Crucial aspect of beam breakup in a steady-state free-electron laser in the microwave regime

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The beam breakup instability caused by the interaction between beam and induction gaps in a steady-state free-electron laser in the microwave regime is considered. The large energy spread induced by free-electron laser performance is theoretically proved not to lead to Landau damping of the beam breakup instability when the synchrotron frequency is of order or larger than the betatron frequency.

I. INTRODUCTION

For many years it has been realized that the major limitation to the transportation of high-current beams in linear accelerators is the growth of a beam-accelerator cavity instability known as beam breakup (BBU). The beam breakup instability in induction accelerators has been extensively studied from theoretical[2,3] and experimental[4,5] points of view, particularly in the Advanced Test Accelerator (ATA). It has been found that optimal cavity design coupled with the proper magnetic transport system can greatly reduce BBU growth, and the phase mix damping provided by its nonlinear focusing can permit acceleration of high currents to arbitrarily high energies without BBU growth. As expected, the latter fact supports a popular theory that the beam breakup instability can be suppressed by Landau damping due to a spread in the betatron wave number.

Meanwhile, there have been neither experimental demonstrations nor theoretical investigations of BBU in a microwave steady-state free-electron laser[6] (FEL). The microwave FEL is motivated by its use in a two-beam accelerator,[7] where BBU may be particularly serious. An essential issue of the two-beam accelerator concept is how far a kiloampere electron beam in a steady-state FEL can be propagated with tolerable loss of beam quality. Therefore it is very important to estimate the characteristic BBU growth distance $L_{\rm BBU}$.

So far the synchrotron oscillation in a large bucket of a steady-state FEL has been supposed to induce a relatively large energy spread associated with a large spread in the betatron wave number k_β, eventually resulting in Landau damping of the BBU. This hypothesis is easily proven to be valid when the synchrotron frequency ν is sufficiently smaller than the betatron frequency. However, a typically proposed scheme does not have this feature. To demonstrate this we note that the ratio of the betatron frequency to the synchrotron frequency is described by (see the Appendix)

$$\frac{k_\beta}{\nu}=\frac{1}{2}\left[\frac{cB_W}{E_s}\right]^{1/2}\left[1+\frac{\lambda_W\lambda_s}{16b^2}\right], \quad (1)$$

where c is the velocity of light, B_W and E_s are the magnetic and signal field amplitudes, respectively, λ_W and λ_s are the wiggler period and the signal wavelength in a vacuum, respectively, and b is the vertical dimension of the waveguide. The second term on the right-hand side is normally much smaller than unity. The power-density requirements ($\sim$GW/m) in the proposed schemes[6] where permanent wiggler magnets with the nominal surface field of ~ 1 T are employed yield $0<k_\beta/\nu\le 2$.

The Landau damping hypothesis apparently fails in the limit of $k_\beta/\nu\simeq 0$ because the averaged betatron frequency for each particle is that of the synchronous particle, i.e., too fast a modulation produces *nothing.*[8] We are here concerned with the case in which the synchrotron frequency is of the same order of magnitude as the betatron frequency; now it is of most interest whether or not BBU is still prevented from occurring by Landau damping.

One of the main purposes of the present paper is to give an answer to this issue from a theoretical point of view. Present analysis for BBU is based on the BBU model developed by Briggs *et al.*,[3] where the induction gaps couple to each other only through their interaction with the beam. The synchrotron oscillation of each particle is included as the periodic energy modulation. Since we restrict present discussions to the case of a steady-state FEL, the synchronous energy is assumed to be constant.

The organization of the paper is as follows. In Sec. II, the above-mentioned BBU model is described. In Sec. III, we analytically evaluate the dispersion relation for BBU by the Green's function method and prove that Landau damping due to a large energy spread is not expected in a steady-state FEL with the synchrotron oscillation frequency of the same order as the betatron frequency.

II. BBU MODEL

An extremely useful model of BBU will now be described. The induction gaps are treated as discretely distributed along the structure with spacing of L_g. The BBU cavity mode is characterized by its angular frequency ω_λ, its Q or quality factor, and its $Z_\perp/Q$ or transverse shunt impedance. The mode is excited by a dipole current source term which is proportional to the product of the beam's current and transverse displacement. The transverse position of the beam centroid is determined by the external linear focusing, the cavity fields, and the planar wiggler field. Here continuous focusing due to quadrupole and wiggler fields is assumed.

