Phase properties of the quantized single-mode electromagnetic field

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The usual mathematical model of the single-mode electromagnetic field is the harmonic oscillator with an infinite-dimensional state space, which unfortunately cannot accommodate the existence of a Hermitian phase operator. Recently we indicated that this difficulty may be circumvented by using an alternative, and physically indistinguishable, mathematical model of the single-mode field involving a finite but arbitrarily large state space, the dimension of which is allowed to tend to infinity *after* physically measurable results, such as expectation values, are calculated. In this paper we investigate the properties of a Hermitian phase operator which follows directly and uniquely from the form of the phase states in this space and find them to be well behaved. The phase-number commutator is not subject to the difficulties inherent in Dirac's original commutator, but still preserves the commutator–Poisson-bracket correspondence for physical field states. In the quantum regime of small field strengths, the phase operator predicts phase properties substantially different from those obtained using the conventional Susskind-Glogower operators. In particular, our results are consistent with the vacuum being a state of random phase and the phases of two vacuum fields being uncorrelated. For higher-intensity fields, the quantum phase properties agree with those previously obtained by phenomenological and semiclassical approaches, where such approximations are valid. We illustrate the properties of the phase with a discussion of partial phase states. The Hermitian phase operator also allows us to construct a unitary number-shift operator and phase-moment generating functions. We conclude that the alternative mathematical description of the single-mode field presented here provides a valid, and potentially useful, quantum-mechanical approach for calculating the phase properties of the electromagnetic field.

I. INTRODUCTION

The single-mode electromagnetic field is a well-known physical system which has been successfully modeled by the quantum harmonic oscillator. Indeed, the success of quantum electrodynamics, based on Dirac's approach,[1] is undeniable. For a long time, however, the nature of the phase of the quantized field has remained an enigma. The oscillator model produced a suitable Hermitian energy operator (or number operator[2] $\hat{N}$) but there was no corresponding Hermitian phase operator.[3–5] This placed the phase in the almost unique[6] position of being a classical observable without a corresponding Hermitian operator counterpart.[3]

While most experiments involved thermal or vacuum fields, the problems associated with quantum optical phase were not important. However, the advent of the maser and, more recently, work on squeezed light[8,9] has renewed interest in the problem.[10] It is tantalizing that a strong coherent field and even a suitably squeezed state with a large coherent amplitude[9] satisfy a phenomenological number-phase uncertainty relation:

$$\Delta N \Delta\phi \geq \tfrac{1}{2} \; . \tag{1.1}$$

This is precisely the relation that has been calculated[11] by using Dirac's quantum relation[1,12]

$$[\hat{\phi},\hat{N}] = -i \; , \tag{1.2}$$

which is now known to be incorrect.[3–5]

Dirac obtained the commutator (1.2) by employing the correspondence between commutators and classical Poisson brackets.[13] The Hermitian number and phase operators were combined in a polar decomposition of the photon annihilation operator:

$$\hat{a} = \exp(i\hat{\phi})\hat{N}^{1/2} \; . \tag{1.3}$$

The assumed Hermiticity of $\hat{\phi}$ implied the unitarity of $\exp(i\hat{\phi})$. The difficulties with this approach, later realized by Dirac himself,[14] were clearly pointed out by Susskind and Glogower.[4] Firstly, the uncertainty relation (1.1) would imply that a well-defined number state would have a phase uncertainty of greater than 2π. This is a symptom of the fact that the commutator (1.2) does not take account of the periodic nature of the phase. Secondly, the commutator (1.2) gives rise to an inconsistency when matrix elements of the commutator are calculated in a number-state basis.[3] Finally, the "exponential" operator $[\exp(i\hat{\phi})]$ derived from this approach is not unitary and so does not define an Hermitian $\hat{\phi}$.[4] The failure of Dirac's approach means that it is now often accepted that a well-behaved Hermitian phase operator does not exist.[3–5,15]

The difficulties associated with the periodicity of the phase operator also arise in the classical context. However, the apparent nonunitarity of $\exp(i\hat{\phi})$ [as derived from Eq. (1.3)] is more serious. The action of $\exp(i\hat{\phi})$ on a number state $|n\rangle$ can be determined from (1.3) except when $n=0$. The action of $\exp(i\hat{\phi})$ on the vacuum is indeterminate. The additional condition imposed in the Susskind-Glogower formalism is that

$$\exp(i\hat{\phi})|0\rangle=0 \ . \tag{1.4}$$

This condition immediately destroys the unitarity of $\exp(i\hat{\phi})$.

A well-behaved Hermitian phase operator $\hat{\phi}$ would lead to a unitary operator $\exp(i\hat{\phi})$ whose action on the vacuum state is well defined. There should be no need for additional constraints like (1.4). Moreover, the commutator of the number and phase operators should not suffer from the inconsistencies of Dirac's commutator (1.2). Because of the failure of Poisson–bracket-commutator correspondence and decomposition of the annihilation operator, an entirely new approach seems to be required.

