma_{\perp} M G(t) \ . \tag{3.11}$$

We introduce the steady-state symmetrized correlation matrix

$$G_{ij}^{(s)}(0) = \tfrac{1}{2}[G_{ij}(0) + G_{ji}(0)] \ . \tag{3.12}$$

The superscript (s) denotes the symmetrized matrix. The $G_{ij}^{(s)}(0)$ satisfies the following equation, by Eq. (3.6):

$$\gamma_{\perp}(\underline{MG}\,{}^{(s)}_{(0)} + \underline{G}\,{}^{(s)}_{(0)}\underline{M}) = -(\underline{L} + \underline{L}^{T}) \ . \tag{3.13}$$

Equation (3.13) defines a set of nine linear equations for the elements of the covariance matrix $G_{ij}^{(s)}(0)$. We solve Eq. (3.13) by tedious calculations; the steady-state correlations are given by the following.

(1) Field-field correlations:

$$\langle \delta\hat{x}^{\dagger} \delta\hat{x} \rangle = \frac{1}{4N_{S,n}}(T-2) \ , \tag{3.14}$$

$$\langle \delta\hat{x}^{2} \rangle = \frac{1}{4N_{S,n}}(T-2M) \ , \tag{3.15}$$

where

$$T = \frac{Q}{P} \ , \tag{3.16}$$

$$P = (1+x_S^{2n})\frac{dy}{dx_s}\{\mu^2 f(f+nB+df) + \mu[f(1+d)^2 + nB(1+dx_S^{2n})] + d(1+d)(1+x_S^{2n})\} \ , \tag{3.17}$$

$$\begin{aligned} Q = nB &\left\{ \mu \left[-\frac{1}{\sigma}(1+x_S^{2n})^2 \left[df + \frac{dy}{dx_S} \right] + \sigma d(1+d) f x_S^{2n} \right] + d(1+d)[(1+d)\sigma x_S^{2n} - \frac{1}{\sigma}(1+x_S^{2n})^2] \right\} \\ &+ (1+2\bar{n})(1+x_S^{2n}) \left[\mu^2 (f+nB+df)\frac{dy}{dx_S} + \mu \left[(1+d)^2 \frac{dy}{dx_S} + dnB(x_S^{2n}-1) \right] + d(1+d)(1+x_S^{2n}) \right] + PM \ , \end{aligned} \tag{3.18}$$

$$M = \frac{-\frac{1}{\sigma} nB(1+x_S^{2n}) + (1+2\bar{n})(1+\mu+\mu B)}{(1+B)[1+\mu(2-f)]} \ , \tag{3.19}$$

$$\frac{dy}{dx_S} = \frac{1}{(1+x_S^{2n})}[f+nB+(1-B)x_S^{2n}] \ , \tag{3.20}$$

$$f = 1 + (n-1)B \ . \tag{3.21}$$

(2) Atom-field correlations:

$$\langle \delta\hat{v} \delta\hat{x} \rangle = \frac{-1}{4N_{S,n} 2C_n x_S^{n-1}}(fT-2M) \ , \tag{3.22}$$

$$\langle \delta\hat{v}^{\dagger} \delta\hat{x} \rangle = \frac{1}{4N_{S,n} 2C_n x_S^{n-1}}[fT - 2(n-1)BM - 2(1+2\bar{n})] \ , \tag{3.23}$$

$$\langle \delta\hat{m} \delta\hat{x} \rangle = \frac{1}{4N_{S,n} 2C_n x_S^{n-1}} \frac{V}{x_S^{n}} \ , \tag{3.24}$$

where

$$\begin{aligned} V = \frac{1}{1+d+\mu f} &\left[[-(1+\mu f)(f+nB) + d(1-B)x_S^{2n}]T + 2M \left[\mu(n-1)^2 B^2 + \mu(1+nB) - \frac{d}{2}(1-B)x_S^{2n} + 1 + nB \right] \right. \\ &\left. + 2(1+2\bar{n})[\mu(n-1)B - \tfrac{1}{2} d x_S^{2n}] \right] \ . \end{aligned} \tag{3.25}$$

(3) Atom-atom correlations:

$$\langle \delta\hat{v}^{\dagger}\delta\hat{v}\rangle = \frac{1}{4N_{S,n}(2C_n x_S^{n-1})^2}\{-nBfT+2(n-1)nB^2M+2nB(1+2\bar{n})+\frac{1}{\mu}[W-2nB-\frac{1}{\sigma}2nB(1+x_S^{2n})]\}\ , \tag{3.26}$$

$$\langle \delta\hat{v}^{2}\rangle = \frac{1}{4N_{S,n}(2C_n x_S^{n-1})^2}\left[-nBfT+2nBM+\frac{W}{\mu}\right]\ , \tag{3.27}$$

$$\langle \delta\hat{m}\,\delta\hat{v}\rangle = \frac{1}{4N_{S,n}(2C_n x_S^{n-1})^2}\frac{1}{\mu x_S^n}(W+dnBx_S^{2n})\ , \tag{3.28}$$

$$\langle \delta\hat{m}^{2}\rangle = \frac{1}{4N_{S,n}(2C_n x_S^{n-1})^2}\left[-nBV+\frac{1}{\mu}\left[-W+d\sigma nB-\frac{1}{\sigma}dnB(1+x_S^{2n})\right]\right]\ , \tag{3.29}$$

$$W=-(d+\mu f)V+dx_S^{2n}[(1-B)(T-M)-(1+2\bar{n})]\ . \tag{3.30}$$

Let $n=1$, $d=2$, $\sigma=-1$, $\bar{n}=0$; Eqs. (3.14)–(3.30) coincide with Ref. 16, Eqs. (3.3a)–(3.8d).