The force exerted on the beam by the mode is due to the transverse magnetic field of the mode. We define the quantity Δ, the z-averaged normalized transverse momentum change of the beam centroid, as $\Delta = \Delta P_\perp / mc$, and the quantity $\xi_{\varepsilon,\varphi}$, the transverse position of each particle. With this definition and the assumption of an isolated deflecting mode the BBU equations are

$$\left[\frac{\partial^2}{\partial t^2} + \frac{\omega_\lambda \partial}{Q\,\partial t} + \omega_\lambda^2\right]\Delta(t,z) = \frac{I_B}{I_0}\omega_\lambda^3 \frac{Z_\perp}{Q}\langle \xi(t,z)\rangle \ , \tag{2a}$$

$$\frac{\partial}{\partial z}\left[[1+\varepsilon\cos(\nu\zeta)]\frac{\partial \xi_{\varepsilon,\varphi}}{\partial z}\right] + k_\beta^2 \xi_{\varepsilon,\varphi} = \frac{\Delta(t,z)}{d\gamma_0}\Theta_I(z) + \frac{\sqrt{2}b_W}{\gamma_0}\cos(k_W z) \ , \tag{2b}$$

with

$$\zeta = z + \frac{\varphi}{\nu} \ ,$$

$$\langle \xi(t,z)\rangle \equiv \frac{1}{2\pi}\int\int_0^{2\pi} d\varepsilon\, d\varphi\, \rho(\varepsilon)\xi_{\varepsilon,\varphi}(t,z) \ ,$$

$$\Theta_I(z) = \begin{cases} 1, & nL_g \le z \le nL_g + d, \quad n = 1,2,3,\dots \\ 0 & \text{elsewhere} \end{cases}$$

$$= \frac{d}{L_g} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[\frac{n\pi d}{L_g}\right]\cos\left[\frac{2n\pi z}{L_g}\right] \ ,$$

where $\langle \xi(t,z)\rangle$ is the displacement of the beam centroid, I_B and I_0 are the beam and Alfvén currents, respectively, ε and φ are the maximum relative energy deviation and the initial phase for each particle, d is the induction gap size, b_W is the normalized wiggler field amplitude, γ_0 is the synchronous energy, and $\rho(\varepsilon)$ is the distribution function. Now, introducing the variable $\tau = t - z/c$ that measures the time delay behind the beam pulse head (we neglect the change in the longitudinal velocity of the particle) and averaging (2b) over a wiggler period and over one period of the induction module (we assume the crude relation among characteristic distances: $L_{\rm BBU} > 2\pi/k_\beta$, $2\pi/\nu > L_g \gg 2\pi/k_W$), the BBU model equations become

$$\left[\frac{\partial^2}{\partial \tau^2} + \frac{\omega_\lambda \partial}{Q\,\partial \tau} + \omega_\lambda^2\right]\Delta(\tau,z) = \frac{I_B}{I_0}\omega_\lambda^3 \frac{Z_\perp}{Q}\langle \xi(\tau,z)\rangle \ , \tag{3a}$$

$$\left[\frac{\partial}{\partial z}[1+\varepsilon\cos(\nu\zeta)]\frac{\partial}{\partial z} + k_\beta^2\right]\xi_{\varepsilon,\varphi}(\tau,z) = \frac{\Delta(\tau,z)}{d\gamma_0}\overline{\Theta}_I(z) \ . \tag{3b}$$

$\overline{\Theta}_I(z) = d/L_g$ is taken in the following derivation. The former represents the time evolution of momentum gain proportional to magnetic wake fields in the accelerating gap located at z after pulse head arrival and the latter represents the orbital evolution of transverse position of the ith particle in the slice at pulse position τ behind the pulse head.

III. DISPERSION RELATION FOR BBU

We now try to solve Eqs. (3a) and (3b). Introducing a new variable

$$\eta_{\varepsilon,\varphi} = [1+\varepsilon\cos(\nu\zeta)]^{1/2}\xi_{\varepsilon,\varphi} \ ,$$

(3b) becomes

$$\left\{\frac{\partial^2}{\partial z^2} + \left[k_\beta^2 + \varepsilon\left[\frac{\nu^2}{2} - k_\beta^2\right]\cos(\nu\zeta)\right]\right\}\eta_{\varepsilon,\varphi} = \frac{\Delta(\tau,z)}{d\gamma_0}\overline{\Theta}_I(z) \ . \tag{4}$$