We have recently indicated a new approach[16] which circumvents the difficulties discussed above. This involves describing the field mode in a finite but arbitrarily large state space of $s+1$ dimensions. Physical properties such as operator expectation values are evaluated in the limit as s tends to infinity. In this paper we explore the details and further consequences of our approach. We conclude that our approach and the conventional infinite state space model are *physically indistinguishable*. However, our method has the additional advantage of being able to incorporate a well-behaved Hermitian phase operator within the formalism. Our phase operator has properties which would normally be associated with phase, both in the classical regime and in the quantum regime of very low intensity fields. The resulting number-phase commutator does not lead to any inconsistencies yet satisfies the condition for Poisson–bracket-commutator correspondence. Periodic operator functions of the Hermitian phase operator can be defined. These are found to have very different properties from the conventional Susskind-Glogower operators[4,5] in the quantum region. For example, our phase operator is consistent with the vacuum being a state of random phase. The Susskind-Glogower operators are *not* consistent with the vacuum being a state of random phase.

II. CLASSICAL PHASE

It is not our intention to derive a phase operator from the commutator–Poisson-bracket correspondence. Nevertheless, it is essential that a meaningful phase operator should reproduce the classical phase in the appropriate limit. In this section we study the behavior of a classical oscillator phase ϕ, and pay particular attention to the problem of the multivalued nature of the phase.

The classical Hamiltonian for a unit mass harmonic oscillator (or single-mode field) is

$$H=\tfrac{1}{2}(p^2+\omega^2x^2), \tag{2.1}$$

where ω is the angular frequency of the oscillator. The phase of the oscillator is

$$\phi=\arctan[p/(\omega x)] \tag{2.2}$$

and is a multivalued property because the arctangent function only defines $\phi \bmod 2\pi$. If we allow ϕ to take a continuous range of values then the rate of change of ϕ has the standard form

$$\frac{d\phi}{dt}=\{\phi,H\}=-\omega \ , \tag{2.3}$$

where $\{\ \}$ denotes a Poisson bracket. However, it is more useful to restrict ϕ to lie within a specified 2π interval, $\theta_0\le\phi<\theta_0+2\pi$. A common choice of θ_0 might be 0 or $-\pi$, but we retain here the general case of arbitrary θ_0. We denote this choice of range by adding a subscript θ to the phase, thus ϕ_θ is restricted to lie in the range $\theta_0\le\phi_\theta<\theta_0+2\pi$. When the phase is restricted in this manner, the Poisson bracket equation of motion for the phase becomes

$$\frac{d\phi_\theta}{dt}=\{\phi_\theta,H\}=-\omega[1-2\pi\delta(\phi-\theta_0)] \ . \tag{2.4}$$

The phase ϕ_θ is a periodic sawtooth function of time: it decreases with slope $-\omega$ until it reaches the value θ_0, where it is immediately stepped by 2π and decreases towards θ_0 again.

Equation (2.4) describes the motion of a single classical oscillator. To describe the behavior of an ensemble of differently phased oscillators we use the phase probability function $P(\phi)$, where $P(\phi)d\phi$ is the probability of finding that the phase of a particular oscillator is in the range ϕ to $\phi+d\phi$. The phase probability function is normalized in the chosen 2π interval:

$$\int_{\theta_0}^{\theta_0+2\pi}P(\phi)d\phi=1 \ . \tag{2.5}$$

In the time interval between t and $t+\delta t$ the associated change $\delta\phi$ for a particular oscillator will have a contribution $-\omega\delta t$ from the first term in Eq. (2.4). If the value of ϕ_θ strays beyond θ_0 during δt, then the second Eq. (2.4) will step the phase up by 2π. The probability of the latter is $P(\theta_0)\omega\delta t$. Dividing by δt and taking the limit as $\delta t\to 0$ gives the rate of change of the expectation value of the phase

$$\frac{d}{dt}\langle\phi_\theta\rangle=-\omega+2\pi P(\theta_0)\omega \ , \tag{2.6}$$

where $P(\theta_0)$ will, in general, be a time-dependent quantity. Equation (2.6) emphasizes the significance of the choice of θ_0, that is, the particular choice of 2π interval in which ϕ_θ is defined.

We have seen that allowing the phase to have a continuous range or restricting it to a specified 2π interval leads to different Poisson brackets. It is not immediately obvious which, or if indeed either, should be used in conjunction with Dirac's commutator–Poisson-bracket correspondence. Dirac[1] originally favored the continuous phase, while Judge and Lewis[17] applied a restricted range to the problem of rotation angle. In view of the long history of difficulties we do not employ commutator–Poisson-bracket correspondence in order to obtain the phase operator. However, we will return to this problem of correspondence once we have derived our phase operator and photon-number-phase commutator by other means.

