Liapunov exponents from a time series of acoustic chaos

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Liapunov exponents have been determined from a time series of acoustic turbulence. It is found that a low-dimensional chaotic attractor with one positive largest exponent rules the motions in phase space in the broadband noise region. The method for calculating the spectrum of Liapunov exponents is analyzed in detail for its applicability in experimental situations. It is shown, in particular, that spurious exponents arising from the embedding procedure do not spoil the calculation and that the calculation of the Liapunov dimension is possible.

INTRODUCTION

The characterization of measured oscillatory time signals is one of the main tasks of experimental physics. Spectral analysis and correlation functions, using the terminology of linear modes, are powerful tools for (multi)periodic time signals. They seem to have certain limitations, however, when the chaotic states of nonlinear dynamical systems are considered. The method of phase-space analysis introduces a variety of possible classifications, namely, the spectrum of Liapunov exponents,[1] the Kolmogorov entropy,[2] the (fractal) dimension,[3-5] and $f(\alpha)$ spectrum.[6] The important link between a single time series and the phase-space analysis is the attractor reconstruction method based on time-shifted samples.[7,8] It opens this interesting field to experimental measurements. Various experimental applications of phase-space analysis exist; especially the algorithms for estimating the dimension of attractors as a measure for the number of degrees of freedom modulating a physical process. These algorithms have widely been accepted.[9] The Liapunov exponents are a basic indicator of chaos. The exponential divergence rate of trajectories in phase space is characterized by at least one positive exponent. It is responsible for the "sensitive dependence on initial conditions" and limits the predictability of the time evolution of a physical system. Few algorithms exist for the calculation of Liapunov exponents from a time series.[10-17] The basic problem with the algorithms approximating the matrix of the linearized flow (local Jacobian) is the appearance of spurious exponents. This problem is due to the embedding procedure of the attractor reconstruction. The attractor is embedded in a phase space, whose embedding dimension may be larger than the (yet unknown) dimension of the attractor, causing exponents that obviously spoil the calculation of the whole spectrum. We show, that with a suitable choice of method-inherent parameters it is possible to determine the "true" Liapunov exponents including negative ones, and to avoid the effect of the spurious exponents, which are more negative than the most negative true exponent. The true exponents indeed converge to their asymptotic values with increasing embedding dimension, the spurious ones staying below. A calculation of the Liapunov dimension is thus possible.

Acoustic turbulence or acoustic chaos, which has been found in water,[18] is subjected to phase-space analysis.[19] The experiment consists of focusing ultrasonic waves of a single frequency and high intensity into water and measuring the sound output of the liquid. Beyond a certain threshold intensity of the sound pressure a cloud of tiny bubbles appears in the liquid. These bubbles are oscillating in the driving sound field and thus producing the acoustic cavitation noise.[20] Broadband noise and a subharmonic route to chaos while increasing the driving sound pressure amplitude have been found by calculating spectral bifurcation diagrams from the cavitation noise data.[21] Attractors in phase space have been reconstructed from experimental time series in the broadband noise region. Evidence for a low-dimensional strange attractor has been achieved by measuring a correlation dimension of about 2.5.[22] Numerical calculations of simple bubble models confirm these findings.[23-26] Measuring the Liapunov exponents could be a further indication of low-dimensional chaos. In fact, a time series analyzed shows that the dynamics in phase space is characterized by one positive exponent.