IV. LINEAR STABILITY ANALYSIS

When these fluctuations are very small, we can neglect the nonlinear terms and obtain Eq. (3.6), which is the basis of the stability analysis of the system. We obtain the eigenvalues of the matrix $\underline{M}$ in Eq. (3.7) by tedious calculations. The eigenvalues are $\{-\mu_i\}$, $i=1,2,\ldots,5$, where

$$\mu_1=-2\varphi\cos\left[\frac{\alpha}{3}\right]+\frac{a}{3}\ , \tag{4.1a}$$

$$\mu_2=2\varphi\cos\left[\frac{\alpha}{3}-\frac{\pi}{3}\right]+\frac{a}{3}\ , \tag{4.1b}$$

$$\mu_3=2\varphi\cos\left[\frac{\alpha}{3}+\frac{\pi}{3}\right]+\frac{a}{3}\ , \tag{4.1c}$$

$$\mu_4=\tfrac{1}{2}(1+\mu(2-f) -\{[1+\mu(2-f)]^2-4\mu(B+1)\}^{1/2})\ , \tag{4.1d}$$

$$\mu_5=\tfrac{1}{2}(1+\mu(2-f) +\{[1+\mu(2-f)]^2-4\mu(B+1)\}^{1/2})\ , \tag{4.1e}$$

where

$$\varphi=\left[\frac{1}{3}\left[\frac{a^2}{3}-b\right]\right]^{1/2}\ , \tag{4.2a}$$

$$\cos\alpha=-\frac{1}{2\varphi^3}(\tfrac{3}{27}a^3-\tfrac{1}{3}ab+c)\ , \tag{4.2b}$$

$$a=1+d+\mu f\ , \tag{4.2c}$$

$$b=d(1+x_S^{2n})+\mu nB+\mu(1+d)f\ , \tag{4.2d}$$

$$c=(1+x_S^{2n})\frac{dy}{dx_S}\mu d\ . \tag{4.2e}$$

Hence the stability condition of the system is

$$\mathrm{Re}\mu_i>0\quad(i=1,2,\ldots,5)\ . \tag{4.3}$$

If Eq. (4.3) is satisfied, from Eqs. (4.1d) and (4.1e) we obtain $f_{max}<2+(1/\mu)$; then from Eqs. (3.21) and (3.5a) we have

$$-\sigma C_n<\frac{n(n-1)^{1/n}}{2(n-1)^2}\left[1+\frac{1}{\mu}\right]\ . \tag{4.4}$$

For the one-photon transitions ($n=1$), whatever the values of C_n, σ, μ may be, Eq. (4.4) is always satisfied. For the multiphoton transitions ($n\geq 2$): (1) in the case of a laser ($\sigma>0$), Eq. (4.4) is always satisfied; (2) in the case of $-1<\sigma<0$, the values of C_n, μ are restricted; for example, in the two-photon optical bistability ($n=2$, $\sigma=-1$), Eq. (4.4) is rewritten

$$C_2<\left[1+\frac{1}{\mu}\right]\ . \tag{4.5}$$

When the value of μ increases, the cooperative parameter C_2 must decrease otherwise the system will be unstable. In the bad-cavity limit ($\mu\gg 1$), the stability condition Eq. (4.5) is written $C_2<1$. Hence, at first, the values of n, C_n, μ, d, σ must satisfy Eq. (4.3) in the following calculations.

Let us discuss the case of μ_i complex.

(1) *μ_2,μ_3 complex conjugates.* Putting

$$\mu_2=\lambda_1-i\lambda_2,\quad \mu_3=\lambda_1+i\lambda_2\ , \tag{4.6}$$

the λ_1,λ_2 are real:

$$\lambda_1=\tfrac{1}{2}(E^++E^-)+\frac{a}{3}\ , \tag{4.7}$$

$$\lambda_2=\frac{\sqrt{3}}{2}(E^+-E^-)\ , \tag{4.8}$$

where

$$E^{\pm}=\left\{-\frac{1}{2}\left[\frac{2a^3}{27}-\frac{ab}{3}+c\right] \pm\left[\frac{1}{4}\left[\frac{2a^3}{27}-\frac{ab}{3}+c\right]^2 +\frac{1}{27}\left[b-\frac{a^2}{3}\right]^3\right]^{1/2}\right\}^{1/3}\ . \tag{4.9}$$

In this case the eigenvalue μ_1 is given by

$$\mu_1 = -(E^+ + E^-) + \frac{a}{3} . \quad (4.10)$$

The stability condition is

$$\mu_1 > 0, \quad \lambda_1 > 0 , \quad (4.11)$$

and Eq. (4.4) is satisfied.

(2) μ_4, μ_5 *complex conjugates.* Putting

$$\mu_4 = \sigma_1 - i\sigma_2, \quad \mu_5 = \sigma_1 + i\sigma_2 , \quad (4.12)$$

where

$$\sigma_1 = \tfrac{1}{2}[1 + \mu(2-f)] , \quad (4.13)$$

$$\sigma_2 = \tfrac{1}{2}\{4\mu(1+B) - [1+\mu(2-f)]^2\}^{1/2} . \quad (4.14)$$

The stability condition is

$$\mathrm{Re}\mu_i > 0, \quad i = 1,2,3 \quad (4.15)$$

and Eq. (4.4) is satisfied.

V. SPECTRUM OF TRANSMITTED LIGHT AND VACUUM RABI SPLITTING

We introduce the matrix $\underline{U}$ to diagonalize the matrix $\underline{M}$ of Eq. (3.7):

$$\underline{UMU}^{-1} = \underline{\Lambda} = \begin{vmatrix} -\mu_1 & 0 & 0 & 0 & 0 \\ 0 & -\mu_2 & 0 & 0 & 0 \\ 0 & 0 & -\mu_3 & 0 & 0 \\ 0 & 0 & 0 & -\mu_4 & 0 \\ 0 & 0 & 0 & 0 & -\mu_5 \end{vmatrix} , \quad (5.1)$$

where $\mu_1 \to \mu_5$, see Eq. (4.1); hence from Eq. (3.10) we obtain

$$\underline{G}(t) = e^{\gamma_\perp \underline{M} t} \underline{G}(0) = \underline{U}^{-1} e^{\gamma_\perp \underline{\Delta} t} \underline{U}\, \underline{G}(0) . \quad (5.2)$$

By substituting into Eq. (5.2) the value of $\underline{G}(0)$ given by Eqs. (3.14)–(3.30), we get after lengthy calculations the results

$$\langle \delta\hat{x}^\dagger(t)\delta\hat{x}(0)\rangle = T_1 e^{-\gamma_\perp\mu_1 t} + T_2 e^{-\gamma_\perp\mu_2 t} + T_3 e^{-\gamma_\perp\mu_3 t} + T_4 e^{-\gamma_\perp\mu_4 t} + T_5 e^{-\gamma_\perp\mu_5 t} \quad (5.3)$$

$$\langle \delta\hat{x}(t)\delta\hat{x}(0)\rangle = T_1 e^{-\gamma_\perp\mu_1 t} + T_2 e^{-\gamma_\perp\mu_2 t} + T_3 e^{-\gamma_\perp\mu_3 t} - T_4 e^{-\gamma_\perp\mu_4 t} - T_5 e^{-\gamma_\perp\mu_5 t} , \quad (5.4)$$

where

$$T_i = \frac{c_i}{2D}[(b_j c_k - c_j b_k)(\langle \delta\hat{v}^\dagger\delta\hat{x}\rangle + \langle \delta\hat{v}\delta\hat{x}\rangle) + 2(c_j - c_k)\langle \delta\hat{m}\delta\hat{x}\rangle + (b_k - b_j)(\langle \delta\hat{x}^\dagger\delta\hat{x}\rangle + \langle \delta\hat{x}^2\rangle)] , \quad (5.5)$$

