Time evolution in stimulated Compton scattering

Salvatore Carusotto
Dipartimento di Fisica, Università di Pisa, I-56100 Pisa, Italy
(Received 21 July 1988)

The time evolution of the electron and field operators in stimulated Compton scattering is considered. The operators are described by equations of motion in which the coordinate and momentum operators of the electron appear separated. These equations are solved by applying a new integration method based on iteration techniques, so the rigorous solutions are obtained. The resultant operators are presented as a Laplace transform and a subsequent inverse Laplace transform of suitable functionals. Moreover, the functionals that describe the creation and annihilation operators are written in ordered form. Finally, applications of these results to particular cases are given.

I. INTRODUCTION

There has been considerable interest recently in the study of the stimulated Compton scattering (SCS), even though this process was extensively analyzed long ago.[1,2] This interest has been motivated by studying the properties of free-electron lasers since SCS is the fundamental process in a free-electron laser working in the Compton regime.[3–8]

It has been shown that, in a moving frame where the frequency of the radiation propagating along one direction is identical to that propagating in the opposite direction, the Hamiltonian for SCS can be written as

$$\hat{H}=\frac{1}{2m}\hat{p}^{\,2}+\hbar\omega(\hat{b}^{\dagger}_{F}\hat{b}_{F}+\hat{b}^{\dagger}_{B}\hat{b}_{B}) +\hbar\Delta[\hat{b}^{\dagger}_{F}\hat{b}_{B}\exp(-2ik\hat{z})+\hat{b}_{F}\hat{b}^{\dagger}_{B}\exp(2ik\hat{z})]\ , \tag{1}$$

where $\hat{z}$ and $\hat{p}$ are the coordinate and momentum operators of the electron, $\hat{b}_F$ and $\hat{b}_B$ ($\hat{b}_F^{\dagger}$ and $\hat{b}_B^{\dagger}$) are the annihilation (creation) operators of the forward and backward propagating fields, $\hbar\omega$ and $\hbar k=\hbar\omega/c$ are the energy and momentum of the photons, and Δ is the coupling constant. This Hamiltonian implies the conservation of total photon number N and and total linear momentum. Consequently, systems obeying the Hamiltonian (1) have been studied by using quantum states $|p,N,n\rangle$ expressed as a product of a plane wave of momentum p for the electron, a Fock state of n photons for the forward propagating radiation, and a Fock state of $N-n$ photons for the backward propagating radiation. In fact, when a system is initially in the state $|p_0,N_0,n_0\rangle$, the time evolution of this state can always be described as a linear combination of the N_0+1 basic state $|p_0,N_0,n\rangle$ with the same total photon number N_0,

$$|p_0,N_0,n_0(t)\rangle=\sum_{n=0}^{N_0}C_n(t)|p_0,N_0,n\rangle\ .$$

Within this framework, from the equation of Schrödinger we can write a recursive differential equation for the probability amplitudes $C_n(t)$, that, at time t, n photons are propagating forward and $N-n$ photons are propagating backward. This equation is given by[9–13]

$$i\frac{d}{dt}C_n(t)=(-2n\delta+n^2\mu)C_n(t) +\Delta[(N-n)(n+1)]^{1/2}C_{n+1}(t) +\Delta[(N-n+1)n]^{1/2}C_{n-1}(t)\ , \tag{2}$$

where

$$\delta=\frac{k}{m}[p_0+2\hbar kn(t=0)],\quad \mu=\frac{2\hbar k^2}{m}\ .$$

Bosco and Dattoli have recognized Eq. (2) as one of the various types of generalized Raman-Nath equations and these authors have called it the spherical Raman-Nath equation.[10,14]

Raman-Nath equations have been used to describe a large number of physical problems, but in only a few limited cases analytical solutions of these equations have actually been found because of the presence of the nonlinear term $n^2\mu$.[8,15] Perturbative solutions in the perturbation parameter μ have been recently obtained. The trouble with these solutions is that the analytical expression is so lengthy that it is very difficult to use it to calculate any observable physical quantity.[10,13]

In this paper we will describe a new method to study the time evolution of a system during the SCS process. In order to investigate the time development of the field and electron operators we apply the mathematical techniques that have recently been used for studying the evolution of a quartic anharmonic oscillator.[16]

First, we write the equation of motion in a form for which the coordinate and momentum operators of the electron appear separated. Then, we look for a solution of these equations by applying iteration methods. But solutions expressed as a power series of time can be written only if a recursive operational relation among the terms of the power series is found and, at the same time, the expansion factor $(n!)^{-1}$ for the generic nth term of this series is taken into account. With the help of some integral operators we are able to overcome these difficulties and to obtain formal solutions of the motion equations. Finally, we condense the resultant power series in integrals of analytical functionals. So the conclusive expressions appear in the shape of a Laplace transform and of a subsequent inverse Laplace transform of suitable operator functionals. Moreover, the operator functionals that describe the creation and annihilation operators of the two fields are presented in normal order by means of associate boson functions.

The results obtained in this paper allow one to describe the properties of the forward- and backward-propagating fields and of the electron in SCS without applying usual calculations techniques.

Section II is devoted to writing the equations that describe the evolution of the field operators in SCS. The exact solutions of these equations for the photons of the forward- and backward-propagating fields are presented in Sec. III. In Sec. IV a concise analysis of the time evolution for the coordinate and momentum operators of the electron is given. Finally, in Sec. V these results are employed to study the time evolution of the photon operators in particular simple cases.

II. EQUATION OF MOTION

In this section we derive the equations that describe the time evolution of the photon operators in the SCS process. We will write these equations in a form upon which we can apply the mathematical method previously used for studying an anharmonic oscillator.

Let us first introduce some theorems of operational calculus. If $\hat{A}$ and $\hat{B}$ are two noncommuting operators, ξ a parameter, and f a function that can be expanded in a power series of the argument, we then have[17]

$$\exp(\xi\hat{A})f(\hat{B})\exp(-\xi\hat{A})=f(\exp(\xi\hat{A})\hat{B}\exp(-\xi\hat{A})) \tag{3}$$

and

$$\exp(\xi\hat{A})\hat{B}\exp(-\xi\hat{A})=\hat{B}+\xi[\hat{A},\hat{B}]+\frac{\xi^2}{2}[\hat{A},[\hat{A},\hat{B}]]+\cdots . \tag{4}$$

From these theorems it follows that

$$\exp(\xi\hat{a}^{\dagger}\hat{a})f(\hat{a},\hat{a}^{\dagger})\exp(-\xi\hat{a}^{\dagger}\hat{a})=f(\hat{a}e^{-\xi},\hat{a}^{\dagger}e^{+\xi}) , \tag{5}$$

$$\exp(\xi\hat{p})\hat{z}\exp(-\xi\hat{p})=\hat{z}-i\hbar\xi , \tag{6}$$

and

$$\exp(\xi\hat{z})\hat{p}\exp(-\xi\hat{z})=\hat{p}+i\hbar\xi , \tag{7}$$

where $\hat{a}$ and $\hat{a}^{\dagger}$ are annihilation and creation boson operators and $\hat{z}$ and $\hat{p}$ are canonical coordinate and momentum operators.

Now, we proceed to study the equation of motion. We begin by analyzing the time evolution operator for SCS

$$\hat{U}(t)=\exp\left[-\frac{it}{\hbar}\hat{H}\right] ,$$

where $\hat{H}$ is the Hamiltonian defined by Eq. (1). When the unitary transformation operator

$$\exp(\xi\hat{L})=\exp(-2ik\hat{z}\hat{b}_F^{\dagger}\hat{b}_F)$$

is applied to the evolution operator, by using Eq. (3) we can write

$$\hat{U}(t)=\exp(\xi\hat{L})\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}\right]\exp(-\xi\hat{L}) , \tag{8}$$

where

$$\hat{\mathcal{H}}=\exp(-\xi\hat{L})\hat{H}\exp(\xi\hat{L}) . \tag{9}$$

Consequently, from Eqs. (5) and (7) we obtain

$$\hat{\mathcal{H}}=\frac{1}{2m}(\hat{p}-2\hbar k\hat{b}_F^{\dagger}\hat{b}_F)^2+\hbar\omega(\hat{b}_F^{\dagger}\hat{b}_F+\hat{b}_B^{\dagger}\hat{b}_B)+\hbar\Delta(\hat{b}_F^{\dagger}\hat{b}_B+\hat{b}_F\hat{b}_B^{\dagger}) .$$