III. HERMITIAN PHASE OPERATOR

We have discussed some of the problems associated with formulating a Hermitian phase operator in the Introduction. Despite these difficulties, states of well-defined phase are known to exist.[18] These phase state are the starting point in our phase operator formalism. The phase state $|\theta\rangle$ is defined as

$$|\theta\rangle = \lim_{s\to\infty}(s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta)|n\rangle\ , \tag{3.1}$$

where $|n\rangle$ are the $(s+1)$ number states which span an $(s+1)$-dimensional state space Ψ. The state of zero phase has been chosen as the state in which all the number state amplitudes are equal. The limiting procedure is necessary in order to normalize the states. These states provide a good description in that their time development is such that at time t the state becomes $|\theta-\omega t\rangle$. Moreover, the expectation value of the electric field is $\pm\infty$ with a divergent variance at all times *except* when $\theta-\omega t$ is an integer multiple of π. At these times the field is precisely zero. These zeros (when the field changes sign) precisely determine the phase of the field.

The phase state is well defined in the space Ψ but care must be taken with the limiting process. As with all limiting procedures, errors can result if s is replaced by infinity prematurely. Our procedure, therefore, will be to work entirely with states and operators in the $(s+1)$-dimensional space Ψ (where s can be arbitrarily large) and then allow s to tend to infinity *after* physical results such as expectation values are calculated. The finiteness of our state space Ψ means that the operators involved may have slightly different properties than those of their infinite space counterparts. For example, the trace of all commutators in the finite space must be zero rather than being undefined.[19] However, we emphasize that such differences will not lead to detectable physical differences when the limit is eventually taken.[20] We shall not be letting s tend to infinity until a later stage and so dispense, for the present, with the limit notation in (3.1).

The parameter θ in the phase state (3.1) can take any real value, although distinct states will only occur for values of θ in a given 2π range. Therefore, there exists an uncountable infinity of different phase states, even in the finite but arbitrarily large state space Ψ. The phase states are necessarily overcomplete and are not in general orthogonal. However, it is not difficult to show that states with values of θ differing by integer multiples of $2\pi/(s+1)$ are orthogonal,[16,21] and consequently, given any reference state $|\theta_0\rangle$ we can find a complete set of $(s+1)$ orthonormal phase states given by

$$|\theta_m\rangle = \exp[i\hat{N}m2\pi/(s+1)]|\theta_0\rangle,\quad m=0,1,\ldots s\ . \tag{3.2}$$

Here, we have used the unitary phase shift operator $\exp(i\hat{N}\gamma)$ which transforms $|\theta\rangle$ to $|\theta+\gamma\rangle$ (as can be seen from the definition of $|\theta\rangle$). If $\gamma=-\omega t$, then this unitary operator is the time-evolution operator and we recover the ideal time development of a phase state. We note that choosing $m\geq(s+1)$ in (3.2) reproduces the states $|\theta_m\rangle$ with values $0\leq m<s$.

The set of phase states $|\theta_m\rangle$ can be used as a basis to span Ψ. The freedom to choose an arbitrary value for θ_0 means that there is an uncountable infinity of such bases. We shall leave θ_0 as an arbitrary phase, allowing the flexibility to choose the most convenient basis to solve a particular problem. The $(s+1)$ values of θ_m are

$$\theta_m = \theta_0 + \frac{2\pi m}{(s+1)}\ , \tag{3.3}$$

which are spread evenly over the range $\theta_0\leq\theta_m<\theta_0+2\pi$. When s tends to infinity, these values correspond to θ_0 plus the rational fractions of 2π. In this limit they form a countable infinity of orthogonal states that exist in a one-to-one correspondence with the countable basis of number states.

The Hermitian phase operator is simply defined in terms of a suitable phase state basis as

$$\begin{aligned}\hat{\phi}_\theta &\equiv \sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m|\\ &= \theta_0 + \frac{2\pi}{s+1}\sum_{m=0}^{s}m|\theta_m\rangle\langle\theta_m|\ .\end{aligned} \tag{3.4}$$

We note that the eigenvalues (θ_m) of $\hat{\phi}_\theta$ are single valued and depend on the chosen value of θ_0. This dependence is denoted by the subscript θ and we will show that the classical analogue of $\hat{\phi}_\theta$ is the single-valued ϕ_θ. The choice of reference state $|\theta_0\rangle$ determines the 2π range which the phase eigenvalues will occupy. This procedure is completely analogous to choosing a window in which to express the inverses of (classical) trigonometric functions as single-value numbers.

