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Evolution of the coupled Bénard-Marangoni convection

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A weakly nonlinear theory that describes the onset of the combined Bénard-Marangoni convection is presented. All relevant transport coefficients are taken to be temperature dependent. When the boundaries are, thermally, nearly insulating, the instability is weak and the perturbed fluid-air interface is found to be proportional to the temperature field $F(\zeta,\tau)$ which evolves according to $F_\tau-\pi_1(F_\zeta)^3_\zeta-\pi_2(F_\zeta)^2_{\zeta\zeta}+\pi_3F_{\zeta\zeta\zeta\zeta}+\pi_4F_{\zeta\zeta}+\pi_5(F_\zeta)^2+\pi_6(F^2)_{\zeta\zeta}+\pi_7(F^2)_\zeta+(\beta_1+\beta_2)F=0$ where β_i are the Biot numbers, and π_i, $i=1,\ldots,7$, are constants.

I. INTRODUCTION

The effects of the buoyancy-driven (Bénard-Rayleigh) convection and the surface-tension-driven (Marangoni) convection are well known and have been extensively described in the literature.[1-7] However, the coupled effect is less understood.

A fluid layer with an upper boundary open to the atmosphere and heated from below exhibits an unstable behavior when a certain critical threshold is crossed. Using the linear approach, Nield[7] showed that this threshold is given by a linear combination of Rayleigh and Marangoni numbers. Therefore both of the mechanisms contribute to the onset of instability. It was shown that the main mechanism of the convection is dependent upon the thickness of the layer.[8]

An asymptotic approach to nonlinear analysis of cellular convection in the case of nearly insulating boundaries was presented in Refs. 9 and 10. It is based on the observation that the characteristic dimensions of the convective cells near the stability limit, are much larger than the thickness of the fluid layer. Thus it is possible to simplify the analysis by separating the spatial variables of the problem and then to reduce its dimension. Using this approach one can develop an evolution equation for the fluid layer temperature and the fluid flow pattern. In Ref. 10 this approach was applied to study the Bénard-Rayleigh convection, including the effects of temperature-dependent thermal conductivity. In Ref. 11 the effect of a deformable free surface was added, however, the fluid properties were taken to be constant. Inertial effects in three-dimensional Bénard-Rayleigh convection were studied in Ref. 12. The same tools were employed in Ref. 9 to study the Marangoni instability.

It is the purpose of this article to unify and generalize previous results. To this end we study the coupled phenomenon of Bénard-Marangoni instability, establish the governing equation for the temperature field and the flow velocities near the onset of instability and elucidate the role of different effects on the convection. Thus previous results should be recovered from ours as particular cases. We consider a fluid layer of thickness d and bounded below by a flat plane while the upper boundary is free and is allowed to deform. This fluid layer is assumed to be heated from below; the kinematic viscosity, the thermal diffusivity, and the surface tension on the interface of the fluid are assumed to be dependent upon its temperature. The boundaries are supposed to be nearly insulating which ensures an onset of a weak instability.

The plan of this paper is as follows; in Sec. II we state the problem. In Sec. III we present an asymptotic analysis that leads to our key result, Eq. (3.24). It describes the spatiotemporal evolution of the temperature and, within a multiplicative constant, see Eq. (3.23b), the evolution of the perturbed interface as well. In Sec. IV we present a brief discussion of our results and future perspectives.

II. FUNDAMENTAL EQUATIONS

We start with the governing equations written in terms of velocity and temperature disturbances u,v and T, respectively. The momentum conservation equations are

$$\frac{\partial u}{\partial t'}+u\frac{\partial u}{\partial X}+v\frac{\partial u}{\partial Y}=-\frac{1}{\rho}\frac{\partial p}{\partial X}+\frac{\partial}{\partial X}\left[\bar{\nu}\frac{\partial u}{\partial X}\right]+\frac{\partial}{\partial Y}\left[\bar{\nu}\frac{\partial u}{\partial Y}\right], \tag{2.1}$$

$$\frac{\partial v}{\partial t'}+u\frac{\partial v}{\partial X}+v\frac{\partial v}{\partial Y}=-\frac{1}{\rho}\frac{\partial p}{\partial Y}+\frac{\partial}{\partial X}\left[\bar{\nu}\frac{\partial v}{\partial X}\right]+\frac{\partial}{\partial Y}\left[\bar{\nu}\frac{\partial v}{\partial Y}\right]+g\alpha T\ . \tag{2.2}$$

The heat equation in the Boussinesq approximation is

$$\frac{\partial T}{\partial t'}+u\frac{\partial T}{\partial X}+v\frac{\partial T}{\partial Y}=\gamma v+\frac{\partial}{\partial X}\left[\bar{\kappa}\frac{\partial T}{\partial X}\right]+\frac{\partial}{\partial Y}\left[\bar{\kappa}\frac{\partial T}{\partial Y}\right], \tag{2.3}$$

where t',X,Y are the time and spatial coordinates, p is the pressure disturbance, α the volume expansion coefficient, g the gravitational acceleration, γ the temperature gradient across the fluid layer, and $\bar{\nu}$ and $\bar{\kappa}$ are the kinematic viscosity and the thermal diffusivity of the fluid, respectively, both temperature dependent. The impact of viscous heating will be considered in Sec. IV.