We point out that the coordinate operator $\hat{z}$ is not present in $\hat{\mathcal{H}}$. Then, we rewrite this Hamiltonian as

$$\hat{\mathcal{H}}=\hat{\mathcal{H}}_0+\hat{\mathcal{H}}_1+\hat{\mathcal{H}}_2 , \tag{10}$$

where

$$\hat{\mathcal{H}}_0=(2m)^{-1}\hat{p}^2 , \tag{11a}$$

$$\hat{\mathcal{H}}_1=\hbar\omega(\hat{b}_F^{\dagger}\hat{b}_F+\hat{b}_B^{\dagger}\hat{b}_B) , \tag{11b}$$

and

$$\hat{\mathcal{H}}_2=\hbar[\hat{\Omega}\hat{b}_F^{\dagger}\hat{b}_F+\mu(\hat{b}_F^{\dagger}\hat{b}_F)^2+\Delta(\hat{b}_F^{\dagger}\hat{b}_B+\hat{b}_F\hat{b}_B^{\dagger})] , \tag{11c}$$

with

$$\hat{\Omega}=-2km^{-1}\hat{p} \tag{12}$$

and

$$\mu=2\hbar m^{-1}k^2 . \tag{13}$$

Since

$$[\hat{\mathcal{H}}_0,\hat{\mathcal{H}}_2]=0,\quad [\hat{\mathcal{H}}_1,\hat{\mathcal{H}}_2]=0 ,$$

we have

$$\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}\right]=\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}_0\right]\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}_2\right]\times\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}_1\right] . \tag{14}$$

Next we recall that the time evolution of an operator $\hat{O}$ is given by the relation

$$\hat{O}(t)=\hat{U}^{\dagger}(t)\hat{O}\hat{U}(t) . \tag{15}$$

When this equation is applied to the annihilation operator $\hat{b}_B$ that describes the backward propagating field in SCS, by making use of Eqs. (8) and (14) we obtain

$$\hat{b}_B(t)=\hat{\mathcal{U}}^{\dagger}(\hat{z},t)\hat{b}_B\hat{\mathcal{U}}(\hat{z},t) , \tag{16a}$$

where

$$\hat{\mathcal{U}}(\hat{z},t)=\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}_2\right]\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}_1\right]\times\exp(2ik\hat{z}\hat{b}_F^{\dagger}\hat{b}_F)\ . \tag{16b}$$

An identical expression describes the time evolution of the creation operator $\hat{b}_B^{\dagger}$. In order to determine the explicit form of $\hat{b}_B(t)$ it is useful to study the operator

$$\hat{b}_B^{(R)}(t)=\exp\left[\frac{it}{\hbar}\hat{\mathcal{H}}_2\right]\hat{b}_B\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}_2\right]\ . \tag{17}$$

From the definition we see that the operator $\hat{b}_B^{(R)}(t)$ would describe the time evolution of the operator $\hat{b}_B$ if the Hamiltonian of the system was $\hat{\mathcal{H}}_2$. Therefore the operator $\hat{b}_B^{(R)}(t)$ obeys the following equation of motion:

$$\frac{d}{dt}\hat{b}_B^{(R)}(t)=-\frac{i}{\hbar}[\hat{b}_B^{(R)}(t),\hat{\mathcal{H}}_2]\ . \tag{18}$$

By analogy we introduce the operator

$$\hat{b}_F^{(R)}(t)=\exp\left[\frac{it}{\hbar}\hat{\mathcal{H}}_2\right]\hat{b}_F\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}_2\right]\ , \tag{19}$$

which will be useful to evaluate $\hat{b}_B^{(R)}$. The operator $\hat{b}_F^{(R)}(t)$ obeys the equation of motion

$$\frac{d}{dt}\hat{b}_F^{(R)}(t)=-\frac{i}{\hbar}[\hat{b}_F^{(R)}(t),\hat{\mathcal{H}}_2]\ . \tag{20}$$

If Eq. (11c) is introduced into Eqs. (18) and (20), we find that the operators $\hat{b}_B^{(R)}(t)$ and $\hat{b}_F^{(R)}(t)$ must satisfy the following equations:

$$\frac{d}{dt}\hat{b}_B^{(R)}(t)=-i\Delta\hat{b}_F^{(R)}(t) \tag{21}$$

and

$$\begin{aligned}\frac{d}{dt}\hat{b}_F^{(R)}(t)=&-i(\hat{\Omega}-\mu)\hat{b}_F^{(R)}(t)\\&-2i\mu\hat{b}_F^{(R)}(t)[\hat{b}_F^{(R)}(t)]^{\dagger}\hat{b}_F^{(R)}(t)\\&-i\Delta\hat{b}_B^{(R)}(t)\ ,\end{aligned} \tag{22}$$

with the initial conditions

$$\hat{b}_B^{(R)}(t=0)=\hat{b}_B,\quad \hat{b}_F^{(R)}(t=0)=\hat{b}_F\ .$$

Here and in the following the boson operators $\hat{b}_B$ and $\hat{b}_F$ are assumed to be expressed in the Schrödinger representation when they are written without any indication of functional dependence. The equations of motion for $[\hat{b}_B^{(R)}(t)]^{\dagger}$ and $[\hat{b}_F^{(R)}(t)]^{\dagger}$ are the complex conjugate of Eqs. (21) and (22), respectively.

When Eq. (21) is solved, we can obtain the explicit expressions of the photon operators $\hat{b}_B(t)$ and $[\hat{b}_B(t)]^{\dagger}$ by introducing the solution into Eq. (16). For the particular symmetry of Hamiltonian $\hat{H}$ the explicit expressions of the photon operators $\hat{b}_F(t)$ and $[\hat{b}_F(t)]^{\dagger}$ can be directly obtained from the expressions of $\hat{b}_B(t)$ and $[\hat{b}_B(t)]^{\dagger}$. To this end we must exchange the subscripts B and F and replace the propagation vector k by $-k$ in the expressions of $\hat{b}_B(t)$ and $[\hat{b}_B(t)]^{\dagger}$. We point out that an equation similar to Eq. (2) can be written when the time evolution operator $\exp[-(it/\hbar)\hat{\mathcal{H}}_2]$ is applied to an initial state $|p_0,N_0,n_0\rangle$.

III. PHOTON OPERATORS

We will premise some mathematical considerations in order to make easier the following study of the SCS process. First, we introduce the integral operator

$$\hat{I}(t;t_1)=\int_0^t dt_1\ , \tag{23}$$

for which we set

$$\hat{I}^{\,n}(t;t_n)=\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\ . \tag{24}$$

Obviously, we have

$$\hat{I}^{\,n}(t;t_n)(t_n)^k=k![(k+n)!]^{-1}t^{k+n} \tag{25a}$$

and, in particular,

$$\hat{I}^{\,n}(t)g\equiv\hat{I}^{\,n}(t;t_n)g=(n!)^{-1}t^n g\ , \tag{25b}$$

provided the function g is chosen so that $dg/dt_i=0$.

Then, we consider the operators $\hat{\mathcal{J}}^{(+)}$ and $\hat{\mathcal{J}}^{(-)}$ defined by the following relations:

$$\hat{\mathcal{J}}^{(+)}(\eta;\zeta)\zeta^n=n!\eta^n \tag{26a}$$

and

$$\hat{\mathcal{J}}^{(-)}(\zeta;\eta)\eta^n=(n!)^{-1}\zeta^n\ . \tag{26b}$$

These operators can be expressed in explicit form by using integral transforms. If

$$\hat{\mathcal{L}}(\eta;\zeta)f(\zeta)=\int_0^\infty d\zeta\exp(-\eta\zeta)f(\zeta)=\varphi(\eta)$$

is the Laplace transform of the function $f(\zeta)$ and

$$\hat{\mathcal{L}}^{-1}(\zeta;\eta)\varphi(\eta)=\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}d\eta\exp(\zeta\eta)\varphi(\eta)=f(\zeta)$$

is the inverse Laplace transform of the function $\varphi(\eta)$, we have

$$\hat{\mathcal{L}}(\eta;\zeta)\zeta^n=n!\eta^{-n-1}$$

and

$$\hat{\mathcal{L}}^{-1}(\zeta;\eta)\eta^{-n-1}=(n!)^{-1}\zeta^n\ .$$

Consequently, for the operators (26) we can write

$$\hat{\mathcal{J}}^{(+)}(\eta;\zeta)=\hat{\mathcal{L}}(\eta^{-1};\zeta)\eta^{-1} \tag{27a}$$

and

$$\hat{\mathcal{J}}^{(-)}(\zeta;\eta)=\hat{\mathcal{L}}^{-1}(\zeta;\eta^{-1})\eta\ . \tag{27b}$$