where $i \to j \to k$ are cyclic values of $1 \to 2 \to 3$, and

$$b_i = \frac{1}{x_S^n}\left[1 - \mu_i + \frac{nB\mu}{\mu f - \mu_i}\right] , \quad (5.6)$$

$$c_i = \frac{-2\mu C_n x_S^{n-1}}{\mu f - \mu_i}, \quad i = 1,2,3 \quad (5.7)$$

$$D = b_1 c_2 - c_1 b_2 + b_2 c_3 - c_2 b_3 + b_3 c_1 - c_3 b_1 , \quad (5.8)$$

$$T_4 = \frac{b_+}{2(b_+ - b_-)}[-b_-(\langle \delta\hat{v}^\dagger\delta\hat{x}\rangle - \langle \delta\hat{v}\delta\hat{x}\rangle) + \langle \delta\hat{x}^\dagger\delta\hat{x}\rangle - \langle \delta\hat{x}^2\rangle] , \quad (5.9)$$

$$T_5 = \frac{-b_-}{2(b_+ - b_-)}[-b_+(\langle \delta\hat{v}^\dagger\delta\hat{x}\rangle - \langle \delta\hat{v}\delta\hat{x}\rangle) + \langle \delta\hat{x}^\dagger\delta x\rangle - \langle \delta\hat{x}^2\rangle] , \quad (5.10)$$

$$b_+ = \frac{2C_n x_S^{n-1}}{nB}(1-\mu_4) , \quad (5.11)$$

$$b_- = \frac{2C_n x_S^{n-1}}{nB}(1-\mu_5) . \quad (5.12)$$

The incoherent spectrum of the transmitted light is given by the Fourier transform of Eq. (5.3) (Ref. 4),

$$S_{\mathrm{inc}}(\omega) = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty dt\, e^{-i(\omega-\omega_0)t}\langle \delta A^\dagger(t)\delta A(0)\rangle . \quad (5.13)$$

From Eqs. (2.10) and (5.3) we have

$$S_{\mathrm{inc}}(\omega) = \frac{N_{S,n}}{\pi}\,\mathrm{Re}\left[\frac{T_1}{\gamma_\perp\mu_1 + i(\omega-\omega_0)} + \frac{T_2}{\gamma_\perp\mu_2 + i(\omega-\omega_0)} + \frac{T_3}{\gamma_\perp\mu_3 + i(\omega-\omega_0)} + \frac{T_4}{\gamma_\perp\mu_4 + i(\omega-\omega_0)} + \frac{T_5}{\gamma_\perp\mu_5 + i(\omega-\omega_0)}\right] . \quad (5.14)$$

We now consider four cases.

(1) μ_i $(i = 1,2,\ldots,5)$ are real. By Eqs. (5.14) and (5.5)–(5.12) we have

$$S_{\mathrm{inc}}(\omega) = \frac{\gamma_\perp N_{S,n}}{\pi}\left[\frac{\mu_1 T_{R_1}}{(\omega-\omega_0)^2 + \gamma_\perp^2\mu_1^2} + \frac{\mu_2 T_{R_2}}{(\omega-\omega_0)^2 + \gamma_\perp^2\mu_2^2} + \frac{\mu_3 T_{R_3}}{(\omega-\omega_0)^2 + \gamma_\perp^2\mu_3^2} + \frac{\mu_4 T_{R_4}}{(\omega-\omega_0)^2 + \gamma_\perp^2\mu_4^2} + \frac{\mu_5 T_{R_5}}{(\omega-\omega_0)^2 + \gamma_\perp^2\mu_5^2}\right] , \quad (5.15a)$$

where

$$T_{R_i}=\frac{1}{2F}(\mu_j-\mu_k)\{-2\mu C_n x_S^{n-1}(\mu_j+\mu_k-\mu f-1)(\langle\delta\hat{v}^{\dagger}\delta\hat{x}\rangle+\langle\delta\hat{v}\delta\hat{x}\rangle)-4\mu C_n x_S^{2n-1}\langle\delta\hat{m}\delta\hat{x}\rangle+[(\mu f-\mu_j)(\mu f-\mu_k)-\mu nB](\langle\delta\hat{v}^{\dagger}\delta\hat{x}\rangle+\langle\delta\hat{x}^2\rangle)\}\ , \tag{5.15b}$$

where $i\to j\to k$ are the cyclic values of $1\to 2\to 3$, and

$$F=(\mu f-\mu_1)(\mu_2-\mu_3)(\mu_2+\mu_3-\mu f-1)+(\mu f-\mu_2)(\mu_3-\mu_1)(\mu_3+\mu_1-\mu f-1)+(\mu f-\mu_3)(\mu_1-\mu_2)(\mu_1+\mu_2-\mu f-1)\ , \tag{5.15c}$$

$$T_{R_4,R_5}=\frac{\pm(1-\mu_{4,5})}{2(\mu_5-\mu_4)}\left[-\frac{2C_n x_S^{n-1}}{nB}(1-\mu_{5,4})(\langle\delta\hat{v}^{\dagger}\delta\hat{x}\rangle-\langle\delta\hat{v}\delta\hat{x}\rangle)+\langle\delta\hat{x}^{\dagger}\delta\hat{x}\rangle-\langle\delta\hat{x}^2\rangle\right]\ , \tag{5.15d}$$

(2) μ_2,μ_3 and μ_4,μ_5 are complex conjugates. By Eqs. (5.14), (5.5)–(5.12), (4.6), and (4.12), we have

$$S_{\rm inc}(\omega)=\frac{N_{S,n}\gamma_\perp}{\pi}\left[\frac{T_{11}\mu_1}{\gamma_\perp^2\mu_1^2+(\omega-\omega_0)^2}+\frac{T_{21}\lambda_1+T_{22}\left[\dfrac{\omega-\omega_0}{\gamma_\perp}-\lambda_2\right]}{(\omega-\omega_0-\gamma_\perp\lambda_2)^2+\lambda_1^2\gamma_\perp^2}+\frac{T_{21}\lambda_1-T_{22}\left[\dfrac{\omega-\omega_0}{\gamma_\perp}+\lambda_2\right]}{(\omega-\omega_0+\gamma_\perp\lambda_2)^2+\gamma_\perp^2\lambda_1^2}+\frac{T_{41}\sigma_1+T_{42}\left[\dfrac{\omega-\omega_0}{\gamma_\perp}-\sigma_2\right]}{(\omega-\omega_0-\gamma_\perp\sigma_2)^2+\gamma_\perp^2\sigma_1^2}+\frac{T_{41}\sigma_1-T_{42}\left[\dfrac{\omega-\omega_0}{\gamma_\perp}+\sigma_2\right]}{(\omega-\omega_0+\gamma_\perp\sigma_2)^2+\gamma_\perp^2\sigma_1^2}\right]\ , \tag{5.16a}$$