The appropriate boundary conditions are taken as follows. For $Y=0$,

$$u=0,\quad v=0,\quad \bar{\kappa}T_Y+h_1T=0\ . \tag{2.4}$$

At the deformed free surface $Y=z(X,t')$,

$$z_{t'}+uz_X=v\ , \tag{2.5a}$$

$$p-p_a=2\mu[z_X^2u_X-z_X(u_Y+v_X)+v_Y](1+z_X^2)^{-1}-\sigma z_{XX}(1+z_X^2)^{-3/2}\ , \tag{2.5b}$$

$$\bar{\kappa}(\mathbf{n}\cdot\nabla)T+h_2T=0\ , \tag{2.5c}$$

p_a being the ambient pressure and σ the temperature-dependent surface tension. An additional boundary condition, expressing the balance for the shear stress on the free surface, is developed in the Appendix in an asymptotic form and will be stated shortly [Eq. (2.11)].

Using the dimensionless variables

$$x=X/d,\quad y=Y/d,\quad t=t'\kappa_0/d^2,\quad \theta=T/\gamma d\ ,$$

and introducing dimensionless stream function Ψ, kinematic viscosity $\nu=\bar{\nu}/\nu_0$, and thermal diffusivity $\kappa=\bar{\kappa}/\kappa_0$, we rewrite Eqs. (2.1)–(2.5) in the form

$$\nabla^2\Psi_t+\frac{1}{\mathcal{N}_{\mathrm{Pr}}}\left[\Psi_y\frac{\partial}{\partial x}(\nabla^2\Psi)-\Psi_x\frac{\partial}{\partial y}(\nabla^2\Psi)\right]=(\nu\Psi_{xx})_{xx}+(\nu\Psi_{yy})_{yy}+2(\nu\Psi_{xy})_{xy}-\mathcal{N}_{\mathrm{Ra}}\theta_x\ , \tag{2.6}$$

$$\theta_t+\Psi_y\theta_x-\Psi_x\theta_y+\Psi_x=(\kappa\theta_x)_x+(\kappa\theta_y)_y\ , \tag{2.7}$$

in the domain $-\infty<x<\infty$, $0<y<Q(x,t)$ and

$$\Psi=0,\quad \Psi_y=0,\quad \theta_y=b_1\theta,\quad y=0 \tag{2.8a}$$

$$\Psi=0,\quad \theta_y=-b_2(1+Q_x^2)^{1/2}\theta+Q_x\theta_x\ , \tag{2.8b}$$

$$p-p_a=2\mathcal{N}_{\mathrm{Pr}}[Q_x^2u_x-Q_x(u_y+v_x)+v_y](1+Q_x^2)^{-1}-\sigma Q_{xx}(1+Q_x^2)^{-3/2} \tag{2.9}$$

at $y=Q(x,t)$. Here $Q(x,t)$ is the location of the fluid-air interface, $\kappa=\kappa(\theta)$, $\nu=\nu(\theta)$, b_1 and b_2 are the Biot numbers referring to the heat transfer on the lower and upper boundaries, respectively,

$$\mathcal{N}_{\mathrm{Ra}}=\frac{g\alpha\gamma d^3}{\nu_0\kappa_0}$$

is the Rayleigh number, $\mathcal{N}_{\mathrm{Pr}}$ is the Prandtl number, and κ_0 are ν_0 are reference values of κ and ν, respectively. Assuming the rigid bottom and the top free surface of the fluid layer to be nearly insulated, we introduce the small perturbation parameter ϵ such that

$$b_i=\epsilon^2\beta_i,\quad i=1,2\ . \tag{2.10}$$

In terms of this small quantity, ϵ, one can rewrite the missing boundary condition on the free surface in an asymptotic form. To a first order we have

$$\mathcal{N}_{\mathrm{Ma}}\theta_{xx}+\epsilon\mathcal{N}_{\mathrm{Ma}}(g_{xx}\theta_y+2g_x\theta_{xy})+\frac{\partial}{\partial x}(\Psi_{yy}-\Psi_{xx})+\epsilon[g_x(\Psi_{yyy}-5\Psi_{xxy})-4g_{xx}\Psi_{xy}]=0\ , \tag{2.11}$$

where it was assumed that the free surface is weakly deformed. Explicitly, its equation is

$$y=Q(x,t)=1+\epsilon g(x,t) \tag{2.12}$$

and $\mathcal{N}_{\mathrm{Ma}}$ is the Marangoni number defined via

$$\mathcal{N}_{\mathrm{Ma}}=\frac{-\left[\dfrac{\partial\sigma}{\partial T}\right]\gamma d}{\rho\nu_0\kappa_0}\ . \tag{2.13}$$