Now, in order to study the time evolution of the operator $\hat{b}_B^{(R)}(t)$, we solve Eqs. (21) and (22). For the integration we will use a mathematical method, previously applied to other problems, which is very convenient to study nonlinear equations. We express the solutions of the equation of motion as a power series of time. So we must find a recursive operational relation among the terms of the power series and, at the same time, take into account of the expansion factor $(n!)^{-1}$ for the generic nth term of the series. We use integral operators that permit us to deal with these difficulties in different phases of the calculations. We begin by writing a formal solution of Eq. (22) that is obtained by iteration techniques. With straightforward argumentations we find for $\hat{b}_F^{(R)}(t)$ the following expression:

$$\hat{b}_F^{(R)}(t)=\hat{\ell}_F(t)+\hat{I}(t)g\{\hat{\ell}_F(t)+\hat{I}(t)g\{\hat{\ell}_F(t)+\hat{I}(t)g\{\cdots\}\}\}\ , \tag{28}$$

where the operator $\hat{I}(t)$ has been introduced by Eqs. (25), the operator $\hat{\ell}_F(t)$ is defined as

$$\hat{\ell}_F(t)=\hat{b}_F-i\Delta\hat{I}(t)\hat{b}_B\ , \tag{29}$$

and the functional $g(\hat{\ell}_F)$ is given by

$$g(\hat{\ell}_F)=-[\Delta^2\hat{I}(t)+i(\hat{\Omega}-\mu)]\hat{\ell}_F-2i\mu\hat{\ell}_F\hat{\ell}_F^\dagger\hat{\ell}_F\ . \tag{30}$$

Then, from Eq. (21) we have for $\hat{b}_B^{(R)}(t)$ that

$$\hat{b}_B^{(R)}(t)=\hat{b}_B-i\Delta\hat{I}(t)\hat{b}_F^{(R)}(t)\ . \tag{31}$$

It is trivial to verify that Eqs. (28) and (31) are solutions of the equations of motion (21) and (22), since the derivative of Eq. (28) gives

$$\frac{d}{dt}\hat{b}_F^{(R)}(t)=-i\Delta\hat{b}_B+g\{\hat{\ell}_F(t)+\hat{I}(t)g\{\hat{\ell}_F(t)+\hat{I}(t)g\{\cdots\}\}\}$$

and, consequently, it is

$$\frac{d}{dt}\hat{b}_F^{(R)}(t)=-i\Delta\hat{b}_B-[\Delta^2\hat{I}(t)+i(\hat{\Omega}-\mu)]\hat{b}_F^{(R)}(t)-2i\mu\hat{b}_F^{(R)}(t)[\hat{b}_F^{(R)}(t)]^\dagger\hat{b}_F^{(R)}(t)\ .$$

Now, we look for the analytical function to which the series (28) converges. For this purpose let us initially consider the simpler series

$$\hat{y}(\tau)=\hat{a}+\hat{I}(\tau)\mathscr{g}\{\hat{a}+\hat{I}(\tau)\mathscr{g}\{\hat{a}+\hat{I}(\tau)\mathscr{g}\{\cdots\}\}\}\ , \tag{32}$$

where $\hat{a}$ is an annihilation boson operator and the functional $\mathscr{g}(\hat{y})$ is given by

$$\mathscr{g}(\hat{y})=-[\Delta^2+i(\Omega-\mu)]\hat{y}-2i\mu\hat{y}\hat{y}^\dagger\hat{y}\ . \tag{33}$$

Here Δ, Ω, and μ are suitable real c-number constants. Our first task is to write the function to which the series (32) converges. We note that $\hat{y}(\tau)$ obeys the following equation of motion:

$$\frac{d\hat{y}(\tau)}{d\tau}=-[\Delta^2+i(\Omega-\mu)]\hat{y}(\tau)-2i\mu\hat{y}(\tau)\hat{y}^\dagger(\tau)\hat{y}(\tau)\ . \tag{34}$$

If we assume that $\hat{y}(\tau)$ is expressed as

$$\hat{y}(\tau)=\exp[\chi(\tau)]\exp[i\hat{\mathscr{K}}(\tau)]\hat{a}\exp[-i\hat{\mathscr{K}}(\tau)]\ , \tag{35a}$$

with

$$\hat{\mathscr{K}}(\tau)=\varphi(\tau)\hat{a}^\dagger\hat{a}+\psi(\tau)(\hat{a}^\dagger\hat{a})^2\ , \tag{35b}$$

from Eq. (34) we see that the real functions $\varphi(\tau)$, $\psi(\tau)$, and $\chi(\tau)$ must satisfy the following relation:

$$i[\hat{a}^\dagger\hat{a},\hat{a}]\frac{d\varphi}{d\tau}+i[(\hat{a}^\dagger\hat{a})^2,\hat{a}]\frac{d\psi}{d\tau}+\hat{a}\frac{d\chi}{d\tau}=-[\Delta^2+i(\Omega-\mu)]\hat{a}-2i\mu\exp(2\chi)\hat{a}\hat{a}^\dagger\hat{a}\ .$$

From this relation we find that Eq. (35a) is solution of Eq. (34) provided

$$\varphi(\tau)=(\Omega-\mu)\tau+\mu(2\Delta^2)^{-1}[1-\exp(-2\Delta^2\tau)]\ ,$$

$$\psi(\tau)=\mu(2\Delta^2)^{-1}[1-\exp(-2\Delta^2\tau)]\ ,$$

and

$$\chi(\tau)=-\Delta^2\tau\ .$$

On making use of the preceding results we can write the functional as

$$\hat{y}(\tau)=\hat{a}\exp\{-[\Delta^2+i(\Omega-\mu)]\tau\}\times\exp\{-i\mu\Delta^{-2}[1-\exp(-2\Delta^2\tau)]\hat{a}^\dagger\hat{a}\}\ , \tag{36}$$

since for boson operators we have

$$f(\hat{a}^\dagger\hat{a})\hat{a}=\hat{a}f(\hat{a}^\dagger\hat{a}-1)\ . \tag{37}$$

For our purposes it is necessary to express the τ dependence of the functional $\hat{y}$ by means of the operator $\hat{I}(\tau)$. So, with the help of operator $\hat{\mathscr{I}}^{(+)}$, we write the following identity:

$$\hat{y}(\tau)=\exp[\hat{I}(\tau)\hat{D}(\eta)]\hat{\mathscr{I}}^{(+)}(\eta;\tau')\hat{y}(\tau=\tau')|_0\ , \tag{38}$$

where, for the sake of brevity, we let

$$\hat{D}(\eta)=\frac{d}{d\eta}\ . \tag{39}$$

We mean by subscript $|_0$ that the functional at the right-hand side of Eq. (38) must be evaluated for $\eta=0$.

Then, we compare the operators $\hat{b}_F^{(R)}(t)$ and $\hat{y}(\tau)$ which have been defined by Eqs. (28) and (32), respectively. We see that $\hat{b}_F^{(R)}(t)$ can be easily obtained from the expression of $\hat{y}(\tau)$. To this end it is sufficient, in Eq. (32), to replace Δ^2 by $\Delta^2\hat{I}$, $\hat{a}$ by $(\hat{b}_F-i\Delta\hat{I}\hat{b}_B)$, $\hat{a}^\dagger$ by $(\hat{b}_F^\dagger+i\Delta\hat{I}\hat{b}_B^\dagger)$, and Ω by $\hat{\Omega}$ and to assume $\tau=t$. If we use Eqs. (36) and (38) to express $\hat{y}(\tau)$, in order to take into account of the preceding substitutions we must replace Δ^2 by $\Delta^2\eta$, $\hat{a}$ by $\hat{b}_F-i\Delta\eta\hat{b}_B$, and $\hat{a}^\dagger$ by $\hat{b}_F^\dagger+i\Delta\eta\hat{b}_B^\dagger$ in Eq. (36). So for $\hat{b}_F^{(R)}(t)$ we obtain the following expression:

$$\hat{b}_F^{(R)}(t)=\exp[\hat{I}(t)\hat{D}(\eta)]\hat{\mathscr{I}}^{(+)}(\eta;\tau')\hat{B}_F^{(R)}(\eta;\tau')|_0\ , \tag{40}$$

where we have set

$$\hat{B}_F^{(R)}(\eta;\tau')=\hat{\mathscr{B}}_F(\eta)\exp\{-[\Delta^2\eta+i(\hat{\Omega}-\mu)]\tau'\}\times\exp\{-i\mu(\Delta^2\eta)^{-1}[1-\exp(-2\Delta^2\eta\tau')]\times\hat{\mathscr{B}}_F^\dagger(\eta)\hat{\mathscr{B}}_F(\eta)\}\ , \tag{41}$$

with

$$\hat{\mathscr{B}}_F(\eta)=\hat{b}_F-i\Delta\eta\hat{b}_B\ . \tag{42}$$

Now, we free Eq. (40) from the operator $\hat{I}(\tau)$ with the help of the operator $\hat{\mathcal{J}}^{(-)}$. From the definitions (25b) and (26b) we find that $\hat{b}_F^{(R)}(t)$ can be expressed as

$$\hat{b}_F^{(R)}(t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\hat{B}_F^{(R)}(\eta;\tau) . \tag{43}$$