From the definition of the phase state (3.1), we can express the projector $|\theta_m\rangle\langle\theta_m|$ in terms of the number state basis:

$$|\theta_m\rangle\langle\theta_m| = (s+1)^{-1}\sum_{n,n'}\exp[i(n'-n)\theta_m]|n'\rangle\langle n|\ , \tag{3.5}$$

with the summations running from 0 to s. Substituting this expression into (3.4) and performing the summation over m yields a number state expansion for the phase operator:

$$\hat{\phi}_\theta = \theta_0 + \frac{s\pi}{s+1} + \frac{2\pi}{s+1}\sum_{n\neq n'}\frac{\exp[i(n'-n)\theta_0]|n'\rangle\langle n|}{\exp[i(n'-n)2\pi/(s+1)]-1}\ . \tag{3.6}$$

We can also express the number operator in the phase space basis:

$$\begin{aligned}\hat{N} &\equiv \sum_{n=0}^{s}n|n\rangle\langle n|\\ &= \frac{s}{2} + \sum_{m\neq m'}\frac{|\theta_{m'}\rangle\langle\theta_m|}{\exp[-i(m'-m)2\pi/(s+1)]-1}\ ,\end{aligned} \tag{3.7}$$

where we have used the result

$$|n\rangle=(s+1)^{-1/2}\sum_{m=0}^{s}\exp(-in\theta_m)|\theta_m\rangle\ , \tag{3.8}$$

which is easily derived from (3.1) using the orthonormality of the basis $\{|\theta_m\rangle\}$. Expressions (3.6) and (3.7) reveal a subtle symmetry between $\hat{\phi}_\theta$ and $\hat{N}$: both the phase and number operators consist of a constant corresponding to the middle of their eigenvalue range plus a sum of off-diagonal projectors. This symmetry is particularly striking if we compare the phase operator with $\theta_0=0$ and the scaled number operator:

$$\hat{\phi}_0=\frac{s\pi}{s+1}+\frac{2\pi}{s+1}\sum_{n\neq n'}\frac{|n'\rangle\langle n|}{\exp[-i(n'-n)2\pi/(s+1)]-1}\ , \tag{3.9a}$$

$$\frac{2\pi\hat{N}}{s+1}=\frac{s\pi}{s+1}+\frac{2\pi}{s+1}\sum_{m\neq m'}\frac{|\theta_{m'}\rangle\langle\theta_m|}{\exp[-i(m'-m)2\pi/(s+1)]-1}\ . \tag{3.9b}$$

IV. PHASE-NUMBER COMMUTATOR

In Sec. III we defined the Hermitian phase operator in terms of a complete set of orthonormal phase states. This technique is independent of commutator–Poisson-bracket correspondence and we are now in a position to *derive* the phase-number commutator. The phase-number commutator is easily calculated using expressions (3.6) and (3.7). Expressed in the number state basis the commutator is

$$[\hat{\phi}_\theta,\hat{N}]=\frac{2\pi}{s+1}\sum_{n\neq n'}\frac{(n-n')\exp[i(n'-n)\theta_0]|n'\rangle\langle n|}{\exp[-i(n'-n)2\pi/(s+1)]-1}\ , \tag{4.1}$$

while in terms of the phase state basis it has the form

$$[\hat{\phi}_\theta,\hat{N}]=\frac{2\pi}{s+1}\sum_{m\neq m'}\frac{(m'-m)|\theta_{m'}\rangle\langle\theta_m|}{\exp[-i(m-m')2\pi/(s+1)]-1}\ . \tag{4.2}$$

These expressions look very different from the Dirac relation (1.2). In particular, our commutator is traceless (as all commutators must be in Ψ). Moreover, it is clear that the vanishing trace follows directly from the fact that the expectation value of the commutator in a number state $|n\rangle$ or phase state $|\theta_m\rangle$ is zero. Our commutator does not suffer from the mathematical inconsistency associated with the Dirac commutator.

The matrix elements of the phase-number commutator in a number state basis are

$$\langle n|[\hat{\phi}_\theta,\hat{N}]|n\rangle=0\ , \tag{4.3a}$$

$$\langle n'|[\hat{\phi}_\theta,\hat{N}]|n\rangle=\frac{2\pi(n-n')\exp[i(n'-n)\theta_0]}{(s+1)\{\exp[i(n'-n)2\pi/(s+1)]-1\}}\ . \tag{4.3b}$$

These expressions are exact but complicated and for many purposes a simpler form can be found. With this aim in mind we define a physically accessible, or preparable, state as one which can be excited from the vacuum state by coupling the mode to a finite energy source for a finite time by means of a finite interaction. Such a physical state has the following properties:[16] firstly, the finite source ensures an upper bound to the number states which have any probability of being excited and secondly, the moments of the energy or photon number distribution $\langle n^q\rangle$ are bounded for any finite number q. The latter condition follows from the finite time and interaction strength and is a weaker condition than the former. Indeed, imposition of the former condition automatically ensures the latter. Examples of states obeying the latter condition include thermal states, coherent states, squeezed states, and single number states. We note that a phase state is *not* a physical state because the expectation value of the photon number diverges as $s\rightarrow\infty$. For a physical state of the field we can make s very much larger than the number n associated with any significant number state component $|n\rangle$ of the state. In this case (with $s\gg n,n'$) expression (4.3) reduces to

$$\langle n'|[\hat{\phi}_\theta,\hat{N}]|n\rangle\approx i(1-\delta_{nn'})\exp[i(n'-n)\theta_0]\ . \tag{4.4}$$