It should be pointed out that Eq. (2.11), being restricted to the zero-order approximation, is identical with the boundary condition given by Pearson[6] for an undeformed free surface and was widely used in recent works.[7,9,13]

III. ASYMPTOTIC ANALYSIS

Let both the Marangoni and Rayleigh numbers be close to their critical values corresponding to the onset of the instability

$$\mathcal{N}_{\mathrm{Ma}}=M(1+\epsilon),\quad \mathcal{N}_{\mathrm{Ra}}=R(1+\epsilon)\ , \tag{3.1}$$

wherein M and R are the aforementioned critical values.

We assume a weak temperature dependence of the fluid properties, explicitly,

$$\kappa=1+\epsilon\kappa_1\ ,\qquad \nu=1+\epsilon\nu_1\ , \tag{3.2}$$

and use the following scaling of the time and space variables[9–11]

$$\zeta=x\sqrt{\epsilon},\quad \eta=y,\quad \tau=\epsilon^2t\ . \tag{3.3}$$

Then, from Eq. (2.11) we have

$$\Psi=\sqrt{\epsilon}\theta\ . \tag{3.4}$$

Using new dependent variables

$$\psi(\zeta,\eta,\tau;\epsilon)=\sqrt{\epsilon}\Psi(x,y,t,\epsilon)\ ,\qquad \Theta(\zeta,\eta,\tau;\epsilon)=\theta(x,y,t,\epsilon)\ , \tag{3.5}$$

and substituting Eqs. (3.1)–(3.5) into Eqs. (2.6)–(2.11), one obtains

$$\epsilon^2\Theta_\tau+\epsilon\psi_\eta\Theta_\zeta-\epsilon\psi_\zeta\Theta_\eta+\epsilon\psi_\zeta=\epsilon\frac{\partial}{\partial\zeta}[(1+\epsilon\kappa_1)\Theta_\zeta]+\frac{\partial}{\partial\eta}[(1+\epsilon\kappa_1)\Theta_\eta]\ , \tag{3.6a}$$

$$\epsilon^2\frac{\partial}{\partial\tau}(\nabla^2\psi)+\frac{\epsilon}{\mathcal{N}_{\rm Pr}}[\psi_\eta(\epsilon\psi_{\zeta\zeta}+\psi_{\eta\eta})_\zeta-\psi_\zeta(\epsilon\psi_{\zeta\zeta}+\psi_{\eta\eta})_\eta]$$
$$=\epsilon^2[(1+\epsilon\nu_1)\psi_{\zeta\zeta}]_{\zeta\zeta}+[(1+\epsilon\nu_1)\psi_{\eta\eta}]_{\eta\eta}$$
$$+2\epsilon[(1+\epsilon\nu_1)\psi_{\zeta\eta}]_{\zeta\eta}-R(1+\epsilon)\Theta_\zeta\ , \tag{3.6b}$$

$$\psi=0,\quad \eta=0 \tag{3.7a}$$
$$\psi_\eta=0,\quad \eta=0 \tag{3.7b}$$
$$\Theta_\eta=\epsilon^2\beta_1\Theta,\quad \eta=0 \tag{3.7c}$$

and

$$p-p_a=2\mathcal{N}_{\rm Pr}[\epsilon^4 g_\zeta^2\psi_{\zeta\eta}-\epsilon^2 g_\zeta-\epsilon^2 g_\zeta(\psi_{\eta\eta}-\epsilon\psi_{\zeta\zeta})-\epsilon\psi_{\zeta\eta}](1+\epsilon^2 g_\zeta^2)^{-1}-6\epsilon^2 g_{\zeta\zeta}(1+\epsilon^2 g_\zeta^2)^{-3/2}\ , \tag{3.8a}$$
$$\epsilon M(1+\epsilon)\Theta_{\zeta\zeta}+\epsilon M(1+\epsilon)(\epsilon g_{\zeta\zeta}\Theta_\eta+2\epsilon g_\zeta\Theta_{\zeta\eta})+\epsilon(\psi_{\zeta\eta\eta}-\epsilon\psi_{\zeta\zeta\zeta})+\epsilon^2[g_\zeta(\psi_{\eta\eta\eta}-5\epsilon\psi_{\zeta\zeta\eta})-4\epsilon^2 g_{\zeta\zeta}\psi_{\zeta\eta}]=0\ , \tag{3.8b}$$
$$\psi_\zeta=0\ , \tag{3.8c}$$
$$\Theta_\eta+\epsilon^2\beta_2\Theta-\epsilon^2 g_\zeta\Theta_\zeta=0,\quad \eta=1+\epsilon g(\zeta,\tau)\ . \tag{3.8d}$$