If we consider Eq. (31), we see that the operator $\hat{b}_B^{(R)}(t)$ can be described by an equation similar to Eq. (43), namely,

$$\hat{b}_B^{(R)}(t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\hat{B}_B^{(R)}(\eta;\tau) , \tag{44}$$

where

$$\hat{B}_B^{(R)}(\eta;\tau)=\hat{b}_B-i\Delta\eta\hat{B}_F^{(R)}(\eta;\tau) . \tag{45}$$

When Eqs. (12) and (41) are introduced into Eq. (45), we find that the operator $\hat{B}_B^{(R)}$ is given by

$$\begin{aligned}\hat{B}_B^{(R)}(\eta;\tau)=\hat{b}_B-i\Delta\eta\hat{\mathcal{B}}_F(\eta)\exp(2ikm^{-1}\hat{p}_0\tau)\\ \times\exp[(i\mu-\Delta^2\eta)\tau]\\ \times\exp\{-i\mu(\Delta^2\eta)^{-1}[1-\exp(-2\Delta^2\eta\tau)]\\ \times\hat{\mathcal{B}}_F^\dagger(\eta)\hat{\mathcal{B}}_F(\eta)\} ,\end{aligned} \tag{46}$$

with

$$\hat{p}_0\equiv\hat{p}(t=0) . \tag{47}$$

These results describe the time evolution of the operator $\hat{b}_B^{(R)}$, which is governed by the equation of motion (18).

In order to obtain the time evolution of $\hat{b}_B$ in SCS we must introduce the expression of $\hat{b}_B^{(R)}(t)$ in Eqs. (16). So we find that

$$\begin{aligned}\hat{b}_B(t)=\exp(-2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\exp\left[\frac{it}{\hbar}\hat{\mathcal{H}}_1\right]\hat{b}_B^{(R)}(t)\\ \times\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}_1\right]\exp(2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F) ,\end{aligned} \tag{48}$$

with

$$\hat{z}_0\equiv\hat{z}(t=0) . \tag{49}$$

While we analyze the consequences of the presence of $\hat{b}_B^{(R)}(t)$ in Eq. (48), we will order the operators present in the resulting functional which gives $\hat{b}_B(t)$. We will write the boson operators in normal order and reorder the electron operators so that all coordinate operators $\hat{z}_0$ are written to the right of all momentum operators $\hat{p}_0$.

We recall that an operator $f(\hat{a},\hat{a}^\dagger)$, a function of boson operators $\hat{a}$ and $\hat{a}^\dagger$, is expressed in normal order when in each term of the series defining the functional f all annihilation operators appear to the right of all creation operators.[18] In order to write the functional $f(\hat{a},\hat{a}^\dagger)$ into normal order we use the associate function $f(\alpha,\alpha^*)$, which is obtained by the diagonal elements of $f(\hat{a},\hat{a}^\dagger)$ in the coherent state representation,

$$f(\alpha,\alpha^*)=\langle\alpha|f(\hat{a},\hat{a}^\dagger)|\alpha\rangle ,$$

with $\hat{a}|\alpha\rangle=\alpha|\alpha\rangle$. We find the normal form of $f(\hat{a},\hat{a}^\dagger)$ if we replace α by $\hat{a}$, α^* by $\hat{a}^\dagger$, and write each term in normal order in the function $f(\alpha,\alpha^*)$.

In the following we will make use of the relation

$$\langle\alpha|\exp(\xi\hat{a}^\dagger\hat{a})|\alpha\rangle=\exp\{[\exp(\xi)-1]\alpha^*\alpha\} , \tag{50}$$

where ξ is a c number.[18] With the help of this relation we can write for two boson operators $\hat{a}_1$ and $\hat{a}_2$ that

$$\begin{aligned}&\langle\alpha_1,\alpha_2|\exp[\xi(\eta_1^*\hat{a}_1^\dagger+\eta_2^*\hat{a}_2^\dagger)(\eta_1\hat{a}_1+\eta_2\hat{a}_2)]|\alpha_1,\alpha_2\rangle\\ &\qquad=\exp(\{\exp[\xi(|\eta_1|^2+|\eta_2|^2)]-1\}(|\eta_1\alpha_1+\eta_2\alpha_2|^2)(|\eta_1|^2+|\eta_2|^2)^{-1}) ,\end{aligned} \tag{51}$$

provided η_1 and η_2 are c numbers and

$$\hat{a}_1\hat{a}_2|\alpha_1,\alpha_2\rangle=\alpha_1\alpha_2|\alpha_1,\alpha_2\rangle . \tag{52}$$

We will now analyze the operator $\hat{b}_B(t)$ described by Eq. (48). We begin by considering the operator

$$\hat{B}_B^{(P)}(t;\eta;\tau)=\exp\left[\frac{it}{\hbar}\hat{\mathcal{H}}_1\right]\hat{B}_B^{(R)}(\eta;\tau)\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}_1\right] , \tag{53}$$

with the Hamiltonian $\hat{\mathcal{H}}_1$ defined by Eq. (11b). We directly obtain from Eqs. (5) and (46) that

$$\hat{B}_B^{(P)}(t;\eta;\tau)=\exp(-i\omega t)\hat{B}_B^{(R)}(\eta;\tau) . \tag{54}$$

If we apply the property (37) to Eq. (46), from Eq. (54) we find that

$$\begin{aligned}\hat{B}_B^{(P)}(t;\eta;\tau)=\exp(-i\omega t)(\hat{b}_B-i\Delta\eta\exp(2ikm^{-1}\hat{p}_0\tau)\exp[(i\mu-\Delta^2\eta)\tau]\\ \times\exp\{-i\mu(\Delta^2\eta)^{-1}[1-\exp(-2\Delta^2\eta\tau)][\hat{\mathcal{B}}_F^\dagger(\eta)\hat{\mathcal{B}}_F(\eta)+\Delta^2\eta^2+1]\}\hat{\mathcal{B}}_F(\eta)) .\end{aligned} \tag{55}$$

Then, we consider the operator

$$\hat{B}_B^{(T)}(t;\eta;\tau)=\exp(-2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\hat{B}_B^{(P)}(t;\eta;\tau)\exp(2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F) . \tag{56}$$

From Eqs. (3) and (6) we have the commutation relation

$$\exp(-2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\exp(2ikm^{-1}\hat{p}_0\tau)=\exp(2ikm^{-1}\hat{p}_0\tau)\exp[-2ik(\hat{z}_0-2\hbar km^{-1}\tau)\hat{b}_F^\dagger\hat{b}_F] \ . \tag{57}$$

Therefore, when we apply the properties (5) and (57) to Eq. (56), for the operator $\hat{B}_B^{(T)}$ we obtain the following expression:

$$\hat{B}_B^{(T)}(t;\eta;\tau)=\exp(-i\omega t)[\hat{b}_B-i\Delta\eta\exp(2ikm^{-1}\hat{p}_0\tau)\exp[(i\mu-\Delta^2\eta)\tau]\exp(4i\hbar k^2m^{-1}\tau\hat{b}_F^\dagger\hat{b}_F)$$
$$\times\exp(-i\mu(\Delta^2\eta)^{-1}[1-\exp(-2\Delta^2\eta\tau)]\{[\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)]^\dagger\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)+\Delta^2\eta^2+1\})\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)] \ , \tag{58}$$

where

$$\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)=\hat{b}_F\exp(2ik\hat{z}_0)-i\Delta\eta\hat{b}_B \ . \tag{59}$$

We point our that Eq. (58) appears to be ordered in the electron operators $\hat{p}_0$ and $\hat{z}_0$. Since the operators $\hat{\mathcal{J}}^{(-)}(t;\eta)$ and $\hat{\mathcal{J}}^{(+)}(\eta;\tau)$ commute with all the operator met in the calculation of $\hat{B}_B^{(T)}$, we can finally write the operator $\hat{b}_B(t)$ as

$$\hat{b}_B(t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\hat{B}_B^{(T)}(t;\eta;\tau) \ . \tag{60}$$