where

$$T_{11}=\frac{\lambda_2}{F_1}\{-2\mu C_n x_S^{n-1}(2\lambda_1-\mu f-1)(\langle\delta\hat{v}^{\dagger}\delta\hat{x}\rangle+\langle\delta\hat{v}\delta\hat{x}\rangle)-4\mu C_n x_S^{2n-1}\langle\delta\hat{m}\delta\hat{x}\rangle+[(\mu f-\lambda_1)^2+\lambda_2^2-\mu nB](\langle\delta\hat{x}^{\dagger}\delta\hat{x}\rangle+\langle\delta\hat{x}^2\rangle)\}\ , \tag{5.16b}$$

$$F_1=2\lambda_2[\lambda_2^2+(\mu_1-\lambda_1)^2]\ , \tag{5.16c}$$

$$T_{21}=\frac{\lambda_2}{2F_1}\{2\mu C_n x_S^{n-1}(2\lambda_1-\mu f-1)(\langle\delta\hat{v}^{\dagger}\delta\hat{x}\rangle+\langle\delta\hat{v}\delta\hat{x}\rangle)+4\mu C_n x_S^{2n-1}\langle\delta\hat{m}\delta\hat{x}\rangle+[(\mu f-\mu_1)(2\lambda_1-\mu f-\mu_1)+n\mu B](\langle\delta\hat{x}^{\dagger}\delta\hat{x}\rangle+\langle\delta\hat{x}^2\rangle)\}\ , \tag{5.16d}$$

$$T_{22}=\frac{1}{2F_1}\{-2\mu C_n x_S^{n-1}[(\lambda_1+\mu_1-\mu f-1)(\lambda_1-\mu_1)-\lambda_2^2](\langle\delta\hat{v}^{\dagger}\delta\hat{x}\rangle+\langle\delta\hat{v}\delta\hat{x}\rangle)+4\mu C_n x_S^{2n-1}(\mu_1-\lambda_1)\langle\delta\hat{m}\delta\hat{x}\rangle+[(\lambda_1-\mu_1)(\mu f-\mu_1)(\mu f-\lambda_1)-\mu nB(\lambda_1-\mu_1)+(\mu f-\mu_1)\lambda_2^2](\langle\delta\hat{x}^{\dagger}\delta\hat{x}\rangle+\langle\delta\hat{x}^2\rangle)\}\ , \tag{5.16e}$$

$$T_{41}=\tfrac{1}{4}(\langle\delta\hat{x}^{\dagger}\delta\hat{x}\rangle-\langle\delta\hat{x}^2\rangle)\ , \tag{5.16f}$$

$$T_{42}=\frac{1}{4\sigma_2}\left[\frac{2C_n x_S^{n-1}}{nB}[\sigma_2^2+(1-\sigma_1)^2](\langle\delta\hat{v}^{\dagger}\delta\hat{x}\rangle-\langle\delta\hat{v}\delta\hat{x}\rangle)-(1-\sigma_1)(\langle\delta\hat{x}^{\dagger}\delta\hat{x}\rangle-\langle\delta\hat{x}^2\rangle)\right]\ . \tag{5.16g}$$

(3) μ_2,μ_3 are complex conjugates; μ_1,μ_4,μ_5 are real. By Eqs. (5.15) and (5.16), $S_{\rm inc}(\omega)$ is expressed by Eq. (5.15a), but the first, second, and third terms are substituted, respectively, by the first, second, and third terms of Eq. (5.16a).

(4) μ_4,μ_5 are complex conjugates; μ_1,μ_2,μ_3 are real. By Eqs. (5.15) and (5.16), $S_{\rm inc}(\omega)$ is expressed by Eq. (5.15a), but the fourth and fifth terms are substituted, respectively, by the fourth and fifth terms of Eq. (5.16a).

In the following we discuss two cases: the spectra of the transmitted light for one- and two-photon transitions.

(1) The spectrum of the transmitted light of a one-photon transition and the vacuum Rabi splitting. For $n=1$, $\sigma=-1$, $C_1=20$, $d=2$, $\mu=1$, $\bar{n}=0$, this is the case of the purely radiative (thermal fluctuations negligible), one-photon optical bistability without adiabatic elimina-

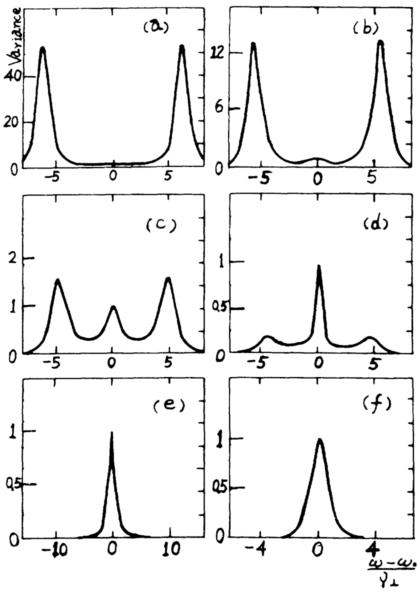

FIG. 1. Variance $S_{\rm inc}(\omega)/S_{\rm inc}(0)$ vs $(\omega-\omega_0)/\gamma_\perp$ for $n=1$, $\sigma=-1$, $C_1=20$, $d=2$, $\bar{n}=0$, $\mu=1$. (a) $x_S=0.01$, (b) $x_S=0.5$, (c) $x_S=0.8$, (d) $x_S=0.95$, (e) $x_S=7$, (f) $x_S=40$.