Here $\varepsilon_{\rm max} = \sigma < 1$ allows the first-order approximation on the left-hand side with respect to ε. For a small amplitude oscillation the BBU exciting term is a perturbation; therefore the modulation term $\varepsilon\cos(\nu\zeta)$ on the right-hand side is neglected to a first-order approximation. We may solve Eqs. (3a) and (4) by first Fourier transforming in the variable τ to ω. Transformed quantities are denoted by tildes. Then we have

$$\left[\omega_\lambda^2 - \omega^2 + i\frac{\omega\omega_\lambda}{Q}\right]\tilde{\Delta} = \omega_\lambda^3 \frac{I_B Z_\perp}{I_0 Q}\langle \tilde{\xi}\rangle \ , \tag{5a}$$

$$\frac{\partial^2 \tilde{\eta}_{\varepsilon,\varphi}}{\partial z^2} + \left[k_\beta^2 + \varepsilon\left[\frac{\nu^2}{2} - k_\beta^2\right]\cos(\nu\zeta)\right]\tilde{\eta}_{\varepsilon,\varphi} = \frac{\tilde{\Delta}}{d\gamma_0}\overline{\Theta}_I(z) \ . \tag{5b}$$

We solve (5b) by using the Green's function method. The exact form of general solution to the homogeneous part of (5b) must be known to make the Green's function. The homogeneous equation to (5b) is a Mathieu equation. Although it may be possible to write solutions in terms of the Mathieu function, they are in general very complicated; therefore we resort to an approximation of a very compact form as described below.

We introduce nonlinearization of the Mathieu coefficient

$$k_\beta^2 + \varepsilon\left[\frac{\nu^2}{2} - k_\beta^2\right]\cos(\nu\zeta) \ .$$

With the variable $Z = \nu\zeta/2$ a Mathieu-like equation

$$d^2x/dz^2 + [a + b\cos(\nu\zeta)]x = 0 \ ,$$

where $a = k_\beta^2$, $b = \varepsilon(\nu^2/2 - k_\beta^2)$, is rewritten as

$$d^2x/dZ^2 + [a' + b'\cos(2Z)]x = 0 \ ,$$

where $a'=(4/\nu^2)a$, $b'=(4/\nu^2)b$. Then the Mathieu coefficient $a'+b'\cos(2Z)$ is nonlinearized in the form

$$a'+b'\cos(2Z)\simeq 1+\frac{(a'-1)}{\left[1+\frac{(-b')}{2(a'-1)}\cos(2Z)\right]^2}\ , \tag{6}$$

which is a sufficiently good approximation to the original form when

$$|(-b'/2(a'-1)|<1\ .$$

The Hill equation with such a nonlinearized Mathieu coefficient as (6) is known to admit exact solutions of the form[9]

$$x_{\pm}(Z)=[1+G\cos(2Z)]^{1/2}\exp\left[\pm i\frac{\sqrt{1+\lambda}}{2}\sin^{-1}\left[(1-G^2)^{1/2}\frac{\sin(2Z)}{1+G\cos(2Z)}\right]\right]\ , \tag{7}$$

where

$$G=\frac{-b'}{2(a'-1)}\ , \tag{8}$$

$$\lambda=\frac{a'-1}{1-G^2}=\frac{4(a'-1)^3}{4(a'-1)^2-b'^2}\ . \tag{9}$$

Expressions (8) and (9) are represented by original quantities as

$$G=\varepsilon\frac{\nu^2-2k_\beta^2}{\nu^2-4k_\beta^2}=\varepsilon g,\quad \nu\neq 2k_\beta \tag{10}$$

$$\lambda=-\frac{1-4(k_\beta^2/\nu^2)}{1-g^2\varepsilon^2}\ . \tag{11}$$

For $k_\beta/\nu<\frac{1}{2}$, $g\sim 1$ and for $k_\beta/\nu>\frac{1}{2}$, $g<\frac{1}{2}$ as shown in Fig. 1; thus $G=g\varepsilon<1$ except for $k_\beta/\nu\simeq\frac{1}{2}$. This enables us to expand (7) to first order in G; we set

$$\sin\varphi=\frac{(1-G^2)^{1/2}\sin(\nu\zeta)}{1+G\cos(\nu\zeta)}\simeq\frac{\sin(\nu\zeta)}{1+G\cos(\nu\zeta)}\ ,$$

then

$$\varphi=\nu\zeta-G\sin(\nu\zeta)\ .$$

Using this result and $\sqrt{\lambda+1}/2\simeq k_\beta/\nu$, we have

$$\begin{aligned}x_{\pm}(Z)&=\eta^{\pm}_{\varepsilon,\varphi}(z)\\&=[1+(G/2)\cos(\nu\zeta)]\\&\quad\times\exp\left[\pm ik_\beta\left[\zeta-\frac{G}{\nu}\sin(\nu\zeta)\right]\right]\ .\end{aligned} \tag{12}$$