We seek an asymptotic solution to Eqs. (3.6)–(3.8) in the form

$$\psi=\psi_0+\epsilon\psi_1+\cdots\ ,$$
$$\Theta=\Theta_0+\epsilon\Theta_1+\cdots\ , \tag{3.9}$$
$$p'=p_0+\epsilon p_1+\cdots\ ,$$

where p' is the deviation of the pressure from its static value p_s,

$$p'=p-p_s=p-G\mathcal{N}_{\rm Pr}^2(1-y)+\tfrac{1}{2}\mathcal{N}_{\rm Pr}\mathcal{N}_{\rm Ra}(1-y)^2 \tag{3.10}$$

and G is the Galileo number $G=gd^3/\nu^2$.

Substituting Eqs. (3.9) into Eqs. (3.6)–(3.8) yields the problem for the zero-order terms

$$\Theta_{0\eta\eta}=0\ ,$$
$$\psi_{0\eta\eta\eta\eta}=R\,\Theta_{0\zeta}\ , \tag{3.11}$$

with the following boundary conditions:

$$\psi_0=0,\quad \psi_{0\eta}=0,\quad \Theta_{0\eta}=0,\quad \eta=0 \tag{3.12}$$

$$\psi_{0\eta\eta\zeta}+M\Theta_{0\zeta\zeta}=0,\quad \psi_0=0,\quad \Theta_{0\eta}=0,\quad \eta=1+\epsilon g(\zeta,\tau)\ . \tag{3.13}$$

The solution of Eqs. (3.11)–(3.13) is

$$\Theta_0=F(\zeta,\tau)\ ,$$
$$\psi_0=\left[\frac{R}{24}\eta^4-\tfrac{1}{4}(M+\tfrac{5}{12}R)\eta^3+\tfrac{1}{4}(M+\tfrac{1}{4}R)\eta^2\right]F_\zeta\ , \tag{3.14}$$

where $F(\zeta,\tau)$ is, so far, an undetermined function. Note that the condition $\psi_0=0$ at $\eta=1+\epsilon g(\zeta,\tau)$ is equivalent to the conservation of mass within the fluid layer. Integrating Eq. (3.6a) over the $0\le\eta\le1$ domain, as in Refs. 9 and 11, one obtains a solvability condition which will be used shortly to derive an equation which governs the evolution of the temperature field $F(\zeta,\tau)$.

The zeroth-order terms of this solvability equation yields the condition that

$$\frac{M}{48}+\frac{R}{320}=1\ . \tag{3.15}$$

Note that for pure Bénard-Rayleigh convection Eq. (3.15) leads to $R=320$ (Ref. 10) and to $M=48$ in the pure Marangoni convection case.[9] To determine $F(\zeta,\tau)$ we use the first-order approximation, written as follows:

$$\psi_{0\eta}\Theta_{0\zeta}-\psi_{0\zeta}\Theta_{0\eta}+\psi_{0\zeta}=\Theta_{0\zeta\zeta}+(\kappa_1\Theta_{0\eta})_\eta+\Theta_{1\eta\eta}\ , \tag{3.16}$$

$$\frac{1}{\mathcal{N}_{\rm Pr}}(\psi_{0\eta}\psi_{0\zeta\eta\eta}-\psi_{0\zeta}\psi_{0\eta\eta\eta})$$
$$=(\nu_1\psi_{0\eta\eta})_{\eta\eta}+\psi_{1\eta\eta\eta\eta}+2\psi_{0\zeta\zeta\eta\eta}-R(\Theta_{0\zeta}+\Theta_{1\zeta})\ , \tag{3.17}$$

with

$$\psi_1=0\ , \tag{3.18a}$$
$$\psi_{1\eta}=0\ , \tag{3.18b}$$
$$\Theta_{1\eta}=0\ , \tag{3.18c}$$

at $\eta=0$, and

$$\psi_0+\epsilon\psi_1=0\ , \tag{3.19a}$$
$$\psi_{1\zeta\eta\eta}+M\Theta_{0\zeta\zeta}+M\Theta_{1\zeta\zeta}+g_\zeta\psi_{0\eta\eta\eta}-\psi_{0\zeta\zeta\zeta}=0\ , \tag{3.19b}$$
$$\Theta_{1\eta}=0\ , \tag{3.19c}$$

at $\eta=1+\epsilon g(\zeta,\tau)$. Again, condition (3.19a) implies conservation of mass and therefore a net zero flux through arbitrary contour connecting any two points on the upper and the lower boundaries.