Let us now introduce the boson associate function of the operators $\hat{b}_B$ and $\hat{B}_B^{(T)}$,

$$\Gamma_B(\beta_B,\beta_F;t)=\langle\beta_B,\beta_F|\hat{b}_B(t)|\beta_B,\beta_F\rangle$$

and

$$\Gamma_B^{(T)}(\beta_B,\beta_F;t;\eta;\tau)=\langle\beta_B,\beta_F|\hat{B}_B^{(T)}(t;\eta;\tau)|\beta_B,\beta_F\rangle \ ,$$

where, as in Eq. (52), we have

$$\hat{b}_B\hat{b}_F|\beta_B,\beta_F\rangle=\beta_B\beta_F|\beta_B,\beta_F\rangle \ .$$

On making use of Eqs. (5), (13), and (51), we find that the associate function $\Gamma_B^{(T)}$ is given by

$$\Gamma_B^{(T)}(\beta_B,\beta_F;t;\eta;\tau)=\exp(-i\omega t)(\beta_B-i\Delta\eta\exp(2ikm^{-1}\hat{p}_0\tau)\exp[(i\mu-\Delta^2\eta)\tau]\exp\{[\exp(2i\mu\tau)-1]|\beta_F|^2\}\Xi(\eta;\tau)$$
$$\times\exp\{[\Xi(\eta;\tau)-1][\bar{\gamma}_F(\beta_B,\beta_F;\hat{z}_0;\eta)]^*(1+\Delta^2\eta^2)^{-1}\}\bar{\gamma}_F(\beta_B,\beta_F;\hat{z}_0;\eta)) \ , \tag{61}$$

where we have put

$$\Xi(\eta;\tau)=\exp\{-i\mu(\Delta^2\eta)^{-1}[1-\exp(-2\Delta^2\eta\tau)](1+\Delta^2\eta^2)\} \ , \tag{62a}$$

$$\gamma_F(\beta_B,\beta_F;\hat{z}_0;\eta;\tau)=\beta_F\exp[2i(k\hat{z}_0-\mu\tau)]-i\Delta\eta\beta_B \ , \tag{62b}$$

and

$$\bar{\gamma}_F(\beta_B,\beta_F;\hat{z}_0;\eta)=\beta_F\exp(2ik\hat{z}_0)-i\Delta\eta\beta_B \ . \tag{62c}$$

From Eq. (60) we see that the associate functions Γ_B and $\Gamma_B^{(T)}$ are connected by the following relation:

$$\Gamma_B(\beta_B,\beta_F;t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\Gamma_B^{(T)}(\beta_B,\beta_F;t;\eta;\tau) \ . \tag{63}$$

Consequently, in order to write the desired associate function $\Gamma_B(\beta_B,\beta_F;t)$ we must introduce Eq. (61) into Eq. (63).

The associate function for the operator $\hat{b}_F(t)$ which describes the forward propagating field in SCS,

$$\Gamma_F(\beta_B,\beta_F;t)=\langle\beta_B,\beta_F|\hat{b}_F(t)|\beta_B,\beta_F\rangle \ ,$$

is easily obtained from the expression of $\Gamma_B(\beta_B,\beta_F;t)$. To this end we must replace the propagation vector k by $-k$ and exchange the subscripts B and F in Eqs. (61) and (63), as it has been previously noticed by analyzing the properties of the operators $\hat{b}_B$ and $\hat{b}_F$. So, for the associate function $\Gamma_B(\beta_B,\beta_F;t)$ we obtain the expression

$$\Gamma_F(\beta_B,\beta_F;t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\Gamma_F^{(T)}(\beta_B,\beta_F;t;\eta;\tau) \ , \tag{64}$$

where it is

$$\Gamma_F^{(T)}(\beta_B,\beta_F;t;\eta;\tau)=\exp(-i\omega t)(\beta_F-i\Delta\eta\exp(-2ikm^{-1}\hat{p}_0\tau)\exp[(i\mu-\Delta^2\eta)\tau]\exp\{[\exp(2i\mu\tau)-1]|\beta_B|^2\}\Xi(\eta;\tau)$$
$$\times\exp\{[\Xi(\eta;\tau)-1][\gamma_B(\beta_B,\beta_F;\hat{z}_0;\eta;\tau)]^*\gamma_B(\beta_B,\beta_F;\hat{z}_0;\eta)(1+\Delta^2\eta^2)^{-1}\}$$
$$\times\bar{\gamma}_B(\beta_B,\beta_F;\hat{z}_0;\eta)) \ , \tag{65}$$

with

$$\gamma_B(\beta_B,\beta_F;\hat{z}_0;\eta;\tau)=\beta_B \exp[-2i(k\hat{z}_0+\mu\tau)]-i\Delta\eta\beta_F \ . \tag{66a}$$

and

$$\gamma_B(\beta_B,\beta_F;\hat{z}_0;\eta)=\beta_B \exp(-2ik\hat{z}_0)-i\Delta\eta\beta_F \ . \tag{66b}$$

The explicit expressions of the boson operators $\hat{b}_B$ and $\hat{b}_F$ are directly obtained from the associate functions $\Gamma_B(\beta_B,\beta_F;t)$ and $\Gamma_F(\beta_B,\beta_F;t)$ when the variables $\beta_B,\beta_B^*,\beta_F$ and β_F^* are replaced with the operators $\hat{b}_B$, $\hat{b}_B^\dagger$, $\hat{b}_F$, and $\hat{b}_F^\dagger$, respectively, provided these operators are written in normal order.

For the sake of completeness we will rewrite Eqs. (63) and (64) by expressing the integral operators in explicit form. If we use the Laplace transform and the inverse Laplace transform, from Eqs. (27) we find that the associate functions $\Gamma_B(\beta_B,\beta_F;t)$ and $\Gamma_F(\beta_B,\beta_F;t)$ are given by

$$\Gamma_B(\beta_B,\beta_F;t)=\hat{\mathcal{L}}^{-1}(t;\chi^{-1})\hat{\mathcal{L}}(\chi^{-1};\tau)\times\Gamma_B^{(T)}(\beta_B,\beta_F;t;\ \eta=\chi^{-1};\ \tau) \tag{67a}$$

and

$$\Gamma_F(\beta_B,\beta_F;t)=\hat{\mathcal{L}}^{-1}(t;\chi^{-1})\hat{\mathcal{L}}(\chi^{-1};\tau)\times\Gamma_F^{(T)}(\beta_B,\beta_F;t;\ \eta=\chi^{-1};\ \tau)\ . \tag{67b}$$

We point out that the integral expressions (67) contain a Laplace transform and a subsequent inverse Laplace transform only. Therefore these expressions can be used to study the properties of the fields in the SCS process or to obtain handy values of quantities in processes which obey Raman-Nath equations.[15] For instance, we recall the analogy between Hamiltonian (11c) and the Hamiltonian governing radiation from a Josephson junction.[8]

IV. ELECTRON OPERATORS

In this section we will suggest a method for evaluating the time development of the coordinate and momentum operators of electrons in the SCS process.

We begin by applying Eq. (15) to the momentum operator $\hat{p}$, so that we have

$$\hat{p}(t)=\hat{U}^\dagger(t)\hat{p}_0\hat{U}(t)\ .$$

From Eqs. (7) and (49) we see that

$$\exp(2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\hat{p}_0\exp(-2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)=\hat{p}_0-2\hbar k\hat{b}_F^\dagger\hat{b}_F\ . \tag{68}$$

Therefore from Eq. (8) we can write

$$\hat{p}(t)=\hat{P}_1(t)-\hat{P}_2(t)\ , \tag{69}$$

where the operators $\hat{P}_1$ and $\hat{P}_2$ are given by

$$\hat{P}_1(t)=[\hat{U}^{(R)}(t)]^\dagger\hat{p}_0\hat{U}^{(R)}(t)$$

and

$$\hat{P}_2(t)=2\hbar k[\hat{U}^{(R)}(t)]^\dagger\hat{b}_F^\dagger\hat{b}_F\hat{U}^{(R)}(t)\ ,$$

with

$$\hat{U}^{(R)}(t)=\exp\left[-\frac{it}{\hbar}\hat{\mathcal{H}}\right]\exp(2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\ .$$

Since the operator $\hat{p}_0$ commutes with the Hamiltonian $\hat{\mathcal{H}}$, we have that

$$\hat{P}_1(t)=\hat{p}_0+2\hbar k\hat{b}_F^\dagger\hat{b}_F\ . \tag{70}$$

On making use of the operator $\hat{b}_F^{(R)}(t)$ defined by Eq. (19), we find from Eq. (14) that

$$\hat{P}_2(t)=2\hbar k\exp(-2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)[\hat{b}_F^{(R)}(t)]^\dagger\times\hat{b}_F^{(R)}(t)\exp(2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\ . \tag{71}$$