Employing terms of the Bessel function, expression (12) is described by

$$\eta^{\pm}_{\varepsilon,\varphi}(z)=\left[1+\frac{G}{2}\cos(\nu\zeta)\right]\times\sum_{n=-\infty}^{\infty}J_n\left[\frac{k_\beta}{\nu}G\right]e^{\pm i(k_\beta-n\nu)\zeta}\ . \tag{13}$$

Meanwhile, a formal solution to Eq. (5a) is written in terms of the initial condition and the Green's function as

$$\tilde{\eta}_{\varepsilon,\varphi}(\omega,z)=\tilde{f}(\omega,z)+\frac{1}{L_g\gamma_0}\int_{-\infty}^{z}G(z,z')\tilde{\Delta}(\omega,z')dz'\ . \tag{14}$$

Multiplying both sides by $[1+\varepsilon\cos(\nu\zeta)]^{1/2}$, substituting the expression

$$\tilde{\Delta}(\omega,z')=h(\omega)\langle\tilde{\xi}(\omega,z')\rangle\ ,$$

where

$$h(\omega)=\frac{\omega_\lambda^3}{\omega_\lambda^2-\omega^2+i\omega\omega_\lambda/Q}\frac{I_B}{I_0}\left[\frac{Z_\perp}{Q}\right]\ ,$$

into the left-hand side, and averaging both sides over the distribution of energy spread and the initial phase, we have

$$\langle\tilde{\xi}(\omega,z)\rangle=\langle[1+\varepsilon\cos(\nu\zeta)]^{1/2}\tilde{f}(\omega,z)\rangle+\frac{h(\omega)}{L_g\gamma_0}\int_{-\infty}^{z}dz'\langle\tilde{\xi}(\omega,z')\rangle\frac{1}{2\pi}\int_0^\sigma\int_0^{2\pi}d\varepsilon\,d\varphi\,\rho(\varepsilon)G(z,z')\ , \tag{15}$$

where σ is the maximum deviation from the synchronous energy. Substitution of expression (13) into the definition form of the Green's function,

$$G(z,z')=\frac{1}{W}[-\eta^{+}_{\varepsilon,\varphi}(z)\eta^{-}_{\varepsilon,\varphi}(z')+\eta^{+}_{\varepsilon,\varphi}(z')\eta^{-}_{\varepsilon,\varphi}(z)]\ ,$$

with the Wronskian,

$$W=\eta^{+}_{\varepsilon,\varphi}(0)\frac{\partial\eta^{-}_{\varepsilon,\varphi}(0)}{\partial z}-\frac{\partial\eta^{+}_{\varepsilon,\varphi}(0)}{\partial z}\eta^{-}_{\varepsilon,\varphi}(0)\ ,$$

yields

$$G(z,z')=\frac{1}{k_\beta}\left[1+\frac{G}{2}[\cos(\nu\zeta)+\cos(\nu\zeta')]\right]\sum_{m,n=-\infty}^{\infty}J_m\left[\frac{k_\beta}{\nu}G\right]J_n\left[\frac{k_\beta}{\nu}G\right]\sin[(k_\beta-m\nu)\zeta-(k_\beta-n\nu)\zeta'] \ . \tag{16}$$

Using (16), the integral in (15) reduces to

$$\frac{1}{2\pi}\int_0^\sigma\int_0^{2\pi}d\varepsilon\, d\varphi\, \rho(\varepsilon)G(z,z')=\frac{1}{k_\beta}\sum_{m,n=-\infty}^{\infty}a_{mn}\left[\frac{k_\beta}{\nu},\sigma\right]\left[\frac{1}{2\pi}\right]\int_0^{2\pi}d\varphi\sin[k_\beta(z-z')-\nu(mz-nz')-(m-n)\varphi]$$

$$=\frac{1}{k_\beta}\sum_{m,n=-\infty}^{\infty}a_{mn}\left[\frac{k_\beta}{\nu},\sigma\right]\sin[k_\beta(z-z')-\nu(mz-nz')]\delta_{mn} \ , \tag{17}$$

where

$$a_{mn}\left[\frac{k_\beta}{\nu},\sigma\right]=\int_0^\sigma\rho(\varepsilon)J_m\left[\frac{k_\beta}{\nu}g\varepsilon\right]J_n\left[\frac{k_\beta}{\nu}g\varepsilon\right]d\varepsilon \ . \tag{18}$$

Here we assume

$$\left[1+\frac{G}{2}[\cos(\nu\zeta)+\cos(\nu\zeta')]\right]\simeq 1 \ ,$$

since the kick term driving BBU is a small perturbation on the homogeneous betatron motion for a finite distance and its ripple can be neglected. Assumption of a flat distribution

$$\rho(\varepsilon)=\begin{cases}1, & \varepsilon\le\sigma\\ 0, & \varepsilon>\sigma\end{cases}$$

leads to

$$a_{mm}\left[\frac{k_\beta}{\nu},\sigma\right]=\frac{1}{\sigma}\int_0^\sigma J_m^2\left[\frac{k_\beta}{\nu}g\varepsilon\right]d\varepsilon$$

$$=\frac{\nu}{\sigma k_\beta g}\int_0^{(k_\beta/\nu)g\varepsilon}J_m^2(q)dq \ . \tag{19}$$