CALCULATION OF THE LIAPUNOV EXPONENTS

The dynamics of a physical system may be described by a differentiable dynamical system

$$\dot{\mathbf{x}}=v(\mathbf{x}) , \tag{1}$$

where $\mathbf{x}$ is a vector in the phase space $\mathbb{R}^n$ and $v(\mathbf{x})$ is the vector field. The vector field v creates a flow $\Phi=\{\Phi^t\}$ on the phase space, where Φ^t maps

$$\mathbf{x}\mapsto\Phi^t(\mathbf{x}), \quad t\in\mathbb{R}, \quad \mathbf{x}\in\mathbb{R}^n . \tag{2}$$

The observed trajectory, starting at $\mathbf{x}_0$, is

$$\{\Phi^t(\mathbf{x}_0)\,|\,t\in\mathbb{R}^+\} . \tag{3}$$

To get information about the time evolution of arbitrarily small perturbed initial conditions, one considers the time evolution of tangent vectors in tangent space, given by the linearization of (1). The Taylor expansion of $v(\Phi^t(\mathbf{x}_0))$ for small $\Delta\mathbf{x}$ is

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$$\dot{\Phi}^t(\mathbf{x}_0)+\dot{\Delta\mathbf{x}}=v(\Phi^t(\mathbf{x}_0))+Dv(\Phi^t(\mathbf{x}_0))\Delta\mathbf{x}+\cdots \quad (4)$$

$Dv(\Phi^t(\mathbf{x}_0))$ is the (local) Jacobian matrix of the vector field v at $\Phi^t(\mathbf{x}_0)$

$$Dv(\Phi^t(\mathbf{x}_0))=\left[\left.\frac{\partial v_i}{\partial x_j}\right|_{\Phi^t(\mathbf{x}_0)}\right] . \quad (5)$$

For $\Delta\mathbf{x}\rightarrow 0$ the first-order approximation holds

$$\dot{\delta\mathbf{x}}=Dv(\Phi^t(\mathbf{x}_0))\delta\mathbf{x} . \quad (6)$$

A solution of the linear variational equation [Eq. (6)] is obtained by

$$\delta\mathbf{x}(t)=D\Phi^t(\mathbf{x}_0)\delta\mathbf{x}_0 , \quad (7)$$

which represents the time dependence of a vector in tangent space. $D\Phi^t(\mathbf{x}_0)$ is the $n\times n$ matrix of the linearized flow and $\delta\mathbf{x}_0$ an initial perturbation, whose time-evolved value is denoted by $\delta\mathbf{x}(t)$. Let $\mathbf{E}:=(\mathbf{e}^i,\ldots,\mathbf{e}^n)$ be an $n\times n$ matrix, where the column vectors are a basis of the tangent space and span the eigenspaces of the limit matrix

$$\Lambda_{\mathbf{x}_0}:=\lim_{t\rightarrow\infty}[D\phi^t(\mathbf{x}_0)^* D\phi^t(\mathbf{x}_0)]^{1/2t} ,$$

given by the theorem of Oseledec.[1] If the limit

$$\lambda_i=\lim_{t\rightarrow\infty}\frac{1}{t}\ln\|D\Phi^t(\mathbf{x}_0)\mathbf{e}^i\| \quad (8)$$

exists, then the λ_i's are called the characteristic Liapunov exponents. They are ordered by their magnitudes $\lambda_1>\lambda_2>\lambda_3>\cdots$. If the limit is independent of $\mathbf{x}_0$, the system is called ergodic.[1] Calculation of the Liapunov exponents using Eq. (8) with an arbitrary set of basis vectors is not possible because, for $t\rightarrow\infty$, all basis vectors grow in the same direction.[27] To inhibit this, one has to use a renormalization procedure after some time Δt. One can write $D\Phi^t(\mathbf{x}_0)$ as a product of $n\times n$ matrices $D\Phi^{\Delta t}(\mathbf{x}_j)$, each representing the linearization of the flow $\Phi^{\Delta t}$, that maps $\mathbf{x}_j\equiv\Phi^{j\Delta t}(\mathbf{x}_0)$ to $\mathbf{x}_{j+1}$:

$$D\Phi^{k\Delta t}(\mathbf{x}_0)=D\Phi^{\Delta t}(\mathbf{x}_{k-1})\circ\cdots\circ D\Phi^{\Delta t}(\mathbf{x}_j)\circ\cdots\circ D\Phi^{\Delta t}(\mathbf{x}_1)\circ D\Phi^{\Delta t}(\mathbf{x}_0) , \quad (9)$$