When Eqs. (70) and (71) are introduced into Eq. (69), for the momentum operator we obtain the expression

$$\hat{p}(t)=\hat{p}_0-2\hbar k\exp(-2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\times\{[\hat{b}_F^{(R)}(t)]^\dagger\hat{b}_F^{(R)}(t)-\hat{b}_F^\dagger\hat{b}_F\}\times\exp(2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\ . \tag{72}$$

Next, for the coordinate operator $\hat{z}$ we can write the well-known equation of motion

$$\frac{d}{dt}\hat{z}(t)=\frac{1}{m}\hat{p}(t)\ , \tag{73}$$

since for Hamiltonian (1) we have

$$[\hat{H},\hat{z}]=-i\hbar m^{-1}\hat{p}\ .$$

The solution of Eq. (73) can be formally expressed as

$$\hat{z}(t)=\hat{z}_0+m^{-1}\hat{I}(t)\hat{p}(t)\ .$$

Finally, by using Eq. (72) we see that the time evolution of the coordinate operator is described by the relation

$$\hat{z}(t)=\hat{z}_0+t(\hat{p}_0+2\hbar k\hat{b}_F^\dagger\hat{b}_F)-2\hbar k\exp(-2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\times\hat{I}(t)\{[\hat{b}_F^{(R)}(t)]^\dagger\hat{b}_F^{(R)}(t)\}\times\exp(2ik\hat{z}_0\hat{b}_F^\dagger\hat{b}_F)\ . \tag{74}$$

Thus we have written the desired expressions that describe the time evolution of the coordinate and momentum operators of electrons in SCS. We point out that by straightforward calculations the operators $\hat{z}(t)$ and $\hat{p}(t)$ can be ordered as it has been made for the boson operators $\hat{b}_B(t)$ and $\hat{b}_F(t)$ in the previous Sec. III, but, for the sake of brevity, the calculations are not present in this paper.

V. APPLICATIONS

We will illustrate the utility of the present approach by applying the results to two simple cases. In these calculations we assume that, at time $t=0$, the electron is described by a state $|p_0\rangle$ with $\hat{p}_0|p_0\rangle=p_0|p_0\rangle$. For the sake of brevity we let

$$\langle E^n(\hat{z}_0)\rangle=\langle p_0|\exp(2ink\hat{z}_0)|p_0\rangle\ .$$

First, we will analyze the time evolution of the photon operators by supposing that the parameter μ defined in

Eq. (13) is negligible. As μ is intimately connected, in the SCS process, to the electron quantum recoil, the case may appear not to be particularly interesting. On the contrary, this problem has been extensively studied since it has been the necessary step for writing the perturbation solution of the equation of motion (2).[19] We consider the associate function (63) that, for $\mu=0$, is given by

$$[\Gamma_B(\beta_B,\beta_F;t)]_\mu=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\times[\Gamma_B^{(T)}(\beta_B,\beta_F;t;\eta;\tau)]_\mu , \tag{75}$$

where

$$[\Gamma_B^{(T)}(\beta_B,\beta_F;t;\eta;\tau)]_\mu=\exp(-i\omega t)\{\beta_B-i\Delta\eta\exp[-(\Delta^2\eta+i\Omega_0)\tau][\beta_F^{(T)}(\hat{z}_0)-i\Delta\eta\beta_B]\} ,$$

with

$$\beta_F^{(T)}(\hat{z}_0)=\langle E(\hat{z}_0)\rangle\beta_F,\quad \Omega_0=-2km^{-1}p_0 .$$

After the integration Eq. (75) becomes

$$[\Gamma_B(\beta_B,\beta_F;t)]_\mu=\exp(-i\omega t)\hat{\mathcal{J}}^{(-)}(t;\eta)\{\beta_B-i\Delta\eta\sum_{n=0}^{\infty}[-(\Delta^2\eta+i\Omega_0)\eta]^n[\beta_F^{(T)}(\hat{z}_0)-i\Delta\eta\beta_B]\} .$$

Then, on performing the inverse Laplace transforms, we find that[20]

$$[\Gamma_B(\beta_B,\beta_F;t)]_\mu=\exp\left[-i\left[\omega+\frac{\Omega_0}{2}\right]t\right](2\chi)^{-1}\left\{\left[(\chi+\Omega_0)\exp\left[i\frac{\chi}{2}t\right]+(\chi-\Omega_0)\exp\left[-i\frac{\chi}{2}t\right]\right]\beta_B + 2\Delta\left[\exp\left[-i\frac{\chi}{2}t\right]-\exp\left[i\frac{\chi}{2}t\right]\right]\beta_F^{(T)}(\hat{z}_0)\right\} , \tag{76}$$

where

$$\chi=(4\Delta^2+\Omega_0^2)^{1/2} .$$

Consequently, the operator $[\hat{b}_B(t)]_\mu$ is given by

$$[\hat{b}_B(t)]_\mu=\exp\left[-i\left[\omega+\frac{\Omega_0}{2}\right]t\right](2\chi)^{-1}\left\{\left[(\chi+\Omega_0)\exp\left[i\frac{\chi}{2}t\right]+(\chi-\Omega_0)\exp\left[-i\frac{\chi}{2}t\right]\right]\hat{b}_B + 2\Delta\left[\exp\left[-i\frac{\chi}{2}t\right]-\exp\left[i\frac{\chi}{2}t\right]\right]\langle E(\hat{z}_0)\rangle\hat{b}_F\right\} . \tag{77a}$$

Likewise, for the operator $[\hat{b}_F(t)]_\mu$ we see that

$$[\hat{b}_F(t)]_\mu=\exp\left[-i\left[\omega-\frac{\Omega_0}{2}\right]t\right](2\chi)^{-1}\left\{\left[(\chi-\Omega_0)\exp\left[i\frac{\chi}{2}t\right]+(\chi+\Omega_0)\exp\left[-i\frac{\chi}{2}t\right]\right]\hat{b}_F + 2\Delta\left[\exp\left[-i\frac{\chi}{2}t\right]-\exp\left[i\frac{\chi}{2}t\right]\right]\langle E^{-1}(\hat{z}_0)\rangle\hat{b}_B\right\} . \tag{77b}$$

So we have found the expressions that describe the time evolution of the photon operators when $\mu\approx0$. These results can be obtained by a direct integration of the equations of motion (21) and (22) also.

Now we will study the time evolution of the photon operators when the field is described, at time $t=0$, by a coherent state $|\psi_B,\psi_F\rangle$ with

$$\hat{b}_F|\psi_B,\psi_F\rangle=\psi_F|\psi_B,\psi_F\rangle,\quad \hat{b}_B|\psi_B,\psi_F\rangle=\psi_B|\psi_B,\psi_F\rangle . \tag{78}$$

Moreover, in order to simplify the following calculations the amplitudes ψ_F and ψ_B are assumed such that

$$|\psi_F|^2\gg1,\quad |\psi_F|^2\gg|\psi_B|^2 . \tag{79}$$

We begin by considering the mean value of the photon operator $\hat{b}_B$,

$$\langle\hat{b}_B(t)\rangle_\psi=\langle\psi_B,\psi_F;p_0|\hat{b}_B(t)|\psi_B,\psi_F;p_0\rangle , \tag{80}$$

which we write as

$$\langle\hat{b}_B(t)\rangle_\psi=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\times\langle\psi_B,\psi_F;p_0|\hat{B}_B^{(T)}(t;\eta;\tau)|\psi_B,\psi_F;p_0\rangle . \tag{81}$$

For the following considerations it is convenient to make use of Eq. (58). Thus we see that

$$\langle p_0|\hat{B}_B^{(T)}(t;\eta;\tau)|p_0\rangle=\exp(-i\omega t)[\hat{b}_B+\hat{B}_F^{(A)}(\eta;\tau)] , \tag{82}$$

where

$$\hat{B}_F^{(A)}(\eta;\tau)=-i\Delta\eta\exp\{-[\Delta^2\eta+i(\Omega_0-\mu)]\tau\}\langle p_0|\exp(2i\mu\tau\hat{b}_F^\dagger\hat{b}_F)\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)$$
$$\times\exp(-i\mu(\Delta^2\eta)^{-1}[1-\exp(-2\Delta^2\eta\tau)]\{[\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)]^\dagger\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)\})|p_0\rangle ,$$

with

$$\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)=\hat{b}_F\exp(2ik\hat{z}_0)-i\Delta\eta\hat{b}_B .$$

When we expand the operator $\hat{B}_F^{(A)}$ as a power series, we obtain

$$\hat{B}_F^{(A)}(\eta;\tau)=-i\Delta\eta\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(n!l!)^{-1}(-1)^l[i\mu(\Delta^2\eta)^{-1}]^{n+l}\exp\{-[(2n+1)\Delta^2\eta+i(\Omega_0-\mu)]\tau\}$$
$$\times\langle p_0|\exp(2i\mu\tau\hat{b}_F^\dagger\hat{b}_F)\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)\{[\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)]^\dagger\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)\}^{n+l}|p_0\rangle .$$