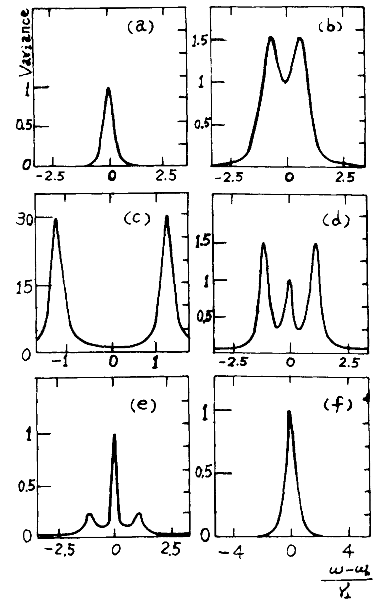

FIG. 3. Variance $S_{\rm inc}(\omega)/S_{\rm inc}(0)$ vs $(\omega-\omega_0)/\gamma_\perp$ for $n=2$, $\sigma=-1$, $C_2=3$, $d=2$, $\bar{n}=0$, $\mu=0.4$. (a) $x_S=0.01$, (b) $x_S=0.4$, (c) $x_S=0.8$, (d) $x_S=1.4$, (e) $x_S=1.5$, (f) $x_S=40$.

tion. As demonstrated in Fig. 1, we chart the curves of $S_{\rm inc}(\omega)$ with different values of x_S. As shown by Figs. 1(a) and 1(b), when the value of x_S is very small, there are two peaks in the spectrum. It is just the vacuum-field Rabi splitting.[19,20] This is a peculiar feature for $\mu\sim1$. Let us discuss it in the following. From Eqs. (4.8) and (4.9), we obtain the distance (the Rabi frequency) between the two peaks,

$$\Omega=\gamma_\perp\sigma_2=\sigma_\perp\lambda_2=\gamma_\perp\left[\frac{8\mu C_1(-\sigma)}{1+x_S^2}-(1-\mu)^2\right]^{1/2}. \tag{5.17}$$

For $\mu\gg1$ or $\mu\ll1$, Ω is an imaginary number. Hence, from the equation for $S_{\rm inc}(\omega)$, there is no vacuum Rabi splitting. Only for $\mu\sim1$, $\Omega=2\gamma_\perp(2C_1)^{1/2}$, comparing with Ref. 16, Eq. (5.6), does vacuum Rabi splitting appear. From Eq. (5.17) when $\sigma>0$ (the laser case), there is also no vacuum Rabi splitting.

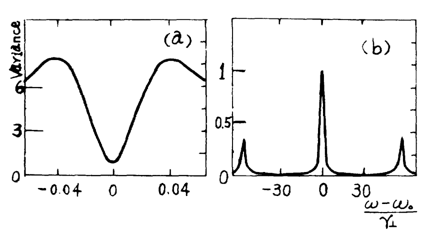

FIG. 2. Variance $S_{\rm inc}(\omega)/S_{\rm inc}(0)$ vs $(\omega-\omega_0)/\gamma_\perp$ for $n=1$, $\sigma=-1$, $C_1=20$, $d=2$, $\bar{n}=0$. (a) $\mu=10^{-3}$, $x_S=0.1$; (b) $\mu=10^3$, $x_S=40$.

When the value of x_S increases, the spectrum becomes three peaked [see Figs. 1(c) and 1(d)]. For $x_S\gg1$, the sidepeaks decrease and finally vanish. We know, for $n=1$, $x_S\gg1$, $\mu\gg1$ (the bad-cavity limit), there are three peaks of resonance fluorescence; for $n=1$, $x_S\gg1$, $\mu\ll1$ (the good-cavity limit), there is one peak. Hence, for $n=1$, $x_S\gg1$, $\mu\sim1$, the spectra as shown by Figs. 1(e) and 1(f) are just between these two limiting cases.

Figure 2 indicates representative spectra for $\mu\gg1$, $\mu\ll1$. The result coincides with Ref. 1, Figs. 29(a) and 30(f).

(2) The two-photon spectrum of the transmitted light. For $n=2$, $\sigma=-1$, $C_2=3$, $\mu=0.4$, $d=2$, $\bar{n}=0$, it is easy to verify that the stability condition Eq. (4.3) is satisfied. As demonstrated in Fig. 3, we chart the curves of $S_{\rm inc}(\omega)$ with different values of x_S. When x_S is very small, there is a single peak only [Fig. 3(a)]. For $x_S\sim1$, the two peaks appear [Figs. 3(b) and 3(c)]. This is Rabi splitting caused by the cavity field. When x_S continues to increase there are three peaks [Figs. 3(d) and 3(e)] and the central peak gradually becomes larger than the sidepeaks. For $x_S\gg1$, the values of the sidepeaks rapidly drop [Fig. 3(f)].

VI. NONCLASSICAL EFFECTS

In this section we discuss squeezing and antibunching effects in the multiphoton case.

(1) *Squeezing effect of the field.* Let

$$A=(A^\dagger+A)/2,\quad A=(A^\dagger-A)/2\,. \tag{6.1}$$

Using Eqs. (3.14) and (3.15), we have

$$\langle(\delta A_1)^2\rangle=\tfrac{1}{4}+\tfrac{1}{2}[\langle\delta A^\dagger\delta A\rangle+\langle(\delta A)^2\rangle]=\tfrac{1}{4}(T-M)\ , \tag{6.2}$$

$$\langle(\delta A_2)^2\rangle=\tfrac{1}{4}+\tfrac{1}{2}[\langle\delta A^\dagger\delta A\rangle-\langle(\delta A)^2\rangle]=\tfrac{1}{4}M\ . \tag{6.3}$$

Where T,M are given by Eqs. (3.16)–(3.19).

(a) Good-cavity limit ($\mu\ll 1$). From Eqs. (6.2) and (6.3), for $\mu\ll 1$, we obtain

$$\langle(\delta A_1)^2\rangle=\frac{1}{4}+\frac{\bar n}{2\dfrac{dy}{dx_S}}+\frac{C_n x_S^{2n-2}}{2(1+x_S^{2n})^3\dfrac{dy}{dx_S}}\{x_S^{4n}(n-\sigma)+x_S^{2n}[\sigma(2n-2)+2n-n\sigma^2(1+d)]+\sigma(2n-1)+n\}\ , \tag{6.4}$$

$$\langle(\delta A_2)^2\rangle=\frac{1}{4}+\frac{\bar n}{2\dfrac{dy}{dx_S}}+\frac{C_n x_S^{2n-2}}{2(1+x_S^{2n})\dfrac{dy}{dx_S}}(nx_S^{2n}+\sigma+n)\ . \tag{6.5}$$

Equations (6.4) and (6.5) are just Ref. 6, Eqs. (9) and (10).