This approximate equality becomes exact for all finite n and n' in the limit as s tends to infinity. However, we stress that the exact expression (4.3) should be used in general with the limit as $s\rightarrow\infty$ being taken only when the final result has been obtained. If our final aim is to calculate the expectation value of the commutator for a physical state, then (4.4) can be used directly. The operator formed from the matrix elements (4.4) has the form

$$\begin{aligned}[\hat{\phi}_\theta,\hat{N}]_p&=i\sum_{n,n'}|n'\rangle\langle n|(1-\delta_{nn'})\exp[i(n'-n)\theta_0]\\&=-i+i\sum_{n'}\exp(in'\theta_0)|n'\rangle\sum_{n}\exp(-in\theta_0)\langle n|\ ,\end{aligned} \tag{4.5}$$

where the subscript p distinguishes this commutator from the precise expression (4.1) and is a reminder that it is only valid when acting on physical states. The first term, which follows from the Kronecker delta, is the Dirac term. The presence of the second term ensures that the trace of the commutator vanishes. It is clear from the definition of the phase state (3.1) that the physical-state commutator reduces to

$$[\hat{\phi}_\theta,\hat{N}]_p=-i[(1-(s+1)|\theta_0\rangle\langle\theta_0|]\ . \tag{4.6}$$

The expectation value of the phase-number commutator in any physical state $|p\rangle$ is

$$\langle p|[\hat{\phi}_\theta,\hat{N}]|p\rangle=-i[1-(s+1)|\langle p|\theta_0\rangle|^2]\ , \tag{4.7}$$

where $|\langle p|\theta_0\rangle|^2$ is the probability that the phase of the state is θ_0. In the continuum limit (as $s\rightarrow\infty$) this may be expressed as $P(\theta_0)2\pi/(s+1)$, where $P(\theta_0)$ is the probability density and $(s+1)/(2\pi)$ is the density of states. With this substitution, the expectation value (4.7) becomes

$$\langle p|[\hat{\phi}_\theta,\hat{N}]|p\rangle = -i[1-2\pi P(\theta_0)] \ . \quad (4.8)$$

Dirac's commutator–Poisson-bracket correspondence requires the form of the commutator and Poisson bracket to be related:[13]

$$[\hat{u},\hat{v}] \leftrightarrow i\hbar\{u,v\} \ . \quad (4.9)$$

The classical expression for the expectation value of the Poisson bracket is [cf. Eq. (2.6)]

$$\langle\{\phi_\theta,H\}\rangle = \int_{\theta_0}^{\theta_0+2\pi} P(\phi)\{\phi_\theta,H\}d\phi = -\omega[1-2\pi P(\theta_0)] \ . \quad (4.10)$$

The quantum Hamiltonian is $(\hat{N}+\frac{1}{2})\hbar\omega$ and so the correspondence between the quantum (4.8) and classical (4.10) expressions is verified. This correspondence is *precise* for all physical states. If the phase probability distribution is very sharply peaked, as for example in the case of a highly excited coherent state,[22] then the expectation value of the commutator reproduces the classical δ-function Poisson bracket (2.4).

From the preceding discussion we see that it is essential for there to be a difference between the Dirac commutator (1.2) and that derived by our approach because the trace of the latter must vanish. This difference removes the inconsistency associated with the number state matrix elements of Dirac's commutator. It also produces the appropriate 2π step in the phase-number commutator for all physical states. This step maintains the phase eigenvalues within the chosen 2π range, that is, it automatically takes care of the periodicity problem. The classical result (2.6) has followed naturally from the quantum-mechanical description of phase applied to physical states. The category of physical states is extremely broad and includes practically all states used so far in quantum electrodynamics, with the phase states being the only notable exception. Nevertheless, (4.6) [and therefore (2.6)] is *not* universally applicable. It is now clear why Dirac's Poisson bracket recipe to extrapolate from (2.3) or even (2.6) to find an Hermitian phase operator, with well-behaved phase eigenstates, had little change of success.

The expectation value (4.8) is a measure of minimum uncertainty in that physical states must satisfy the relation

$$\Delta N \Delta\phi_\theta \geq \tfrac{1}{2}|1-2\pi P(\theta_0)| \ . \quad (4.11)$$

This uncertainty relation depends on θ_0, that is, the choice of range for the phase eigenvalues. This phenomenon persists in classical mechanics where the expectation value and variance of a classical phase distribution will depend on the range of phase values employed. The quantum (or classical) probability distributions $P(\theta)$ will be periodic with period 2π. If $P(\theta)$ is sharply peaked at and approximately symmetric about $\theta=\beta+2n\pi$, then a 2π window which totally encloses a peak [that is, chosen so that $P(\theta_0)$ is small] will yield a mean phase of approximately β and a small variance. If, however, the window is chosen such that $P(\theta_0)$ is large (for example, $\theta_0=\beta$) then the distribution in the 2π window has one peak at β and another at $2\pi+\beta$. In this case the calculated mean will be about $\beta+\pi$ and the variance will be large. This effect is explored in detail elsewhere,[22] but we note here that the mean and variance of the phase only have meaning if the particular window of phase eigenvalues is specified. For many distributions $P(\theta)$ is sufficiently small over a range of θ for the mean and variance to be reasonably insensitive to the precise choice of θ_0 [provided θ_0 is sufficiently different from the peak of $P(\theta)$].