Thus the integral equation (15) is identified as a Volterra equation of the second kind,

$$X(z)=Q(z)+\frac{h(\omega)}{L_g\gamma_0 k_\beta}\sum_{m=-\infty}^{\infty}a_{mm}\int_{-\infty}^{z}dz'X(z')\sin[(k_\beta-\nu m)(z-z')] \ , \tag{20}$$

where the abbreviations $X(z)=\langle\tilde{\xi}(\omega,z)\rangle$ and $Q(z)=\langle[1+\varepsilon\cos(\nu\zeta)]^{1/2}\tilde{f}(\omega,z)\rangle$ are used. Utilizing a Faltung theorem, Eq. (20) can be solved by the Laplace transformation. If we write $X(p)$ as the transform of $X(z)$ and $Q(p)$ as the transform of $Q(z)$, the transformed equation of (20) is written as

$$X(p)=Q(p)+\frac{h(\omega)}{L_g\gamma_0 k_\beta}\sum_{m=-\infty}^{\infty}a_{mm}\frac{k_\beta-\nu m}{p^2+(k_\beta-\nu m)^2}X(p) \ . \tag{21}$$

The Laplace inverse transform of $X(p)$ gives the solution

$$X(z)\equiv\langle\tilde{\xi}(\omega,z)\rangle=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{Q(p)e^{pz}}{\left[1-\dfrac{h(\omega)}{L_g\gamma_0 k_\beta}\displaystyle\sum_{m=-\infty}^{\infty}\frac{a_{mm}(k_\beta-\nu m)}{p^2+(k_\beta-\nu m)^2}\right]}dp \ , \tag{22}$$

where c is positive and must be larger than all the real parts of the poles of the integrand. From the theory of residue, the integral is evaluated in the form

$$X(z)=\sum_j A_j(\omega)\exp[p_0^j(\omega)z] \ , \tag{23}$$

where $p_0^j(\omega)$ is the zero point of the denominator in the above integrand. Finally, the Fourier inverse transform of $X(z)$ gives

$$\langle\xi(z)\rangle=\sum_j\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A_j(\omega)e^{p_0^j(\omega)z}e^{+i\omega\tau}d\omega \ . \tag{24}$$

The integral may be evaluated by the method of steepest descents to yield

$$\langle\xi(z)\rangle\propto A_j(\omega_s)\exp[p_0^j(\omega_s)z+i\omega_s\tau] \ , \tag{25}$$

where the saddle point ω_s satisfies the following equation:

$$\frac{dp_0^j(\omega_s)}{d\omega}+i\tau=0 \ . \tag{26}$$

The real part of $p_0^j(\omega_s)$ will determine the stability for BBU.

We need only consider nontrivial poles of the zero points of the denominator in (22); the problem finally reduces to a mathematical problem of solving the so-called dispersion relation

$$1-\frac{h(\omega)}{L_g\gamma_0 k_\beta}\sum_{m=-\infty}^{\infty}\frac{a_{mm}(k_\beta-\nu m)}{p^2+(k_\beta-\nu m)^2}=0\ , \tag{27}$$

where

$$a_{mm}=\frac{1}{\sigma}\int_0^\sigma J_m^2\left[\frac{k_\beta}{\nu}g\varepsilon\right]d\varepsilon\ . \tag{28}$$

It hardly seems easy to evaluate zero points from (28) for an arbitrary ν because the finite integral (28) is not represented in terms of elementary functions. However, we are concerned with the case of $k_\beta/\nu \simeq 1-3$ as stated in the Introduction. From the assumption of $k_\beta/\nu \simeq 1-3$, the parameter $(k_\beta/\nu)g\sigma$ is apparently less than unity; therefore summing the first few terms $(m,n=0,\pm 1)$ in Eq. (20) leads to a sufficiently good approximation because of $J_m(z) \ll 1$ when $z<1$ for $|m|\geq 2$. Using the expressions $J_0(z)\simeq 1-(z^2/4)$ and $J_1(z)\simeq z/2$ and performing integrations, we have

$$a_{00}=1-\tfrac{1}{6}\delta^2,\quad \delta\equiv\frac{k_\beta}{\nu}g\sigma$$

$$a_{11}=a_{-1-1}=\tfrac{1}{12}\delta^2\ .$$

Then, we have a dispersion relation

$$1-\frac{h(\omega)}{L_g\gamma_0}\left[\frac{1}{p^2+k_\beta^2}+\frac{\delta^2}{12}\left[\frac{-2}{p^2+k_\beta^2}+\frac{1-\nu/k_\beta}{p^2+(k_\beta-\nu)^2}+\frac{1+\nu/k_\beta}{p^2+(k_\beta+\nu)^2}\right]\right]=0\ . \tag{29}$$