with $k\Delta t\equiv t$. After every time step of the evolution time Δt any renormalization method can be applied. We use Householder's QR decomposition method:[28] Each invertible $n\times n$ matrix can be split uniquely into a product of an upper triangular matrix R with non-negative diagonal elements and an orthogonal matrix Q, such that

$$D\Phi^{\Delta t}(\mathbf{x}_j)\mathbf{E}_j=Q_jR_j=\mathbf{E}_{j+1}R_j , \quad (10)$$

with $\mathbf{E}_j:=(\mathbf{e}_j^1,\ldots,\mathbf{e}_j^n)$. The matrix Q_j serves as the new basis $\mathbf{E}_{j+1}$ and the logarithms of the diagonal elements of R_j are (local) expanding coefficients, whose time-averaged values are the Liapunov exponents. Using

$$D\Phi^{k\Delta t}(\mathbf{x}_0)\mathbf{E}_0=\prod_{j=0}^{k-1}D\Phi^{\Delta t}(\mathbf{x}_j)\mathbf{E}_0=Q_{k-1}\prod_{j=0}^{k-1}R_j \quad (11)$$

in (8) gives

$$\lambda_i=\lim_{k\rightarrow\infty}\frac{1}{k\Delta t}\sum_{j=0}^{k-1}\ln r_{ii}^j , \quad (12)$$

where r_{ii}^j are the diagonal elements of the matrix R_j.

The matrix of the linearized flow $D\Phi^{\Delta t}(\mathbf{x}_j)$ can be approximated from a single trajectory by using the recurrent structure of strange attractors. This is done by averaging over the time evolution of difference vectors between $\mathbf{x}_j$ and points of the same trajectory on the attractor, that are within a small distance r. The set of N difference vectors in a cube centered at $\mathbf{x}_j=\Phi^{j\Delta t}(\mathbf{x}_0)$, with $j\Delta t\equiv t$ is

$$B(r)=\{\Phi^t(\mathbf{x}_0)-\Phi^{t+t_i}(\mathbf{x}_0)\mid\|\Phi^t(\mathbf{x}_0)-\Phi^{t+t_i}(\mathbf{x}_0)\|\le r,\ t_i\ge -t,\ i=1,\ldots,N\} \quad (13)$$

and shall be denoted $\{\mathbf{y}^i|i=1,\ldots,N\}$. After the evolution time of Δt it is mapped to the set $\{\Phi^{t+\Delta t}(\mathbf{x}_0)-\Phi^{t+t_i+\Delta t}(\mathbf{x}_0)\}\equiv\{\mathbf{z}^i|i=1,\ldots,N\}$. Now an $n\times n$ matrix A_j is determined, such that the average of the squared error norm of all possible mappings $\{\mathbf{y}^i\}\mapsto\{\mathbf{z}^i\}$ takes a minimum[11]

$$\min_{A_j}S=\min_{A_j}\frac{1}{N}\sum_{i=1}^{N}\|\mathbf{z}^i-A_j\mathbf{y}^i\|^2 . \quad (14)$$

This equation can be solved for the matrix A_j,

$$A_j=CV^{-1}\ \text{ with } ((V))_{kl}=\frac{1}{N}\sum_{i=1}^{N}y_k^iy_l^i ,\quad ((C))_{kl}=\frac{1}{N}\sum_{i=1}^{N}z_k^iy_l^i . \quad (15)$$

If the cube length r and the evolution time Δt are short enough to represent a mapping in tangent space [Eq. (7)], A_j should be a good approximation of the matrix of the linearized flow $D\Phi^{\Delta t}(\mathbf{x}_j)$. Note that the evolution times Δt in the renormalization and the approximation process do not necessarily have to be the same, but are chosen equal for convenience. This approximation method seems to be the most flexible one in analyzing a data set, because several parameters [i.e., the evolution time Δt or the time interval T (see below)] can be controlled separately.