V. CREATION AND ANNIHILATION OPERATORS

The phase operator was originally intended to combine with the square root of the number operator in a polar decomposition of the creation and annihilation operators.[1] However, the failure of early attempts to construct a Hermitian phase operator,[3] or even a unitary exponential phase operator,[4,5] suggested that this procedure was unsatisfactory. It seemed to be clear that the zero in the eigenvalue range of $\hat{N}$ precluded any possibility of constructing a unitary exponential phase operator $[\exp(\pm i\hat{\phi})]$.[23] In this section we reexamine the problem in using our Hermitian phase operator.

We can construct a unitary operator $\exp(i\hat{\phi}_\theta)$ from the Hermitian phase operator. This operator function may be defined by its series expansion and is guaranteed to be unitary by the Hermiticity of $\hat{\phi}_\theta$. the unitary operator (*and* its conjugate operator) will have the phase states as eigenstates:

$$\exp(\pm i\hat{\phi}_\theta)|\theta_m\rangle = \exp(\pm i\theta_m)|\theta_m\rangle \ . \quad (5.1)$$

The properties of the unitary operator are demonstrated by considering its action on the photon number states:

$$\exp(i\hat{\phi}_\theta)|n\rangle = \exp\left[i\sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m|\right]|n\rangle = (s+1)^{-1/2}\sum_{m=0}^{s}\exp[-i(n-1)\theta_m]|\theta_m\rangle \ , \quad (5.2)$$

where we have used expression (3.8). For $n>0$, a comparison of (5.2) with (3.8) shows that

$$\exp(i\hat{\phi}_\theta)|n\rangle = |n-1\rangle \ . \quad (5.3)$$

For the vacuum state ($n=0$) the resulting state is

$$(s+1)^{-1/2}\sum_m \exp(i\theta_m)|\theta_m\rangle = (s+1)^{-1/2}\exp[i(s+1)\theta_0]\sum_m \exp(-is\theta_m)|\theta_m\rangle = \exp[i(s+1)\theta_0]|s\rangle \ . \quad (5.4)$$

Therefore, the number state representation of $\exp(i\hat{\phi}_\theta)$ is

$$\exp(i\hat{\phi}_\theta) = |0\rangle\langle 1| + |1\rangle\langle 2| + \cdots + |s-1\rangle\langle s| + \exp[i(s+1)\theta_0]|s\rangle\langle 0| \ . \quad (5.5)$$

This operator resembles the one introduced by Susskind and Glogower,[4,5] but with the vital difference that the action on the vacuum state is not indeterminate and so cannot be arbitrarily set to zero. The result of acting on the vacuum with $\exp(i\hat{\phi}_\theta)$ is *uniquely and precisely determined* to be the state $\exp[i(s+1)\theta_0]|s\rangle$. The expansion (5.5) is fully consistent with $\exp(i\hat{\phi}_\theta)$ being a unitary operator. The conjugate operator $\exp(-i\hat{\phi}_0)$ is also unitary and clearly commutes with $\exp(i\hat{\phi}_\theta)$.

We can define cosine and sine operators from the unitary operators. These will correspond to cosine and sine series in the Hermitian phase operator. These operators are more consistent than their counterparts formed from the Susskind-Glogower operators. In particular, we find

$$\cos^2\hat{\phi}_\theta+\sin^2\hat{\phi}_\theta=1 \ , \tag{5.6}$$

$$[\cos\hat{\phi}_\theta,\sin\hat{\phi}_\theta]=0 \ , \tag{5.7}$$

$$\langle n|\cos^2\hat{\phi}_\theta|n\rangle=\langle n|\sin^2\hat{\phi}_\theta|n\rangle=\tfrac{1}{2} \ . \tag{5.8}$$

The last of these results is consistent with the phase of the vacuum being random and is in marked contrast with the result obtained using the Susskind-Glogower operators. In the Susskind-Glogower formulation the vacuum expectation values of $\cos^2\hat{\phi}_\theta$ and $\sin^2\hat{\phi}_\theta$ are $\frac{1}{4}$. The major difference between our operators and those of Susskind and Glogower involves the action on the vacuum state. It should not be surprising that very different results will occur in the quantum regime involving field states with a significant vacuum component. Our operators are in *no sense* an approximation to the Susskind-Glogower operators and have similar properties to the latter only for fields with large energies where both sets of operators give results in accord with classical behavior. Moreover, our cosine and sine operators are derived as operator functions of an Hermitian phase operator. No analogous procedure is possible in the Susskind-Glogower formulation.