We seek the poles from this dispersion relation which should be close to the poles in the unmodulated system, $p_0(\sigma=0)$. Setting $w=w_0+u$, where $w=p^2$ and $w_0=p_0^2(\sigma=0)$, substituting this relation into Eq. (29), and remaining at first order for u and δ^2, we have

$$1-\frac{\bar{h}(\omega)}{w_0+k_\beta^2}=0,\quad \bar{h}(\omega)\equiv\frac{h(\omega)}{L_g\gamma_0} \tag{30a}$$

$$u=\frac{\delta^2}{6}\bar{h}(\omega)\left[-1+\bar{h}(\omega)\left[\frac{\bar{h}(\omega)-2\nu^2}{[\bar{h}(\omega)+\nu^2]^2-4k_\beta^2\nu^2}\right]\right]\ . \tag{30b}$$

An assumption of strong focus ($\bar{h}/k_\beta^2$, $\bar{h}/\nu^2 \ll 1$) leads to

$$p_0(\omega)=-ik_\beta+\frac{i\bar{h}(\omega)}{2k_\beta}\left[1-\frac{\delta^2}{6}\right]\ . \tag{31}$$

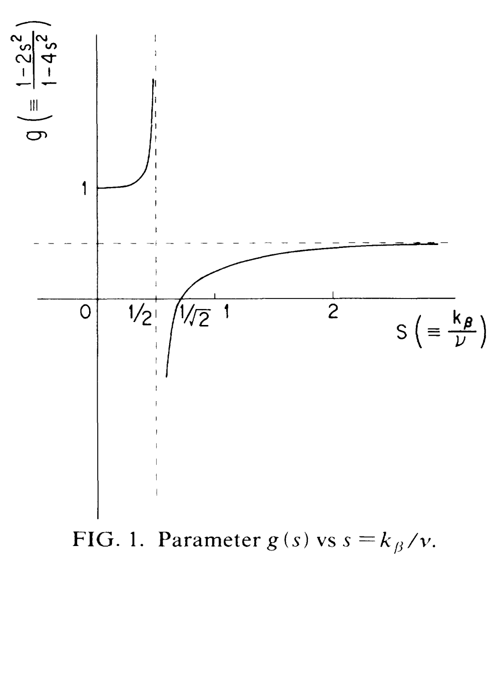

FIG. 1. Parameter $g(s)$ vs $s=k_\beta/\nu$.

Now we set the index of the exponential term in (24) to be $W(\omega)$,

$$W(\omega)=p_0(\omega)z+i\omega\tau\ . \tag{32}$$

Substitution of (31) into (32) yields

$$W(\omega)=-ik_\beta z+\frac{ih(\omega)}{2L_g k_\beta\gamma_0}\left[1-\frac{\delta^2}{6}\right]z+i\omega\tau\ . \tag{33}$$

From our recognition that an ample contribution of the integral (24) comes from the vicinity of $\omega=\omega_\lambda$, we introduce a convenient variable

$$\Omega=\omega-\omega_\lambda \tag{34}$$

and rewrite (33) in terms of Ω,

$$W(\Omega)=-ik_\beta z+i\omega_\lambda\tau+i\Omega\tau+iK\left[1-\frac{\delta^2}{6}\right]z\ , \tag{35}$$

where the abbreviation $K\equiv h(\omega)/2L_g k_\beta\gamma_0$ is used. Then, the Fourier inverse transformation is written as

$$\langle\xi(\tau,z)\rangle \propto \exp(-ik_\beta z+i\omega_\lambda\tau)$$
$$\times\int A(\Omega+\omega_\lambda)$$
$$\times\exp[i\Omega\tau+iK(1-\delta^2/6)z]d\Omega\ . \tag{36}$$

Since K is a function of ω or Ω, it is straightforward to seek a saddle point K_s instead of ω_s or Ω_s. Employing the approximate form of $h(\omega)$ in the vicinity of $\omega=\omega_\lambda$,

$$h(\omega)=\frac{1}{2}\frac{I_B}{I_0}\left[\frac{Z_\perp}{Q}\right]\frac{\omega_\lambda^2}{-\Omega+i\alpha},\quad \alpha\equiv\frac{\omega_\lambda}{2Q} \tag{37}$$