The connection between fractal dimensions and the Liapunov exponents is given by the Liapunov dimension[29,30]

$$D_L = m + \frac{\sum_{i=1}^{m} \lambda_i}{|\lambda_{m+1}|} , \tag{16}$$

with $m \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} \lambda_i \geq 0 \text{ and } \sum_{i=1}^{m+1} \lambda_i < 0 .$$

It should have a value close to the fractal dimension.

To reconstruct an attractor in phase space from a single time series of an observable, we use the method of time-shifted samples.[7,8] Let an observable be a function p, that maps any point $\Phi^t(\mathbf{x}_0)$ in the phase space to a (measurable) real value $p(\Phi^t(\mathbf{x}_0))$. It has been shown[7] for compact manifolds of dimension m, that the set

$$\{p(\Phi^t(\mathbf{x}_0)), p(\Phi^{t+T}(\mathbf{x}_0)), \ldots, p(\Phi^{t+2mT}(\mathbf{x}_0)) \,|\, T \in \mathbb{R}^+ \setminus \{0\},\ t \to \infty\} \tag{17}$$

is diffeomorphic to the positive limit set of $\Phi^t(\mathbf{x}_0)$ under generic conditions. T is called the time "delay."

NUMERICAL TESTS

The values of the various free parameters of the method along with the applicability to experimental data is tested with a time series obtained from a differential equation. The $x(t)$ variable of the Duffing oscillator as a model for a simple dissipative periodically driven nonlinear oscillator is used,

$$\begin{aligned} &\ddot{x} + D\dot{x} + x + x^3 = F\cos(\omega t) , \\ &D = 0.2 , \quad F = 40 , \\ &\omega = 2\pi/T_0 = 1 . \end{aligned} \tag{18}$$

Integrating the differential equation along with Eq. (6) (Refs. 10, 27, and 31) yields $\lambda_1 = 1.0$, $\lambda_2 = 0.0$, and $\lambda_3 = -2.8$ bits/T_0 as the values for the Liapunov exponents, identifying a chaotic attractor in phase space. The $x(t)$ coordinate has been sampled with a sampling time of $t_s \approx T_0/52.36$, with $T_0 = 2\pi$ being the duration of one period of the driving force. The sampling time t_s has to be incommensurable to the period of the driving force T_0, because otherwise hyperplanes of dimension $D-1$ are generated in phase space and no information about the flow direction can be extracted. $\lambda_2 = 0.0$ would be shifted to very negative values. The precision of the samples is $1{:}2^{10} = 10$ bits. The number of data points is 20 000. The radius of the cubes needed for the approximation of the matrix A_j is kept as small as possible and is enlarged whenever the number of data points inside a cube is less than $n+2$ (n = embedding dimension) or a singularity problem arises in the QR decomposition.

Figure 1 shows Liapunov exponents for different embedding dimensions. The evolution time equals $1t_s$, the time delay for the reconstruction $T = 2t_s$, where t_s is the sampling time. The values for the exponents converge quite well to the correct values with increasing matrix dimension. In Fig. 1 the main difficulty of the method becomes visible: the existence of "spurious exponents." Because of the embedding procedure and the QR decomposition of the $n \times n$ matrix A_j one gets n exponents. The determination of the right ones is easy here: The spurious exponents are the most negative ones. Calculation of, e.g., the Liapunov dimension (16) is now possible and gives a measure for the relevant number of degrees of freedom of an unknown physical system.