The creation and annihilation operators can now be constructed by definition:

$$\hat{a}\equiv\exp(i\hat{\phi}_\theta)\hat{N}^{1/2} \tag{5.9}$$

$$=|0\rangle\langle 1|+2^{1/2}|1\rangle\langle 2|+\cdots+s^{1/2}|s-1\rangle\langle s| \ , \tag{5.10}$$

with $\hat{a}^\dagger$ being the Hermitian conjugate of $\hat{a}$. As s tends to infinity the action of $\hat{a}$ becomes that of the conventional annihilation operator. Note that the Hermitian amplitude operator $\hat{N}^{1/2}$ removes the $|s\rangle\langle 0|$ projector term when acting on (5.5) to give (5.10). Thus, while $\exp(i\hat{\phi}_\theta)$ uniquely determines $\hat{a}$, it is now clear why the reverse procedure (involving a decomposition of $\hat{a}$) cannot produce the vital $|s\rangle\langle 0|$ projector term in (5.5). Such a procedure can at best only produce an indeterminate result, requiring an *extra* assumption as used by Susskind and Glogower. The $|s\rangle\langle 0|$ projector *never* occurs in any coupling involving the electromagnetic field as these take place via $\hat{a}$ and $\hat{a}^\dagger$.[16]

It is known[5] that an expression of the form (5.9) (as proposed by Dirac[1]) is inconsistent with the conventional commutation relation of $\hat{a}$ and $\hat{a}^\dagger$ if $\exp(i\hat{\phi}_\theta)$ is unitary. If it were possible, we would require that a unitary transformation $\exp(i\hat{\phi}_\theta)\hat{N}\exp(-i\hat{\phi}_\theta)$ would simply add the unit operator to $\hat{N}$. However, our approach does not suffer from such problems because of the finite (but arbitrarily large) nature of the state space. An extra term was necessary in Dirac's commutator (1.2) and we should also anticipate an additional term in the $\hat{a},\hat{a}^\dagger$ commutator, even if only to ensure that its trace vanishes. On calculating the commutator of $\hat{a}$ and $\hat{a}^\dagger$, as defined by (5.9), we indeed obtain a result that is traceless:

$$[\hat{a},\hat{a}^\dagger]=1-(s+1)|s\rangle\langle s| \ . \tag{5.11}$$

The last term exactly compensates the trace obtained from the first term in the same way as the additional part to Dirac's term in $[\hat{\phi}_0,\hat{N}]$. However, while the addition to the $\hat{\phi}_\theta,\hat{N}$ commutator has direct physical consequences, the additional term in (5.11) has no effect when $[\hat{a},\hat{a}^\dagger]$ acts on any physical state and so has no physically observable consequences. The physical-state commutator

$$[\hat{a},\hat{a}^\dagger]_p=1 \tag{5.12}$$

is sufficient when operating on physical states.

This is most easily seen for states excited from the vacuum by a finite energy source, but is also true for states which have no upper bound to $|n\rangle$ but which have finite energy moments. As a definite example, consider a coherent state $|\alpha\rangle$. If q is any chosen positive integer then we have

$$[\hat{a},\hat{a}^\dagger]^q=1-|s\rangle\langle s|[1-(-s)^q] \ . \tag{5.13}$$

The coherent state expectation value of this operator is

$$\langle\alpha|[\hat{a},\hat{a}^\dagger]^q|\alpha\rangle=1-\exp(-|\alpha|^2)\frac{|\alpha|^{2s}}{s!}[1-(-s)^q] \ . \tag{5.14}$$

As s tends to infinity the last term vanishes for any given (finite) q. For a more general physical state $|p\rangle$ we obtain

$$\langle p|[\hat{a},\hat{a}^\dagger]^q|p\rangle=1-|c_s|^2[1-(-s)^q] \ , \tag{5.15}$$

where $c_s=\langle s|p\rangle$. The requirement that the moment $\langle N^q\rangle$ be convergent as s tends to infinity ensures that $|c_s|^2s^q$ must tend to zero and the second term vanishes.

We emphasize again that when states other than the physical states (for example the phase states) are used in a calculation, then (5.11) must be used in place of (5.12) and the limit must be found at the end of the calculation.