Before proceeding further we shall explicitly assume that

$$\nu_1=\mu(F+A-\eta),\quad \mu, A \text{ constants}$$

and

$$\kappa_1=K_0(F+B-\eta),\quad K_0, B \text{ constants}\ .$$

Substituting Eqs. (3.14) into Eqs. (3.16)–(3.19) one obtains

$$\Theta_1=F_\zeta^2\left[\frac{R\eta^5}{120}-5L\eta^4+4m\eta^3\right]$$
$$+F_{\zeta\zeta}\left[\frac{R\eta^6}{720}-L\eta^5+m\eta^4-\frac{\eta^2}{2}\right]+H(\zeta,\tau)\ , \tag{3.20}$$

$$\psi_{1\zeta}=f_1(\eta)F_{\zeta\zeta\zeta\zeta}+f_2(\eta)(F_\zeta F_{\zeta\zeta})_\zeta+f_3(\eta)F_{\zeta\zeta}$$
$$+[\mu RF_\zeta(F+A)-RF_\zeta-RH_\zeta]_\zeta f_4(\eta)$$
$$+\frac{\omega_5}{2}(gF_\zeta)_\zeta(\eta^3-3\eta^2)+\frac{\omega_7}{2}g_\zeta F_\zeta(\eta^2-\eta^3)$$
$$-\frac{M}{4}H_{\zeta\zeta}(\eta^3-\eta^2)\ , \tag{3.21}$$

where $H(\zeta,\tau)$ is an arbitrary function and

$$L=\tfrac{1}{80}(M+\tfrac{5}{12}R),\quad m=\tfrac{1}{48}(M+\tfrac{1}{4}R)\ ,$$

$$f_1(\eta)=\frac{R^2\eta^{10}}{3\,628\,800}-\frac{RL\eta^9}{3024}+\frac{Rm\eta^8}{1680}-\frac{R\eta^6}{240}+2L\eta^5$$
$$-2m\eta^4+\tfrac{1}{2}(\omega_3-\omega_1)\eta^3-\tfrac{1}{2}(3\omega_3-\omega_1)\eta^2\ ,$$

$$f_2(\eta)=\frac{R^2\eta^9}{181\,440}-\frac{RL\eta^8}{168}+\frac{Rm\eta^7}{105}$$
$$+\frac{1}{\mathcal{N}_{\mathrm{Pr}}}\left[\frac{R^2\eta^9}{72\,576}-\frac{5RL\eta^8}{336}+\frac{Rm+1200L^2}{210}\eta^7\right.$$
$$\left.-8Lm\eta^6+\frac{24m^2}{5}\eta^5\right]$$
$$+\tfrac{1}{2}(\omega_4-\omega_2)\eta^3-\tfrac{1}{2}(3\omega_4-\omega_2)\eta^2\ ,$$

$$f_3(\eta)=\mu\left[\frac{R\eta^5}{40}-10L\eta^4+8m\eta^3\right]$$
$$+\frac{1}{2}\left[\mu\omega_6-\mu\omega_8-\frac{M}{2}\right]\eta^3$$
$$+\frac{1}{2}\left[\mu\omega_8-3\mu\omega_6+\frac{M}{2}\right]\eta^2\ ,$$

$$f_4(\eta)=-\tfrac{1}{24}\eta^4+\tfrac{5}{48}\eta^3-\tfrac{1}{16}\eta^2\ , \tag{3.22}$$

$$\omega_1=\frac{R^2}{80\,640}-\frac{RL}{84}+\frac{Rm}{60}-\frac{R}{16}+20L-12m$$
$$+\frac{M}{2}\left[\frac{R}{720}-L+m-\tfrac{1}{2}\right],$$

$$\omega_2=\frac{R^2}{5040}-\frac{RL}{6}+\frac{Rm}{5}$$
$$+\frac{1}{\mathcal{N}_{\mathrm{Pr}}}\left[\frac{R^2}{2016}-\frac{5RL}{12}+\frac{Rm+1200L^2}{10}\right.$$
$$\left.-120Lm+48m^2\right]$$
$$+M\left[\frac{R}{120}-5L+4m\right],$$

$$\omega_3=\frac{R^2}{3\,628\,800}-\frac{RL}{3024}+\frac{Rm}{1680}-\frac{R}{240}+2L-2m\ ,$$

$$\omega_4=\frac{R^2}{181\,440}-\frac{RL}{168}+\frac{Rm}{105}$$
$$+\frac{1}{\mathcal{N}_{\mathrm{Pr}}}\left[\frac{R^2}{72\,576}-\frac{5RL}{336}+\frac{Rm+1200L^2}{210}-8Lm\right.$$
$$\left.+\frac{24m^2}{5}\right],$$