K becomes

$$K\simeq\frac{B\omega_\lambda}{2(\Omega-i\alpha)},\quad B\equiv\frac{\omega_\lambda}{2L_g k_\beta\gamma_0}\frac{I_B}{I_0}\left[\frac{Z_\perp}{Q}\right] \tag{38}$$

and then

$$\Omega=\frac{B\omega_\lambda}{2K}+i\alpha \ . \quad (39)$$

The index of the exponential term in (36) is represented as a function of K by

$$\Phi(K)=i\Omega\tau+iK\left[1-\frac{\delta^2}{6}\right]z$$
$$=-\alpha\tau+i\frac{B\omega_\lambda}{2K}\tau+iK\left[1-\frac{\delta^2}{6}\right]z \ . \quad (40)$$

The saddle point K_s satisfies $\partial\Phi/\partial K=0$ (Ref. 10) and is written as

$$K_s=-i\left[\frac{|B|\omega_\lambda\tau}{2z(1-\delta^2/6)}\right]^{1/2} \ . \quad (41)$$

Substituting (41) into (40), we have

$$\Phi(K_s)=-\alpha\tau+\left[2|B|\omega_\lambda\tau\left[1-\frac{\delta^2}{6}\right]z\right]^{1/2} \ . \quad (42)$$

Accordingly, the asymptotic form of integral (36) becomes

$$\langle\xi(\tau,z)\rangle\propto\exp(-ik_\beta z+i\omega_\lambda\tau)\exp\{-\alpha\tau+[2|B|\omega_\lambda\tau(1-\delta^2/6)z]^{1/2}\} \ . \quad (43)$$

The last exponential term represents the BBU growth in a steady-state FEL. If $\delta=0$, this is in agreement with Eq. (5.13) in Ref. 2, where the BBU growth rate for a conventional linac has been derived by a different technique from that presented here. Equation (43) describes the horizontal excursion of the centroid of the beam segment placed at τ behind the beam head propagating through a steady-state FEL by z.

The nonoscillating phase of (43) represented by $\Psi(\tau)$ takes its maximum value

$$\Psi(\tau_{\max})=\frac{|B|\omega_\lambda}{2\alpha}\left[1-\frac{\delta^2}{6}\right]z \quad (44)$$

for

$$\tau_{\max}=\frac{1}{2\alpha^2}\left[|B|\omega_\lambda\left[1-\frac{\delta^2}{6}\right]z\right] \ .$$

Eventually, we can evaluate the BBU e-folding distance as follows:

$$L_{\rm BBU}=\frac{z}{\Psi(\tau_{\max})}$$
$$=\frac{2L_g k_\beta\gamma_0}{\omega_\lambda Z_\perp}\left[\frac{I_0}{I_B}\right]\frac{1}{1-\delta^2/6} \ . \quad (45)$$

If Eq. (43) is written in terms of $L_{\rm BBU}(\sigma=0)$ and original parameters, we have

$$L_{\rm BBU}(\sigma)=\frac{L_{\rm BBU}(\sigma=0)}{1-\frac{1}{6}[(k_\beta/\nu)g]^2\sigma^2},\quad g\equiv\frac{\nu^2-2k_\beta^2}{\nu^2-4k_\beta^2} \ . \quad (46)$$

For the present case, the BBU growth distance falls in the range below,

$$L_{\rm BBU}(\sigma=0)<L_{\rm BBU}(\sigma)\le\tfrac{6}{5}L_{\rm BBU}(\sigma=0) \ , \quad (47)$$

because $[(k_\beta/\nu)g]^2\sigma^2\sim 1$ at most.

This result should not be so surprising. Equation (13) indicates that the frequency modulated betatron oscillation involves an infinite number of eigenmodes with the frequency $|k_\beta\pm n\nu|$ and the relative strength of these modes is determined by the Bessel-function term which is a function of the betatron and synchrotron frequencies and the energy spread. This discrete spectrum of oscillation mode tends to localize at $k_\beta(1-\varepsilon/2)$ in the limit of $k_\beta/\nu\rightarrow\infty$, yielding an effective spread in the betatron frequency of the beam. The spread leads to Landau damping of the BBU. When $k_\beta/\nu\sim 1$, on the other hand, there are only three dominant modes of k_β, $|k_\beta\pm\nu|$ as derived above. It is easily supposed that interference among different spectra consisting of three lines is quite weak. In fact, we have mathematically proved it.