(b) Bad-cavity limit ($\mu\gg 1$). From Eqs. (6.2) and (6.3), for $\mu\gg 1$, we obtain

$$\langle(\delta A_1)^2\rangle=\frac{1+2\bar n}{4}\frac{1}{1+(n-1)B}+\frac{1}{\mu}J\ , \tag{6.6}$$

$$\langle(\delta A_2)^2\rangle=\frac{1+2\bar n}{4}\frac{1}{1-(n-1)B}+\frac{1}{\mu}\frac{2\bar n-B\left[1+\dfrac{1}{\sigma}n(1+x_S^{2n})\right]}{4(1+B)[1-(n-1)B]}\ , \tag{6.7}$$

where

$$J=\frac{nB}{f(f+nB+df)(1+x_S^{2n})\dfrac{dy}{dx_S}}\left[\sigma d(1+d)fx_S^{2n}-\frac{(1+x_S^{2n})^2}{\sigma}\left[df+\frac{dy}{dx_S}\right]+(1+2\bar n)(1+x_S^{2n})\left[d(x_S^{2n}-1)-\frac{1}{f}(1+dx_S^{2n})\frac{dy}{dx_S}\right]\right]\ . \tag{6.8}$$

For $n=1$, $\bar n=0$, Eq. (6.6) is just Ref. 5, Eq. (9). For $n\geq 2$ the first terms of Eqs. (6.6) and (6.7) are

$$(1+2\bar n)/4[1\pm(n-1)B]\ ,$$

which indicate that there is a fixed squeezing value in the cavity for the bad-cavity case. When $n\geq 2$, $\bar n=0$, the minima of the $\langle(\delta A_1^2\rangle$ and $\langle(\delta A_2)^2\rangle$ are

$$\langle(\delta A_{1,2})^2\rangle=\frac{1}{4}\frac{1}{1\pm(-2\sigma C_n)\frac{1}{n}(n-1)^{2-1/n}}+O\left[\frac{1}{\mu}\right]\ .$$

For $\sigma<0$, we obtain $\langle(\delta A_1)^2\rangle\langle\frac{1}{4}$, and A_1 is squeezed. For $\sigma>0$, we obtain $\langle(\delta A_2)^2\rangle\langle\frac{1}{4}$; hence for the bad-cavity limit, in the multiphoton laser the A_2 can be squeezed. From the stability condition Eq. (4.4) the value of σC_n is limited for $\mu\ll 1$, so the minimum squeezing of $\langle(\delta A_1)^2\rangle$ or $\langle(\delta A_2)^2\rangle$ is $\frac{1}{8}$.

From Eqs. (6.4)–(6.7), when thermal fluctuations exist ($\bar n\neq 0$), the squeezing effects weaken or disappear.

Figures 4 and 5, respectively, show $\langle(\delta A_1)^2\rangle$ as a function of changes of x_S with μ for $n=1$ and 2. In Fig. 4, for $\mu\sim 1$ the squeezing effect is the most remarkable; for $\mu\ll 1$ $\langle(\delta A_1)^2\rangle$ slightly increases. When μ increases, $\langle(\delta A_1)^2\rangle$ rapidly tends to $\frac{1}{4}$. In Fig. 5 the more μ increases, the more $\langle(\delta A_1)^2\rangle$ decreases. For $\mu\gg 1$ $\langle(\delta A_1)^2\rangle_{\min}$ tends to the limit $1/[4(1+C_2)]=0.139$. From Fig. 5 the more μ increases, the larger the range of values x_S in the case of squeezing.

(2) *Squeezing in the atomic system.* Let

$$R_1=\tfrac{1}{2}(R^++R^-),\quad R_2=\frac{1}{2i}(R^+-R^-)\ . \tag{6.9}$$

One has squeezing if

$$\langle(\delta R_1)^2\rangle\langle\tfrac{1}{2}|\langle R_3\rangle|\quad\text{or}\quad\langle(\delta R_2)^2\rangle\langle\tfrac{1}{2}|\langle R_3\rangle|\ . \tag{6.10}$$

Substituting Eqs. (3.26), (3.27), and (2.3) into Eq. (6.10), we have

$$\begin{aligned}\langle(\delta R_1)^2\rangle-\tfrac{1}{2}|\langle R_3\rangle|&=\tfrac{1}{2}[\langle\delta R^+\delta R^-\rangle+\langle(\delta R^+)^2\rangle-\langle R_3\rangle-|\langle R_3\rangle|]\\&=\frac{N}{4}\frac{1}{2C_n x_S^{2n-2}}\left[\mu B\{-f[T-M]+(1+2\bar n)\}+\frac{W}{\mu}-\frac{B}{\sigma}(1+x_S^{2n})-|B|\right]\ ,\end{aligned} \tag{6.11a}$$

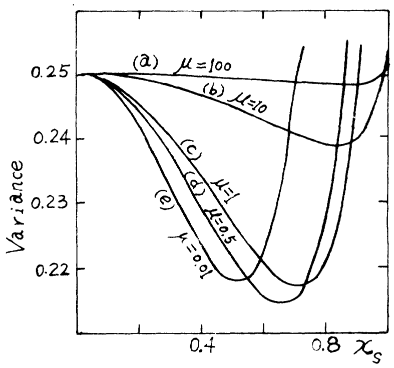

FIG. 4. Variance $\langle(\delta A_1)^2\rangle$ as a function of the transmitted field x_S, for $n=1$, $\sigma=-1$, $C_1=20$, $d=2$, $\bar{n}=0$. (a) $\mu=100$, (b) $\mu=10$, (c) $\mu=1$, (d) $\mu=0.5$, (e) $\mu=0.01$.

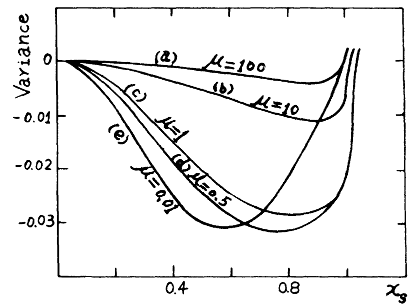

FIG. 6. Variance $\langle(\delta R_1)^2\rangle-\frac{1}{2}|\langle R_3\rangle|$ as a function of the transmitted field x_S, for $n=1$, $\sigma=-1$, $C_1=20$, $d=2$, $\bar{n}=0$. (a) $\mu=100$, (b) $\mu=10$, (c) $\mu=1$, (d) $\mu=0.5$, (e) $\mu=0.01$.

$$\langle(\delta R_2)^2\rangle-\tfrac{1}{2}|\langle R_3\rangle|=\tfrac{1}{2}[\langle\delta R^+\delta R^-\rangle-\langle(\delta R^+)^2\rangle-\langle R_3\rangle-|\langle R_3\rangle|]$$

$$=\frac{N}{4}\frac{1}{2C_n x_S^{2n-2}}\left[\mu B[(2-f)M+1+2\bar{n}]-\frac{B}{\sigma}(1+x_S^{2n})-|B|\right]. \tag{6.11b}$$

Figures 6 and 7 show $\langle(\delta R_1)^2\rangle-\frac{1}{2}|\langle R_3\rangle|$ as a function of x_S with different values of μ, for $n=1$ and 2. In Fig. 6, $n=1$, the larger μ, the weaker the squeezing. In Fig. 7, $n=2$, the larger μ, the more remarkable the squeezing. So in the bad-cavity limit, the atom-atom correlations strengthen the squeezing for the two-photon case,[15] and weaken the squeezing for the single-photon case.[5]

In the following we discuss the limiting cases.