In order to give a handy form to the mean value

$$\Psi_F^{(S)}(\psi_B,\psi_F;\eta;\tau)=\langle\psi_B,\psi_F;p_0|\exp(2i\mu\tau\hat{b}_F^\dagger\hat{b}_F)\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)\{[\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)]^\dagger\hat{\mathcal{B}}_F^{(C)}(\hat{z}_0;\eta)\}^s|\psi_B,\psi_F;p_0\rangle \tag{83}$$

we introduce some approximations. Since the conditions (79) allow us to neglect the terms of superior order with respect to $|\psi_B/\psi_F|$, we can write Eq. (83) as

$$\Psi_F^{(S)}(\psi_B,\psi_F;\eta;\tau)\approx\langle p_0|\exp(2i\mu\tau|\psi_F|^2)(1+i\Delta\eta\{\psi_B^*\exp[2i(k\hat{z}_0-\mu\tau)]\hat{D}(\psi_F^*)$$
$$-\psi_B\exp(-2ik\hat{z}_0)\hat{D}(\psi_F)\})\exp(2ik\hat{z}_0)\psi_F^{*s}\psi_F^{s+1}|p_0\rangle . \tag{84}$$

If we make use of Eq. (84), we see that

$$\hat{\mathcal{J}}^{(+)}(\eta;\tau)\langle\psi_B,\psi_F|\hat{B}_F^{(A)}(\eta;\tau)|\psi_B,\psi_F\rangle=\langle E(\hat{z}_0)\rangle\gamma_0^{(1)}(\psi_F;\eta)+i\Delta\langle E^2(\hat{z}_0)\rangle\psi_B^*\hat{D}(\psi_F^*)\gamma_2^{(2)}(\psi_F;\eta)-i\Delta\hat{D}(\psi_F)\gamma_0^{(2)}(\psi_F;\eta) , \tag{85}$$

where

$$\gamma_j^{(v)}(\psi_F;\eta)=-i\Delta\eta^v\sum_{n,l}\sum_{h=0}^{\infty}(n!l!)^{-1}(-1)^l[i\mu(\Delta^2\eta)^{-1}]^{n+l}(-\{(2n+1)\Delta^2\eta+i[\Omega_0+\mu(j-1)-2\mu|\psi_F|^2]\}\eta)^h|\psi_F|^{2(n+1)}\psi_F . \tag{86}$$

Now it is useful to write some properties of the operator $\hat{\mathcal{J}}^{(-)}$. If $f(\eta)$ is an arbitrary function, from the definition (26b) we have

$$\hat{\mathcal{J}}^{(-)}(t;\eta)[\eta^{-v}f(\eta)]=\hat{D}^v(t)[\hat{\mathcal{J}}^{(-)}(t;\eta)f(\eta)] \tag{87a}$$

and

$$\hat{\mathcal{J}}^{(-)}(t;\eta)[\eta f(\eta)]=\int_0^t dt_1[\hat{\mathcal{J}}^{(-)}(t_1;\eta)f(\eta)] . \tag{87b}$$

Consequently, from Eq. (86) we find that the functions

$$G_j^{(v)}(\psi_F,\psi_F^*;t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\gamma_j^{(v)}(\psi_F;\eta)$$

are given by

$$G_j^{(v)}(\psi_F,\psi_F^*;t)=\psi_F\exp[-i\mu\Delta^{-2}|\psi_F|^2\hat{D}(t)]\sum_{n=0}^{\infty}(n!)^{-1}[i\mu\Delta^{-2}|\psi_F|^2\hat{D}(t)]^nF_{j,n}^{(v)}(t) , \tag{88}$$

where

$$F_{j,n}^{(v)}(t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\left[-i\Delta\eta^v\sum_{h=0}^{\infty}(-\{(2n+1)\Delta^2\eta+i[\Omega_0+\mu(j-1)-2\mu|\psi_F|^2]\}\eta\right]^h .$$

If we operate as for Eq. (76), we see that for $v\leq 2$, we have

$$F_{j,n}^{(v)}(t)=\Delta(\chi_j^{(n)})^{-1}\left[[2i(\chi_j^{(n)}+\Omega_j)^{-1}]^{v-1}\exp\left[-\frac{i}{2}(\chi_j^{(n)}+\Omega_j)^t\right]\right.$$
$$\left.-[2i(\Omega_j-\chi_j^{(n)})^{-1}]^{v-1}\exp\left[\frac{i}{2}(\chi_j^{(n)}-\Omega_j)t\right]+2i\chi_j^{(n)}[(\Omega_j)^2-(\chi_j^{(n)})^2]^{-1}\delta_{v2}\right], \tag{89}$$

where

$$\chi_j^{(n)}=\{4(2n+1)\Delta^2+[\Omega_0+\mu(j-1)-2\mu|\psi_F|^2]^2\}^{1/2}$$

and

$$\Omega_j=\Omega_0+\mu(j-1)-2\mu|\psi_F|^2 .$$

Then, we introduce Eq. (89) into Eq. (88) and after a little algebra we obtain

$$G_j^{(v)}(\psi_F,\psi_F^*;t)=\left\{\sum_{n=0}^{\infty}(n!)^{-1}(2i)^{v-1}\Delta(\chi_j^{(n)})^{-1}[\mu|\psi_F|^2(2\Delta^2)^{-1}]^n\exp\left[-\frac{i}{2}(t-i\mu\Delta^{-2}|\psi_F|^2)\Omega_j\right]\right.$$
$$\times\left[(\chi_j^{(n)}+\Omega_j)^{n+1-v}\exp\left[-\frac{i}{2}(t-i\mu\Delta^2|\psi_F|^{-2})\chi_j^{(n)}\right]\right.$$
$$\left.\left.-(\Omega_j-\chi_j^{(n)})^{n+1-v}\exp\left[\frac{i}{2}(t-i\mu\Delta^2|\psi_F|^{-2})\chi_j^{(n)}\right]\right]+2i\chi_j^{(n)}[(\Omega_j)^2-(\chi_j^{(n)})^2]^{-1}\delta_{v2}\right\}\psi_F . \tag{90}$$

If we consider Eq. (85), we see from Eq. (90) that the mean value

$$\langle\hat{B}_F^{(A)}(t)\rangle_\psi=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\tau)\langle\psi_B,\psi_F|\hat{B}_F^{(A)}(\eta;\tau)|\psi_B,\psi_F\rangle$$

can be written in the present approximation as

$$\langle\hat{B}_F^{(A)}(t)\rangle_\psi\approx\langle E(\hat{z}_0)\rangle G_0^{(1)}(\psi_F,\psi_F^*;t)+\langle E^2(\hat{z}_0)\rangle G_2^{(2)}(\psi_F,\psi_F^*+i\Delta\psi_B^*;t)-G_0^{(2)}(\psi_F-i\Delta\psi_B,\psi_F^*;t) . \tag{91}$$

Finally, when Eq. (91) is introduced into Eq. (82), from Eq. (81) we find that

$$\langle\hat{b}_B(t)\rangle_\psi=\exp(-i\omega t)[\psi_B+\langle\hat{B}_F^{(A)}(t)\rangle_\psi] .$$

Thus the expression that describes the time evolution of the operator $\hat{b}_B$ is obtained.

Now we will study the mean value of the operator $\hat{b}_F$,

$$\langle\hat{b}_F(t)\rangle_\psi=\langle\psi_B,\psi_F;p_0|\hat{b}_F(t)|\psi_B,\psi_F;p_0\rangle . \tag{92}$$

On the analogy with the evaluation of the mean value $\langle\hat{b}_B(t)\rangle_\psi$, we consider the expression

$$\langle p_0|\hat{B}_F^{(T)}(t;\eta;\tau)|p_0\rangle=\exp(-i\omega t)[\hat{b}_F+\hat{B}_B^{(A)}(\eta;\tau)] , \tag{93}$$

where

$$\hat{B}_B^{(A)}(\eta;\tau)=-i\Delta\eta\sum_{n,l}(n!l!)^{-1}(-1)^l[i\mu(\Delta^2\eta)^{-1}]^{n+l}\exp\{-[(2n+1)\Delta^2\eta-i(\Omega_0+\mu)]\tau\}$$
$$\times\langle p_0|\exp(2i\mu\tau\hat{b}_B^\dagger\hat{b}_B)\hat{\mathcal{B}}_B^{(C)}(\hat{z}_0;\eta)\{[\hat{B}_B^{(C)}(\hat{z}_0;\eta)]^\dagger\hat{B}_B^{(C)}(\hat{z}_0;\eta)\}^{n+l}|p_0\rangle .$$

with

$$\hat{\mathcal{B}}_B^{(C)}(\hat{z}_0;\eta)=\hat{b}_B\exp(-2ik\hat{z}_0)-i\Delta\eta\hat{b}_F .$$