The dependence on the choice of the time delay T is shown in Fig. 2. It shows the result for $T = 7t_s$. The convergence of the two non-negative exponents is good, whereas the value for the third exponent after being close to the right value approaches 0 for higher matrix dimensions. An explanation for this behavior can be that with a larger embedding time $(n-1)T$ the attractor in the reconstruction space gets folded more and more, such that the folded trajectory reenters the noninfinitesimal

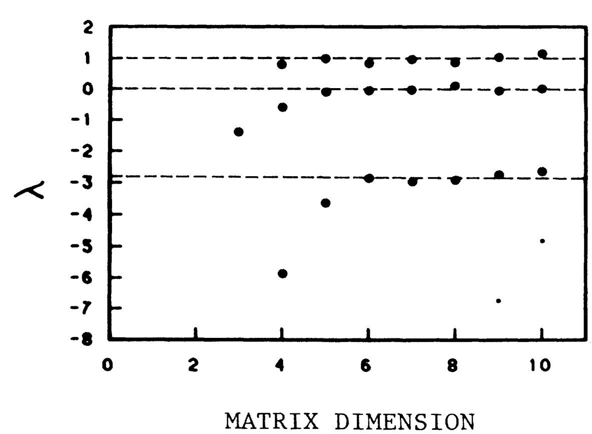

FIG. 1. Liapunov exponents from a time series of the Duffing attractor as a function of the dimension of the matrix of the linearized flow. See the text for the parameter settings. The three largest exponents, indicated by the bigger points, converge to the correct values.

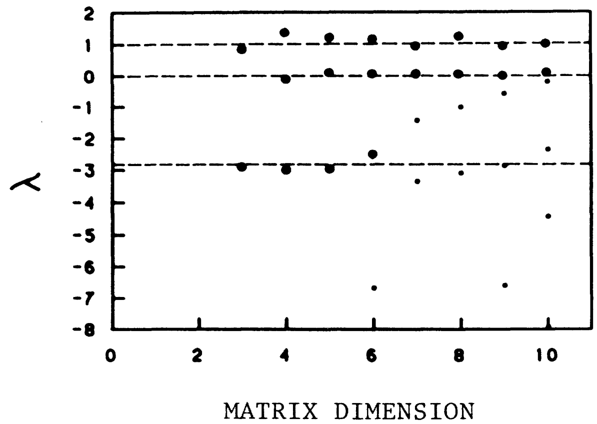

FIG. 2. Liapunov exponents of the Duffing attractor. A too large time delay for the reconstruction has been chosen.

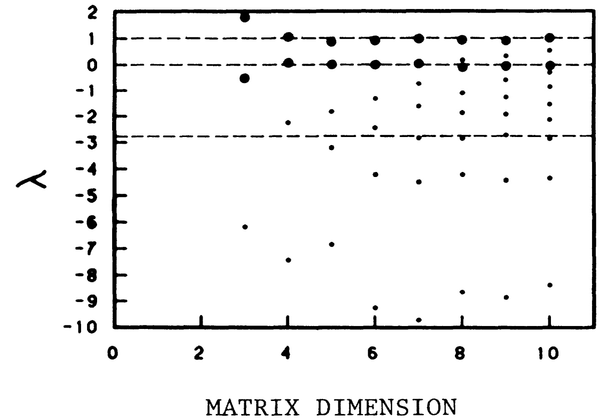

FIG. 4. Liapunov exponents of the Duffing attractor. The evolution time in the approximation process is too large.

cubes, causing a higher number of calculated nonnegative exponents. The same problem arises in dimension calculations. We claim that the embedding time should be less than $(n_D - 1)T_m$. T_m is the first minimum of the mutual information content[32] calculated from the data. It is equal to the first zero crossing of the autocorrelation function for the test data and approximately $T_0/4$. n_D is the minimum embedding dimension required by the embedding theorem ($\approx 2D_f + 1$, D_f the fractal dimension, 2.3 in our case). The formula then yields an embedding time that is slightly larger than T_0. All the "plateaus" of correct Liapunov exponents have embedding times smaller than T_0.

Figure 3 shows the calculation for a lower number of data points. The effect for the third exponent is the same as in Fig. 2; it deviates from its correct value with increased matrix dimension, while λ_1 and λ_2 show good convergence. With just 5000 data points spread on the whole attractor, the cubes needed for the approximation of the A_j's have to become quite large to contain enough points, thus violating the local concept of the method [Eqs. (6 and 7)].