(a) $\mu\ll 1$, good-cavity limit. From Eq. (6.11), we have

$$\langle(\delta R_1)^2\rangle-\tfrac{1}{2}|\langle R_3\rangle|$$

$$=\frac{N}{4}\left[1-\frac{|\sigma|}{(1+x_S^{2n})^2}[1+(1+d|\sigma|)x_S^{2n}]\right], \tag{6.12a}$$

$$\langle(\delta R_2)^2\rangle-\tfrac{1}{2}|\langle R_3\rangle_S|=\frac{N}{4}\left[1-\frac{|\sigma|}{1+x_S^{2n}}\right]. \tag{6.12b}$$

It coincides with Ref. 14, Eq. (6.8). The minimum is

$$[\langle(\delta R_1)^2\rangle-\tfrac{1}{2}|\langle R_3\rangle|]_{\min}=\frac{N}{4}\left[1-\frac{(1+d|\sigma|)^2}{4d}\right]. \tag{6.12c}$$

It indicates that the minimum squeezing is independent of the number of photons n. We can see that also by the curves of $\mu\ll 1$ in Figs. 6 and 7.

(b) The bad-cavity limit. For $\mu\gg 1$, from Eqs. (6.11), (3.16)–(3.19), (3.25), and (3.30) we have

$$\langle(\delta R_1)^2\rangle-\tfrac{1}{2}|\langle R_3\rangle|$$

$$=\frac{N}{4}\frac{1}{2C_n x_S^{2n-2}}\left[\frac{1+2\bar{n}}{f}B-|B|+\frac{f^2}{n}J\right], \tag{6.13a}$$

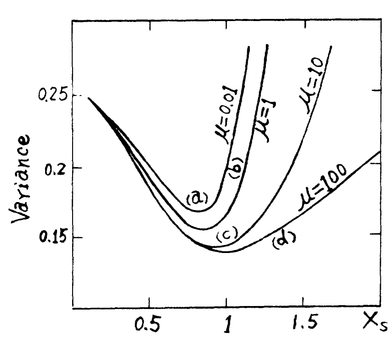

FIG. 5. Variance $\langle(\delta A_1)^2\rangle$ as a function of the transmitted field x_S, for $n=2$, $C_2=0.8$, $\sigma=-1$, $d=2$, $\bar{n}=0$. (a) $\mu=0.01$, (b) $\mu=1$, (c) $\mu=10$, (d) $\mu=100$.

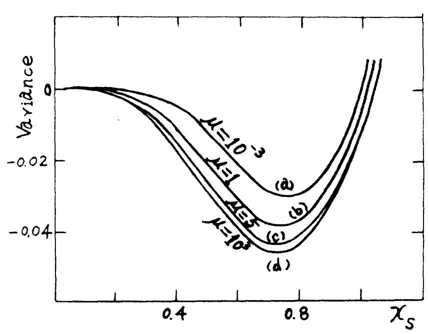

FIG. 7. Variance $\langle(\delta R_1)^2\rangle-\frac{1}{2}|\langle R_3\rangle|$ as a function of the transmitted field x_S, for $n=2$, $\sigma=-1$, $C_2=0.8$, $d=2$, $\bar{n}=0$. (a) $\mu=10^{-3}$, (b) $\mu=1$, (c) $\mu=5$, (d) $\mu=10^3$.

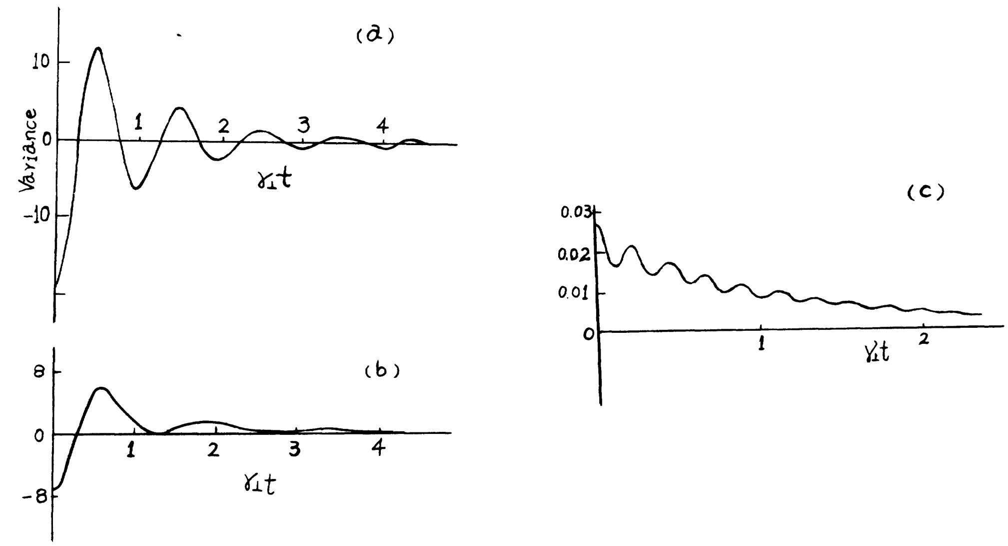

FIG. 8. Variance $N[g^{(2)}(t)-1]$ as a function of the $\gamma_\perp t$ for $n=1$, $\sigma=-1$, $C_1=20$, $d=2$, $\bar{n}=0$. (a) $x_S=0.01$, (b) $x_S=0.5$, (c) $x_S=20$.

$$\langle(\delta R_2)^2\rangle-\tfrac{1}{2}|\langle R_3\rangle| = \frac{N}{4}\frac{1}{2C_n x_S^{2n-2}}\left[\frac{B}{1+B}\left[-\frac{1}{\sigma}(1+x_S^{2n})(1+B+nB)-B+2\bar{n}\right]-|B|\right], \tag{6.13b}$$

where J is given by Eq. (6.8). When $\bar{n}=0$, $n=1$, these expressions coincide with Ref. 5, Eq. (37). When $n=2$, they coincide with Ref. 15, Eq. (44), (45).