IV. SUMMARY

The integral equation for BBU growth has been evaluated in a compact form, introducing a novel technique of nonlinearization of the Mathieu coefficient, and the dispersion relation which is almost valid except for the vicinity of a singular point $k_\beta/\nu=\frac{1}{2}$ has been derived. In the region of $k_\beta\sim\nu$ of particular interest, we have calculated saddle points from the dispersion relation and finally arrived at a formula for a characteristic BBU growth distance $L_{\rm BBU}$ which is a function of the synchrotron and betatron frequencies and the energy spread. From the expression of $L_{\rm BBU}$, we realize that enlargement in $L_{\rm BBU}$ due to the energy spread is quite small. Accordingly, we conclude that a large energy spread particularly for a FEL in the microwave regime does not contribute to Landau damping of the BBU.

The above singularity has been artificially introduced in the process of Mathieu coefficient nonlinearization; it is not intrinsic to the nature of the frequency modulating system. The present result, $L_{\rm BBU}(\sigma)\sim L_{\rm BBU}(\sigma=0)$, is expected to be valid over a whole range of $\nu\sim k_\beta$ including $k_\beta/\nu=\frac{1}{2}$.

The BBU growth formula gives $L_{\rm BBU}=71$ m with typical parameters[11] of $I_B=2$ kA, $k_\beta=2\pi/3$ m^{-1}, $L_g=2$ m, $\gamma=40$, $\omega_\gamma Z_\perp=0.4$ cm^{-1}. This value is crucial for a steady-state FEL employed in a two-beam accelerator. One would have hoped for beam transport over a greater distance for higher conversion efficiency from beam power to microwave power. This requirement may be satisfied in two possible ways. One of those is to use induction gaps with the same accelerating voltage but slightly different deflecting mode frequencies; Landau damping of the BBU can be expected because of dephasing by the frequency spread. The other is to introduce a sufficient spread in the betatron number caused by nonlinearity as seen in the ion focusing regime.[5] The latter[12] has been proposed in Ref. 11, where a possibility of ion channel guiding is theoretically anticipated.

The present theory is general for the beam breakup instability in a frequency modulated system. For instance, the present conclusion can be applied to the case of a relativistic klystron[13] (RK), which also is motivated by its use in a two-beam accelerator, if it is driven with a low energy. Unlike a steady-state free-electron laser, however, k_β/ν in a relativistic klystron is proportional to $\gamma^{1/2}$; therefore Landau damping will be expected when a RK is operated with a sufficiently large γ. $L_{\rm BBU}(\sigma)$ in such a case must be analytically derived by solving the original dispersion relation (26) or obtained by computer simulations. However, both are out of the present scope.

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APPENDIX

Following the usual definition for FEL quantities, the synchrotron frequency is represented by

$$\nu=\gamma_0^{-1}\left[\frac{2e_s b_W \cos(\phi_r)\omega_s(k_W-\delta k_s)}{c k_s k_W}\right]^{1/2}, \qquad (A1)$$

where γ_0 is the resonance energy in the rest mass unit, e_s and b_W are the normalized signal field and wiggler field, k_s and k_W are the wave numbers of the signal field in a vacuum and of the wiggler field, δk_s is the difference between k_s and the wave number in a waveguide, ω_s is the angular frequency of the signal field, and ϕ_r is the resonant phase $\sim 0°$. Meanwhile, the betatron wave number is given by

$$k_\beta=\frac{b_W}{\sqrt{2}\gamma_0} . \qquad (A2)$$

Here we assume the intrinsic focusing force originating from the spatially varying wiggler field is reduced by the external quadrupole focusing force by a factor of 2. From (A1) and (A2), we have

$$\begin{aligned}\frac{k_\beta}{\nu}&=\frac{1}{2}\left[\frac{b_W}{e_s}\right]^{1/2}\left\{1-\left[\frac{\lambda_W}{\lambda_s}\right]+\left[\left[\frac{\lambda_W}{\lambda_s}\right]^2-\left[\frac{\lambda_W}{2b}\right]^2\right]^{1/2}\right\}^{-1/2}\\ &\simeq\frac{1}{2}\left[\frac{b_W}{e_s}\right]^{1/2}\left[1+\frac{\lambda_s\lambda_W}{16b^2}\right]\\ &=\frac{1}{2}\left[\frac{cB_W}{E_s}\right]^{1/2}\left[1+\frac{\lambda_W\lambda_s}{16b^2}\right], \end{aligned} \qquad (A3)$$

where the meanings of E_s, B_W, λ_s, λ_W, and b are given in the text. Substitution of numerical parameters $E_s=100$ MV/m and $B_W=3$ kG into (A3) yields

$$\frac{k_\beta}{\nu}\sim\frac{1}{2} . \qquad (A4)$$

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