$$\omega_5=\frac{R}{6}-60L+24m\ ,$$

$$\omega_6=\frac{R}{40}-10L+8m\ ,$$

$$\omega_7=\frac{R}{2}-60L\ ,$$

$$\omega_8=\frac{R}{4}-60L+24m\ .$$

We recall that the deformed free surface is still undetermined. In order to find its form, the zero-order approximation of the pressure disturbance p has to be evaluated and the boundary condition for the normal stress should be applied. Noting the fact that the Galileo number G is usually very large, we assume it to be of the order of ϵ^{-1} (cf. Ref. 11), i.e.,

$$G=G'\epsilon^{-1},\quad G'=O(1)\ .$$

Then we obtain

$$p_0=\mathcal{N}_{\mathrm{Pr}}(R\eta-120L)F(\zeta,\tau)\ , \tag{3.23a}$$

$$g(\zeta,\tau)=\frac{2\omega_7}{\mathcal{N}_{\mathrm{Pr}}G'}F(\zeta,\tau)\ . \tag{3.23b}$$

Turning back to the solvability condition and keeping only terms to the order of ϵ^2 we obtain an evolution equation for the temperature-disturbance amplitude,

$$F_\tau-\pi_1(F_\zeta)^3_\zeta-\pi_2(F_\zeta)^2_{\zeta\zeta}+\pi_3F_{\zeta\zeta\zeta\zeta}+\pi_4F_{\zeta\zeta}$$
$$+\pi_5(F_\zeta)^2+\pi_6(F^2)_{\zeta\zeta}+\pi_7(F^2)_\zeta+\beta F=0\ , \tag{3.24}$$

where $\beta\equiv\beta_1+\beta_2$,

$$\pi_1=\frac{R^2}{5184}+\frac{400L^2}{7}+\frac{144m^2}{5}-\frac{5}{24}RL+\frac{Rm}{7}-80Lm\ ,$$

$$\pi_2=\frac{-31R^2}{1\,814\,400}+\frac{RL}{504}-\frac{9mR}{560}-\frac{3\omega_4}{16}$$
$$+\frac{\omega_2}{48}-\frac{R}{480}-\frac{25L^2}{4}-4m^2+10Lm-L+\frac{m}{2}$$
$$+\frac{1}{\mathcal{N}_{\mathrm{Pr}}}\left[\frac{R^2}{1\,451\,520}-\frac{5RL}{6048}+\frac{Rm+1200L^2}{3360}\right.$$
$$\left.-\frac{4Lm}{7}+\frac{2m^2}{5}\right],$$

$$\pi_3=R\left[\frac{-1}{1260}-\frac{L}{30\,240}+\frac{m}{15\,120}+\frac{R}{39\,916\,800}\right]$$
$$+\frac{L}{2}-\frac{3m}{5}+\frac{1}{6}+\frac{\omega_1}{24}-\frac{3\omega_3}{8}\ ,$$

$$\pi_4 = 1 + \mu\left[\frac{5-12A}{3840} - \frac{3M}{320}\right] - K_0(B - \tfrac{1}{2}) ,$$

$$\pi_5 = \frac{\omega_7}{12\mathcal{N}_{\mathrm{Pr}}G'}(\omega_7 - 24), \quad \pi_7 = \frac{\omega_7}{4\mathcal{N}_{\mathrm{Pr}}G'}\left[M + \frac{R}{12}\right] ,$$

$$\pi_6 = \frac{2\omega_7}{\mathcal{N}_{\mathrm{Pr}}G'}\left[-\frac{1}{2} - \frac{3}{16}\left[\frac{R}{6} - 60L + 24m\right]\right] - \frac{\mu R}{640} - \frac{K_0}{2} . \tag{3.25}$$

Once the function $F(\zeta,\tau)$ is determined via Eq. (3.24), the temperature field of the fluid layer is given as

$$\overline{T}(x,y,t) = T_0 + (T_1 - T_0)[F(x\sqrt{\epsilon}, \epsilon^2 t) - y] , \tag{3.26}$$

and the flow field is determined from

$$u = \Psi_y = \sqrt{\epsilon}\psi_\eta, \quad v = -\Psi_x = -\epsilon\psi_\zeta . \tag{3.27}$$

Equation (3.24) is our main result. Equation (3.24) can be reduced to the corresponding equation in Ref. 9 when the following effects are neglected: the buoyancy-driven convection, the deformation of the free surface, the temperature-dependence of the fluid transport, and when only the reduced version of the boundary condition for the shear stress is used.

A comparison with the second limiting case, discussed in Ref. 11, shows that apart the new terms $\pi_5 F_\zeta^2$ and $\pi_7(F^2)_\zeta$ in Eq. (3.24), there is also a difference in the numerical value of several coefficients. The reason for this discrepancy is not completely clear and probably has to do with some misprints and computational errors.