(3) *Second-order correlation function; antibunching.* The second-order correlation function[16]

$$g^{(2)}(t)=1+\frac{1}{x_S^2}[\langle\delta\hat{x}^{\dagger}(t)\delta\hat{x}(0)\rangle+\langle\delta\hat{x}^{\dagger}(0)\delta\hat{x}(t)\rangle +2\,\mathrm{Re}(\langle\delta\hat{x}(t)\delta\hat{x}(0)\rangle)] . \tag{6.14}$$

Using Eqs. (5.3) and (5.4), we obtain

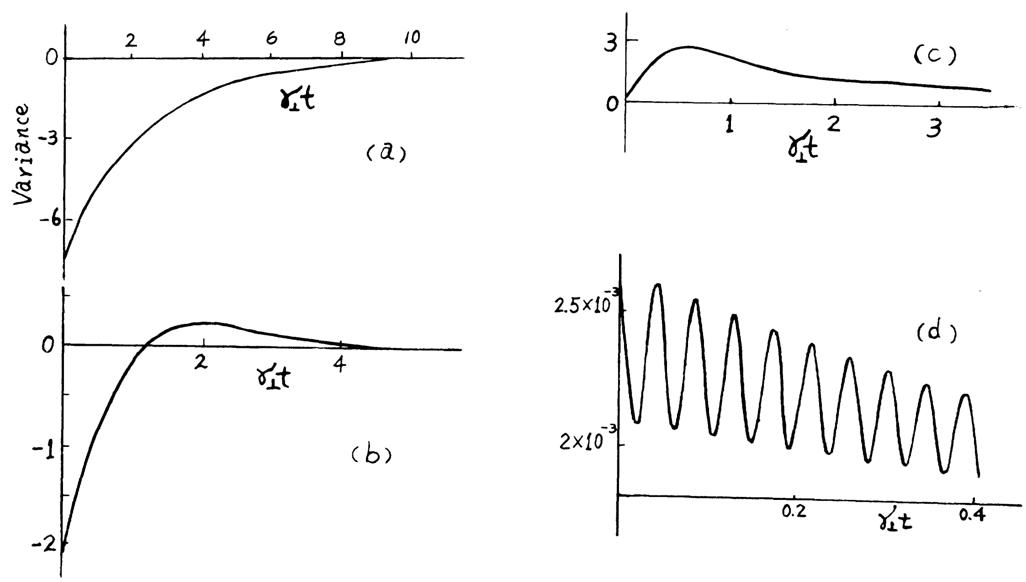

FIG. 9. Variance $N[g^{(2)}(t)-1]$ as a function of the $\gamma_\perp t$ for $n=2$, $\sigma=-1$, $C_2=0.8$, $d=2$, $\bar{n}=0$. (a) $x_S=0.01$, (b) $x_S=0.5$, (c) $x_S=1.2$ (d) $x_S=20$.

$$g^{(2)}(t)=1+\frac{2}{x_S^2}(T_{R_1}e^{-\gamma_1\mu_1 t}+T_{R_2}e^{-\gamma_1\mu_2 t}+T_{R_3}e^{-\gamma_1\mu_3 t}) , \quad (6.15)$$

where μ_i $(i=1,2,3)$ are real. Or

$$g^{(2)}(t)=1+\frac{2}{x_S}\{T_{11}e^{-\gamma_1\mu_1 t} +2e^{-\gamma_1\lambda_1 t}[T_{21}\cos(\gamma_1\lambda_2 t) +T_{22}\sin(\gamma_1\lambda_2 t)]\} , \quad (6.16)$$

where μ_1 is real, μ_2,μ_3 are complex conjugates. One has bunching for $g^{(2)}>1$ and antibunching for $g^{(2)}<1$. Figure 8 shows the $g^{(2)}(t)\sim t$ for the one-photon case. In Fig. 8(a) the oscillatory behavior is a new feature for $\mu=1$; this is the vacuum-field Rabi oscillation ($x_S \ll 1$). When x_S increases, the antibunching is weakened [Fig. 8(b)]. For $x_S \gg 1$, the oscillatory behavior is intensive and bunching appears [Fig. 8(c)].

Figure 9 shows $g^{(2)}(t)\sim t$ for the two-photon case; when $x_S \ll 1$ the antibunching is the most remarkable [Fig. 9(a)]. For $x_S \gg 1$, the oscillatory behavior is intensive and bunching appears [Figs. 9(c) and 9(d)].

VII. CONCLUSION

In this paper a multiphoton quantum-statistical theory without adiabatic elimination has been developed for the driven optical system. This theory has very wide application. We can study various physical processes of the driven optical system by choosing every parameter $(n,\mu,\sigma,d,C_n,\bar{n})$ at different values. For example, by choosing every parameters at an appropriate value, the best squeezing may be obtained in the best optical device. This theory coincides with the adiabatic elimination theory for the good- and bad-cavity limits.

[1]L. A. Lugiato, in *Progress in Optics XXI,* edited by E. Wolf (North-Holland, Amsterdam, 1984), p. 71, and references therein.

[2]L. A. Lugiato, F. Casagrande, and L. Pizzuto, Phys. Rev. A **26**, 3438 (1982).

[3]F. Casagrande and L. A. Lugiato, in *Quantum Optics*, edited by C. A. Engelbrcht (Springer-Verlag, Berlin, 1982), p. 53.

[4]L. A. Lugiato, Nuovo Cimento B **50**, 89 (1979).

[5]L. A. Lugiato, Phys. Rev. A **33**, 4079 (1986).

[6]L. A. Lugiato and G. Strini, Opt. Commun. **41**, 374 (1982); **41**, 477 (1982).

[7]F. Casagrande, E. Eshenazi, L. A. Lugiato, and G. Strini, in *Coherence and Quantum Optics V,* edited by L. Mandel and F. Wolf (Plenum, New York, 1984), p. 637.

[8]D. F. Walls, Phys. Rev. A **24**, 2029 (1981).

[9]D. F. Walls and M. D. Reid, Phys. Rev. A **33**, 3282 (1986).

[10]R. Loudon, Opt. Commun. **49**, 67 (1984).

[11]G. S. Agrawal and S. Singh, in Ref. 7, p. 885, and references therein.

[12]G. S. Agarwal and G. P. Hildred, Opt. Commun. **58**, 287 (1986).

[13]G. S. Agarwal, Opt. Commun. **62**, 190 (1987).

[14]R. M. Lin, S. X. Huang, and L. Zhang, Acta Phys. Sin. **37**, 573 (1988).

[15]L. Zhang, R. M. Lin, and S. X. Huang, Acta Phys. Sin. **37**, 1438 (1988).

[16]H. J. Carmichael, Phys. Rev. A **33**, 3262 (1986).

[17]J. P. Gordon, Phys. Rev. **161**, 367 (1967).

[18]F. Casagrande, E. Eschenazi, and L. A. Lugiato, Phys. Rev. A **29**, 239 (1984).

[19]J. J. Sanchez-Mondrago, N. B. Narozhny, and J. H. Eberly, Phys. Rev. Lett. **51**, 550 (1983).

[20]G. S. Agarwal, Phys. Rev. Lett. **53**, 1732 (1984).