In Fig. 4 the value of the evolution time Δt is considered. With $\Delta t = 5t_s$ which is approximately one-tenth of the period of the driving force and a time delay of $T = 5t_s$ even another positive exponent arises in the calculation with increasing matrix dimension. Also, all spurious exponents are shifted to higher values and interfere with the determination of the true exponents. Just the largest exponent is extractable. A mapping in tangent space as assumed for calculating the matrices A_j is no longer valid here. Thus a calculation of the right exponents is only possible when using a small enough sampling time t_s.

To clarify the influence of external noise, Gaussian noise with a standard deviation of 0.5% of the maximal attractor extent has been added to the data. Figure 5 shows the result. The parameter settings are the same as

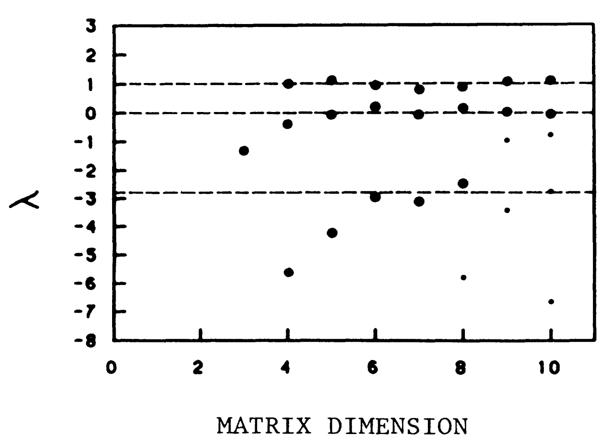

FIG. 3. Liapunov exponents of the Duffing attractor. A too small number of data points has been used.

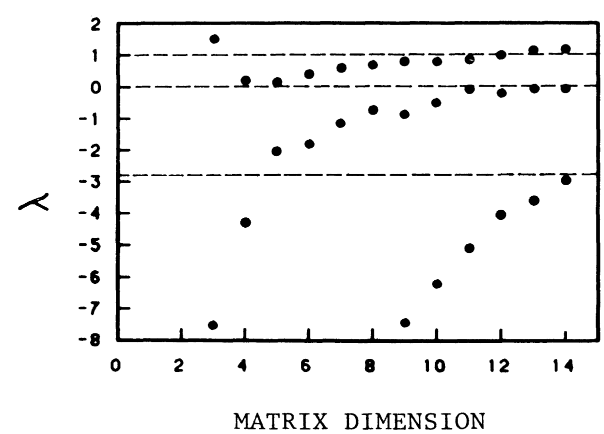

FIG. 5. Liapunov exponents of the Duffing attractor. A large amount of noise has been added to the data.

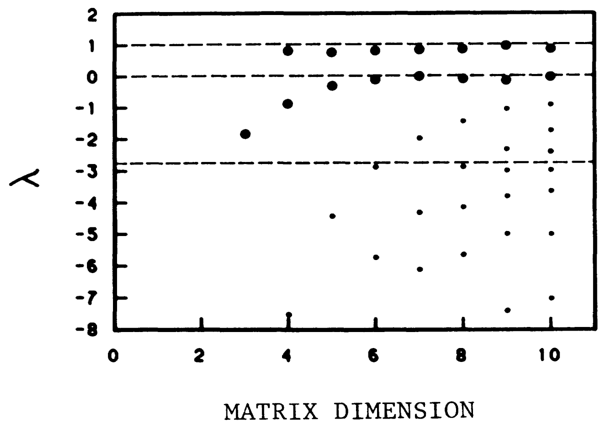

FIG. 6. Liapunov exponents of the Duffing attractor. The effects of the noise have been reduced by enlarging evolution and reconstruction time.

in Fig. 1. It is seen that the values for the two non-negative exponents slowly converge. They are close to their right values only for high embedding dimensions (>10), whereas for the negative exponent this is not justified. It is shifted into the very negative. Noise tends to decrease the values for the exponents, including the spurious ones. This gives the possibility of enlarging the values of the evolution and reconstruction time. In Fig. 6 good convergence of the non-negative exponents is achieved by setting $\Delta t = 3t_s$ and $T = 3t_s$. The information about the negative exponent is lost, however. Noise might introduce severe problems with flat attractors in determining a very negative exponent.