IV. DISCUSSION

Using a weakly nonlinear analysis, we have derived an equation describing the nonlinear evolution of a weak instability. Since previous works covered some of the relevant effects but left out others, we expected, and posteriori verified it to be true, that inclusion of all the aformentioned effects will lead to a generalized evolution equation which incorporates previous results as particular cases. Indeed, transport coefficients π_i $(i=1,\ldots,7)$ of Eq. (3.24) reflect in a lumped manner the various effects. Whether they are stabilizing or destabilizing depends on numerical values of often competing factors. The complexity of the problem makes it clear that a complete catalog of the problem as predicted by Eq. (3.24) is possible, at least at this stage, only via numerically aided study of Eq. (3.24). This is now underway. Nevertheless, we can derive some conclusions from Eq. (3.24). First we rewrite it as

$$F_\tau + \{[-\pi_1(F_\zeta)^2 + \pi_4 + \pi_6 F]F_\zeta\}_\zeta + \pi_3 F_{\zeta\zeta\zeta\zeta} + \beta F + [\pi_7(F^2)_\zeta + \pi_5(F_\zeta)^2 - \pi_2(F_\zeta^2)_{\zeta\zeta}] = 0 . \tag{4.1}$$

A direct calculation shows that

$$\pi_1 > 0, \quad \pi_3 > 0 ,$$

for the whole range of critical values of Marangoni and Rayleigh numbers. Since μ is, in the case of liquids, negative $\pi_4 > 0$ as well. Though $K_0 > 0$ the temperature dependence of thermal diffusivity is not expected to be large to the extent as to overcome the other two factors entering into the definition of π_4 [see Eq. (3.25)]. With the sign of π_1, π_3, and π_4 at hand we can make the following observations; the main source of instability is due to the backward part of the diffusion coefficient: $\pi_4 + \pi_6 F$. This part is always destabilizing independent of the sign of π_6. Indeed, since F describes a perturbed temperature it may be both positive or negative. Thus $\pi_6 F > 0$ ensures backward diffusion. Note, that for $\pi_6 F < 0$ is destabilizing only if $|F| > \pi_1 / |\pi_6|$. Stabilizing competition is provided by the third term (remember, $\pi_3 > 0$) and the pseudo-diffusive-component $-\pi_1 F_\zeta^2$. In fact if π_6 is absent the instability fed by π_4 (>0) will be ultimately counterbalanced by $-\pi_1 F_\zeta^2$ to generate a bounded amplitude pattern. However, with $\pi_6 F$ intact, it is *a priori* unclear whether the gradients of F will grow fast enough to counterbalance the backward diffusion enhanced by $\pi_6 F$. It is here that the last two terms in Eq. (4.1), which mainly affect the behavior of gradients of F, are expected to play an important role; they provide the necessary mechanism to generate high wave numbers and thus to increase the role of the pseudodiffusive part; $\pi_1(F_\zeta)^2$. But whether this term is always sufficient to stop the growth of the instability remains to be seen.

In this context it is of interest to note that for $\pi_1 = \pi_2 = 0$ Eq. (3.24) was derived in Refs. 14 and 15 to describe the evolution of solidification front of dilute binary alloy. However, in Ref. 16 it was shown that such an equation describes either perturbations that quench or explode in finite time. No bounded pattern is possible. The remedy in the solidification context was to introduce a new asymptotic expansion which resulted with $F_{\zeta\zeta\zeta\zeta}$ being replaced by $(F_{\zeta\zeta}/C)_{\zeta\zeta}$, where $C = (1 + F_\zeta^2)^{3/2}$ describes the stabilizing effect of the curvature.

Now a methodological remark is in order. While the Newell-Segal-Whitehead method[17,18] expands in a wavelength band around a finite wavelength k, the present one depends very much on the long-wavelength instability or, more precisely, it expands around $k=0$. This distinction has consequences extending beyond the technical differences of the two methods, namely, their structural stability. While the expansion around finite k appears to be robust to small changes in the physical setting of the problem,[19] our experience with the presented problem reveals that it is very sensitive to structural changes. It necessitates a quite tight scaling condition of dimensionless quantities; a change of scale modifies the resulting amplitude equation. Admittingly, it is a troublesome feature of the method. Whether it is an inherent penalty that one has to pay for the benefit of the method's simplicity, or only a removable, technical difficulty, is still to be seen.

A case in point is the scaling of G. In our work G was assumed to be large. This was done to assure consistence with the scaling of the other quantities. In a recent work[20] a similar problem was addressed. There, however, G was assumed to be small or, at most, of order 1.

This condition enforced a different scaling of the stream function and the temperature. Their work resulted with a different amplitude equation. There is no direct transition from our amplitude equation to theirs or vice versa. We of course can, in principle, compare the resulting motion, but the tightness of the scaling involved in the asymptotic expansion, precludes the possibility of embedding one equation within the other.