On the present conditions we can approximate the mean value

$$\Psi_B^{(S)}(\psi_B,\psi_F;\eta;\tau)=\langle\psi_B,\psi_F;p_0|\exp(2i\mu\tau b_B^\dagger\hat{b}_B)\hat{\mathcal{B}}_B^{(C)}(\hat{z}_0;\eta)\{[\hat{\mathcal{B}}_B^{(C)}(\hat{z}_0;\eta)]^\dagger\hat{\mathcal{B}}_B^{(C)}(\hat{z}_0;\eta;\tau)\}^s|\psi_B,\psi_F;p_0\rangle$$

as follows:

$$\Psi_B^{(S)}(\psi_B,\psi_F;\eta;\tau)\approx\langle p_0|\{-i(1-i(\Delta\eta)^{-1}\{\psi_B^*\exp[2i(k\hat{z}_0+\mu\tau)]\hat{D}(\psi_F^*)$$
$$-\psi_B\exp(-2ik\hat{z}_0)\hat{D}(\psi_F)\})(\Delta^2\eta)^{2s+1}\psi_F^{*s}\psi_F^{s+1}\}|p_0\rangle .$$

From this relation we obtain

$$\hat{\mathcal{J}}^{(+)}(\eta;\tau)\langle\psi_B,\psi_F|\hat{B}_F^{(A)}(\eta;\tau)|\psi_B,\psi_F\rangle=\bar{\gamma}_0^{(0)}(\psi_F;\eta)+\langle E(\hat{z}_0)\rangle\psi_B^*\hat{D}(\psi_F^*)\bar{\gamma}_1^{(1)}(\psi_F;\eta)-\langle E^{-1}(\hat{z}_0)\rangle\hat{D}(\psi_F)\bar{\gamma}_0^{(1)}(\psi_F;\eta)\ ,$$

where

$$\bar{\gamma}_j^{(v)}(\psi_F;\eta)=\sum_{n,l}(n!l!)^{-1}(-1)^l\mathcal{F}_{j,nl}^{(v)}(\eta)|\psi_F|^{2(n+l)}\psi_F\ , \tag{94}$$

with

$$\mathcal{F}_{j,nl}^{(v)}(\eta)=i\Delta\eta[(i\Delta\eta-1)\delta_{v0}+1]\sum_{h=0}^{\infty}(i\mu\eta)^{n+l}(-\{(2n+1)\Delta^2\eta-i[\Omega_0+\mu(j+1)]\}\eta)^h\ .$$

If we set

$$\bar{F}_{j,nl}^{(v)}(t)=\hat{\mathcal{J}}^{(-)}(t;\eta)[\mathcal{F}_{j,nl}^{(v)}(\eta)]\ ,$$

from Eq. (87b) we see that the functions $\bar{F}_{j,nl}^{(v)}$ are given by

$$\bar{F}_{j,nl}^{(v)}(t)=\Delta(\bar{\chi}_j^{(n)})^{-1}(i\mu)^{n+l}[(1-i\Delta)\delta_{v0}-1]\Phi_{n+l-v+1}(-\bar{\chi}_j^{(n)}-\bar{\Omega}_j;t)-\Phi_{n+l-v+1}(\bar{\chi}_j^{(n)}-\Omega_j;t)\ , \tag{95}$$

where

$$\Phi_m(x;t)=\left[\frac{i}{2}x\right]^{-m}\left[\exp\left[\frac{i}{2}xt\right]-Y_m\left[\frac{i}{2}xt\right]\right]$$

with

$$Y_s(x)=\sum_{h=0}^{s}(h!)^{-1}x^h\ .$$

Here it is

$$\bar{\chi}_j^{(n)}=\{4(2n+1)\Delta^2+[\Omega_0+\mu(j+1)]^2\}^{1/2}$$

and

$$\bar{\Omega}_j=-[\Omega_0+\mu(j+1)]\ .$$

When we introduce this result into Eq. (94), from Eq. (93) we see that the mean value of the operator $\hat{b}_B$ can be written as

$$\langle\hat{b}_F(t)\rangle_\psi\approx\exp(-i\omega t)\left[\psi_F+\sum_{n,l}(n!l!)^{-1}(-1)^l[\bar{F}_{0,nl}^{(0)}(t)\psi_F^{*n+l}\psi_F^{n+l+1}+\langle E(\hat{z}_0)\rangle\bar{F}_{1,nl}^{(1)}(t)(\psi_F^*+\psi_B^*)^{n+l}\psi_F^{n+l+1}\right.$$
$$\left.-\langle E^{-1}(\hat{z}_0)\rangle\bar{F}_{0,nl}^{(1)}(t)\psi_F^{*n+l}(\psi_F+\psi_B)^{n+l+1}]\right]\ . \tag{96}$$

The presence of the functions Y_s in the expressions (95), which gives the functions $\bar{F}_{j,nl}^{(v)}$, does not permit to compact Eq. (96) further, nevertheless, in specific problems the physical conditions should have to simplify Eq. (96) suitably. To conclude, we have obtained the desired expressions that give the time evolution of the photon operators when the conditions (78) and (79) are satisfied. In a forthcoming paper we shall show how the above method turns out to be very useful to study the statistical properties of the field after the SCS process.

VI. CONCLUSIONS

We have studied the time evolution of the operators in the SCS process. The equations of motion have been solved by using iteration methods and the solutions have been presented as a Laplace transform and a subsequent inverse Laplace transform of suitable functionals. The solutions written in this form permit one to analyze the properties of the SCS and of other systems that are described by particular Raman-Nath equations. In fact, the inverse Laplace transform as well as the Laplace transform can be evaluated by using the convolution law or one of the many approximate methods of calculation reported in literature. So many special problems can be studied without applying the usual perturbation techniques.

In order to verify the feasibility of the present method we have analyzed the time evolution of the photon operators in particular cases for which we have been able to write the resulting functionals in analytical form.

ACKNOWLEDGMENTS

This research was supported by the Consiglio Nazionale delle Ricerche (CNR) (Italy) through the Gruppo Nazionale di Elettronica Quantistica e Plasmi and by the Ministero della Pubblica Istruzione (Italy).

[1]E. Schrödinger, Ann. Phys. (Leipzig) **82**, 257 (1927).
[2]P. L. Kapitza and P.A.M. Dirac, Proc. Cambridge Phys. Soc. **29**, 297 (1933).
[3]R. H. Pantell, G. Soncini, and H. E. Puthoff, IEEE J. Quantum Electron. **QE-4**, 905 (1968).
[4]J. M. J. Madey, J. Appl. Phys. **42**, 1906 (1971).
[5]A. Bambini and S. Stenholm, Opt. Commun. **30**, 391 (1979).
[6]G. Dattoli, A. Renieri, F. Romanelli, and R. Bonifacio, Opt. Commun. **34**, 240 (1980).
[7]W. Becker and M. S. Zubairy, Phys. Rev. A **25**, 956 (1982).
[8]W. Becker, M. O. Scully, and M. S. Zubairy, Phys. Rev. Lett. **48**, 475 (1982).
[9]G. Dattoli and A. Renieri, in *Laser Handbook,* edited by M. L. Stitch and M. Bass (North-Holland, Amsterdam, 1985), Vol. 4, p. 1.
[10]P. Bosco and G. Dattoli, J. Phys. A **16**, 4409 (1983).
[11]P. Bosco, G. Dattoli, and M. Richetta, J. Phys. A **17**, 1333 (1984).
[12]C. T. Lee, Phys. Rev. A **31**, 1213 (1985); **38**, 1230 (1988).
[13]C. T. Lee, J. Phys. A **18**, L1139 (1985); **20**, 5473 (1987).
[14]C. W. Raman and N. S. Nath, Proc. Indian Acad. Sci. **2**, 406 (1936).
[15]See, for example, J. H. Eberly, B. W. Shore, Z. Bialynicka-Birula, and I. Bialynicki-Birula, Phys. Rev. A **16**, 2038 (1977); B. W. Shore and J. H. Eberly, Opt. Commun. **24**, 83 (1978).
[16]S. Carusotto, Phys. Rev. A **38**, 3249 (1988).
[17]W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973), Chap. 3.
[18]See Ref. 17, p. 155.
[19]P. Bosco, J. Gallardo, and G. Dattoli, J. Phys. A **17**, 2739 (1984).
[20]A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms* (McGraw-Hill, New York, 1954), Vol. 1, p. 229.