EXPERIMENTAL RESULTS

Using the results of the numerical tests, a data set of acoustic cavitation noise has been analyzed. The frequency of the driving sound field was $f_0 = 22.9$ kHz, the sampling frequency $t_s = 1\ \mu\text{s} \approx T_0/43.7$. The total number of data points is 10 000. In Fig. 7(a) a part of the at-

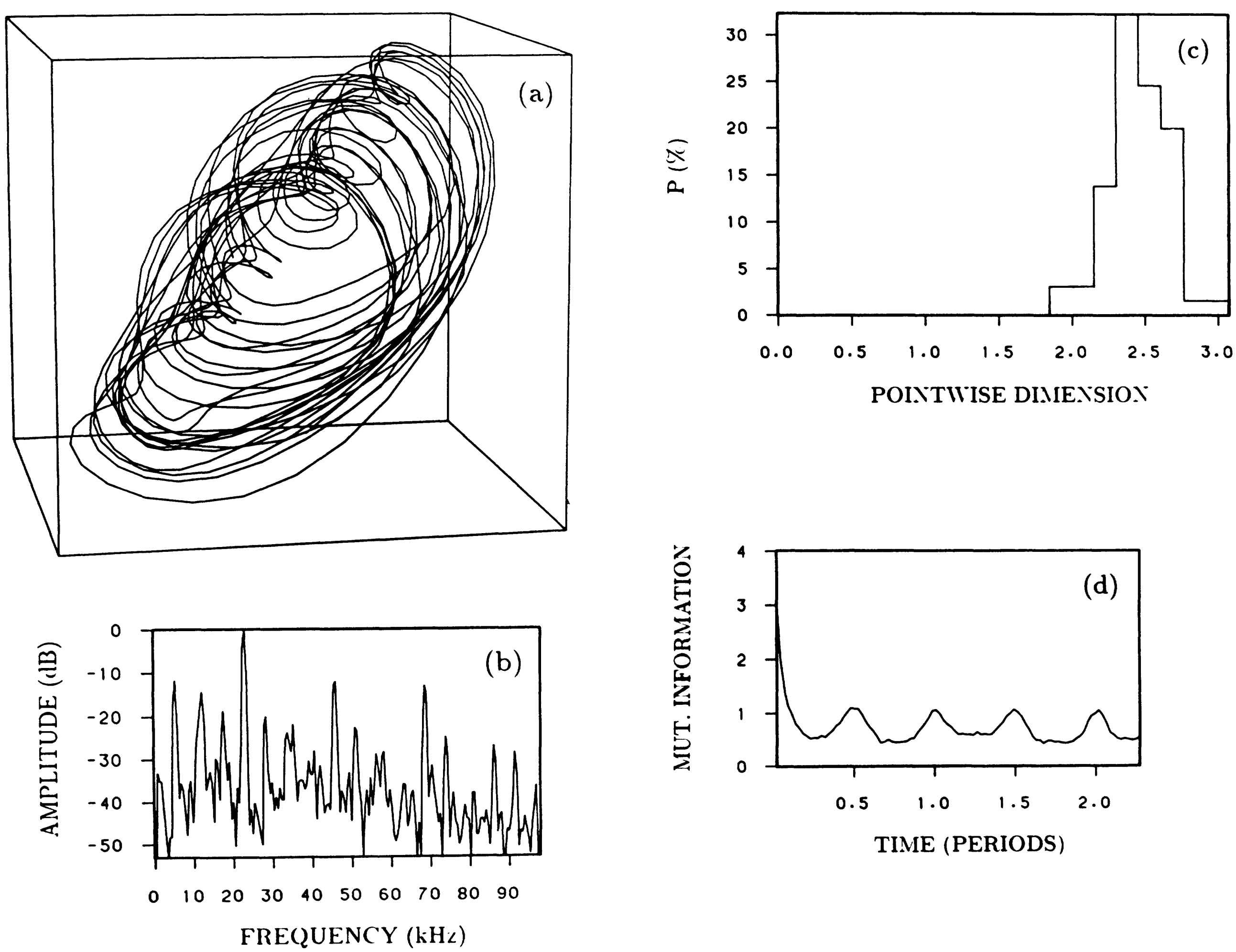

FIG. 7. Acoustic cavitation noise: Analysis of the sound pressure vs time series. (a) Projection of a three-dimensional reconstruction of the trajectory in phase space. (b) Power spectrum. (c) Histogram of 200 pointwise dimensions. (d) Mutual information content.

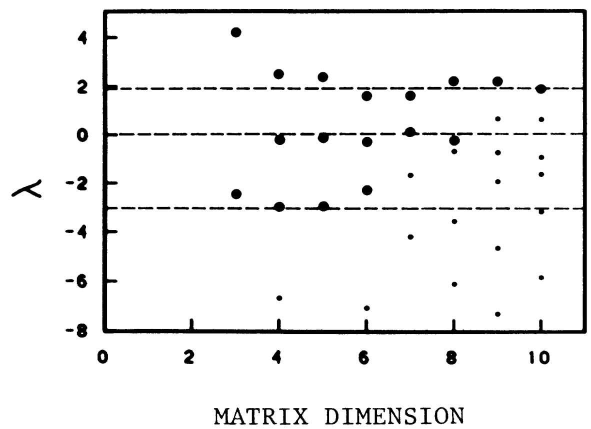

FIG. 8. Liapunov exponents for the acoustic cavitation noise data.

tractor is shown. A 3-tuple with a time shift of $T=4$ μs is used for the reconstruction. Figure 7(b) shows its power spectrum with a large amount of broadband noise and superimposed broad lines at f_0, $f_0/2$, $f_0/4$, and their harmonics. Figure 7(c) is a histogram of 200 pointwise dimensions,[33] calculated from the attractor data. It yields an average value of 2.6 ± 0.2, which is an indication for a strange attractor. The mutual information content of the data is plotted in Fig. 7(d). The first minimum is at $T_m=T_0/4\approx 11t_s$. The "maximal" embedding time would be around $5/4T_0$. The calculation of the Liapunov exponents is done with an evolution time of $\Delta t=1t_s$ and a time delay of $T=2t_s$ (Fig. 8). With increasing matrix dimension the largest exponent converges to an approximate value of $\lambda_1=1.9$ bits/T_0. The second exponent reaches a large plateau at $\lambda_2=0$ bits/T_0, while the third exponent has a small plateau at embedding dimensions $n=3$, 4, 5 at $\lambda_3\approx-3$ bits/T_0 before going off to higher values. Calculating the Liapunov dimension yields a value of $D_L\approx 2.6$, which is equal to the averaged pointwise dimension and confirms the values of the exponents.

CONCLUSIONS

To summarize the numerical method one can say that in principle all Liapunov exponents of a low-dimensional system can be extracted by careful choice of parameters. The upper bound of a maximal embedding time should be kept to avoid multiple foldings and increased curvature of the reconstructed attractor. The overall concept of locality must be obeyed in taking small enough cubes and a small evolution time. Also a small amount of noise is needed (finite precision) to keep spurious exponents below the most negative one. Shortening the evolution time or adding noise have the same effect of lowering spurious exponents. The application of the numerical procedure to experimental data of acoustic cavitation noise leads to encouraging results as fractal dimension estimation and the Liapunov dimension yield the same values. This gives further evidence that acoustic cavitation noise is a deterministically chaotic system.

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