Finally, we consider the effect of viscous heating on the evolution of the interface. Its sole effect is to modify the numerical value of parameter π_5 as follows:

$$\pi_5 \rightarrow \pi_5 - \omega_9 \left[\frac{R^2}{20} + 4800L^2 + 576m^2 - 15RL + 4Rm - 1440Lm \right] ,$$

where

$$\omega_9 = \frac{\nu_0 \kappa_0}{c_p \gamma d^3} .$$

Consistent with the scaling used in this work it is assumed that $\omega_9 = O(\epsilon)$.

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APPENDIX: DERIVATION OF THE BOUNDARY CONDITION FOR THE TANGENTIAL COMPONENT OF THE STRESS ON A FREE SURFACE

We start with Eq. (29) of Ref. 21 which, in the case of an interface of negligible mass and surface dilatational and shear viscosities, is written as

$$-\mathbf{F} = \nabla_s \sigma - \sigma(\nabla_s \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \tag{A1}$$

where $\mathbf{F}$ is the traction on the interface, σ is the surface tension, and ∇_s is a surface gradient operator given by

$$\nabla_s = \hat{\mathbf{e}}^{\alpha} \frac{\partial}{\partial u^{\alpha}} ,$$

$\hat{\mathbf{e}}^{\alpha}$ being the reciprocal base vectors of the surface coordinates and $\hat{\mathbf{n}}$ is the normal unit vector. On the other hand, the traction $\mathbf{F}$ on the interface can be written as

$$-\mathbf{F} = -p\hat{\mathbf{n}} + \mu \hat{\mathbf{n}} \cdot [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] , \tag{A2}$$

where p is the pressure and μ is the fluid viscosity.

Equating the surface divergences of Eqs. (A1) and (A2) we obtain

$$\nabla_s \cdot (\nabla_s \sigma) - \nabla_s \cdot (\sigma \hat{\mathbf{n}} \nabla_s \cdot \hat{\mathbf{n}}) = -\nabla_s \cdot (p\hat{\mathbf{n}}) + \mu \nabla_s \cdot \{\hat{\mathbf{n}} \cdot [\nabla \mathbf{v} + (\nabla \mathbf{v})^T]\} . \tag{A3}$$

We apply this equation for a perturbed free surface given by the equation

$$y = 1 + \epsilon g(x,z,t) + o(\epsilon) . \tag{A4}$$

To first order the surface gradient and surface divergence terms were shown to be

$$\nabla_s a = \left[\frac{\partial a}{\partial x} + \epsilon g_x \frac{\partial a}{\partial y} \right] \hat{\mathbf{i}} + \left[\epsilon g_x \frac{\partial a}{\partial z} + \epsilon g_z \frac{\partial a}{\partial x} \right] \hat{\mathbf{j}} + \left[\frac{\partial a}{\partial z} + \epsilon g_z \frac{\partial a}{\partial y} \right] \hat{\mathbf{k}} + o(\epsilon) \tag{A5}$$

and

$$\nabla_s \mathbf{b} = \frac{\partial \mathbf{b}^1}{\partial x} + \frac{\partial \mathbf{b}^3}{\partial z} + \epsilon \left[g_x \frac{\partial \mathbf{b}^1}{\partial y} + g_z \frac{\partial \mathbf{b}^3}{\partial y} \right] + o(\epsilon) , \tag{A6}$$

respectively. Here a is an arbitrary function and $\mathbf{b}$ is an arbitrary vector whose components in the fixed vector base $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are $\mathbf{b}^1, \mathbf{b}^2, \mathbf{b}^3$. Using Eqs. (A5) and (A6) for the calculation for the terms in Eq. (A3), and exploiting the two dimensionality ($\omega = 0$, $\partial_z \equiv 0$) leads to the following equation:

$$\mathcal{N}_{\mathrm{Ma}} \theta_{xx} + \epsilon \mathcal{N}_{\mathrm{Ma}} (g_{xx} \theta_y + 2g_x \theta_{xy}) + \frac{\partial}{\partial x}(u_y + v_x) + \epsilon \left[g_x \frac{\partial}{\partial y}(u_y + v_x) - 4 \frac{\partial}{\partial x}(g_x u_x) \right] = 0 , \tag{A7}$$

where we have used the linear dependence of the surface tension upon the perturbed temperature.[6] Equation (A7) can be rewritten in terms of the stream function Ψ,

$$\mathcal{N}_{\mathrm{Ma}} \theta_{xx} + \epsilon \mathcal{N}_{\mathrm{Ma}} (g_{xx} \theta_y + 2g_x \theta_{xy}) + \frac{\partial}{\partial x}(\Psi_{yy} - \Psi_{xx}) + \epsilon \left[g_x (\Psi_{yyy} - \Psi_{xxy}) - 4 \frac{\partial}{\partial x}(g_x \Psi_{xy}) \right] = 0 . \tag{A8}$$

This equation in the zeroth-order approximation is identical to the well-known boundary condition deduced in Ref. 6 and widely used in other works. The first-order terms in ϵ account for the deformable free surface.

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