VI. PHASE PROPERTIES OF A GENERAL STATE

For completeness and for future reference, we show how the phase operator can be used to examine the phase properties of a field state. Consider a general pure state of the field mode[24]

$$|f\rangle=\sum_{n=0}^{s}c_n|n\rangle \ . \tag{6.1}$$

This may be reexpressed in the phase state basis using (3.8):

$$|f\rangle=(s+1)^{-1/2}\sum_n\sum_m c_n\exp(-in\theta_m)|\theta_m\rangle \ . \tag{6.2}$$

The phase probability distribution is

$$|\langle \theta_m | f \rangle|^2 = (s+1)^{-1} \left| \sum_n c_n \exp(-i\theta_m) \right|^2 , \tag{6.3}$$

with an expectation value and variance

$$\langle \hat{\phi}_\theta \rangle = \sum_m \theta_m |\langle \theta_m | f \rangle|^2 , \tag{6.4}$$

$$\Delta \hat{\phi}_\theta^2 = \sum_m (\theta_m - \langle \hat{\phi}_\theta \rangle)^2 |\langle \theta_m | f \rangle|^2 . \tag{6.5}$$

Any discussion of phase-number minimum uncertainty states requires the expectation value of the phase-number commutator:

$$\langle f | [\hat{\phi}_\theta, \hat{N}] | f \rangle = \frac{2\pi}{s+1} \sum_{n \neq n'} \frac{c_n^* c_{n'} (n-n') \exp[i(n-n')\theta_0]}{1 - \exp[i(n-n')2\pi/(s+1)]} . \tag{6.6}$$

It is the purpose of this paper to look at phase properties of a single-mode field from as general a perspective as possible. Therefore, we shall not pursue the many applications of these formulas to specific field states. We note, however that when $|f\rangle$ is a single number state the expectation value and variance in the phase are $\theta_0 + \pi$ and $\pi^2/3$, respectively (as $s \to \infty$). These values correspond to the mean and variance of a classical phase with a random value between θ_0 and $\theta_0 + 2\pi$. The details of this calculation will be presented elsewhere[22] along with other special cases including coherent states. In this paper we focus our attention on a class of states that we call partial phase states.

VII. PARTIAL PHASE STATES

The form of the phase state (3.1) and the phase probability (6.3) suggest that interesting phase properties are to be expected when the state is of the form

$$|b\rangle = \sum_{n=0}^{s} b_n \exp(in\beta) |n\rangle , \tag{7.1}$$

where b_n is real and positive. Obviously, the phase state is a special example of this with $b_n^2 = (s+1)^{-1}$. The states $|b\rangle$, which we shall refer to as partial phase states, will not normally be eigenstates of phase. A very important subset of these states will be the physical partial phase states, of which the coherent state is a particular example. The phase states are themselves unphysical and so the best attempt at a physical phase measurement will only *project* the system into a physical partial phase state.

The phase probability distribution for a partial phase state is given [from (6.3) and (7.1)] by

$$|\langle \theta_m | b \rangle|^2 = (s+1)^{-1} \left| \sum_n b_n \exp[in(\beta - \theta_m)] \right|^2 = \frac{1}{s+1} + \frac{2}{s+1} \sum_{n > n'} b_n b_{n'} \cos[(n-n')(\beta - \theta_m)] . \tag{7.2}$$

The mean and variance of $\hat{\phi}_\theta$ will depend on the chosen value of θ_0. We note that all values are equally valid but choose θ_0 in the most convenient and physically transparent way. For the partial phase state $|b\rangle$ we set

$$\theta_0 = \beta - \frac{\pi s}{s+1} \tag{7.3}$$

and define a new phase label

$$\mu = m - \frac{s}{2} . \tag{7.4}$$

From (3.3), Eq. (7.2) becomes

$$|\langle \theta_m | b \rangle|^2 = \frac{1}{s+1} + \frac{1}{s+2} \sum_{n > n'} b_n b_{n'} \cos[(n-n')\mu 2\pi/(s+1)] , \tag{7.5}$$

with μ ranging in integer steps from $-s/2$ to $s/2$. This distribution is symmetric in μ. Using Eqs. (7.3)–(7.5), we find that

$$\langle b | \hat{\phi}_\theta | b \rangle = \beta . \tag{7.6}$$

This is a very important and general result for partial phase states which can be applied immediately, for example, to any coherent state.

The choice of θ_0 means that the variance in the phase probability distribution has a particularly simple form:

$$\Delta \phi_\theta^2 = \frac{4\pi^2}{(s+1)^2} \sum_{\mu = -s/2} |\langle \theta_m | b \rangle|^2 \mu^2 . \tag{7.7}$$

The summation in (7.7) is most easily performed in the limit as s tends to infinity by transforming it into an integral. We replace $\mu 2\pi/(s+1)$ by θ, $2\pi/(s+1)$ by $d\theta$, and integrate from $-\pi$ to π to obtain

$$\Delta \phi_\theta^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} \left[1 + 2 \sum_{n > n'} b_n b_{n'} \cos[(n-n')\theta] \right] \theta^2 d\theta = \frac{\pi^2}{3} + 4 \sum_{n > n'} b_n b_{n'} (-1)^{(n-n')} (n-n')^{-2} . \tag{7.8}$$

We see that for the extreme case of a single number state (where only one of the b_n is nonzero) the last term vanishes and the variance is $\pi^2/3$. At the other extreme is the phase state [with $b_n = (s+1)^{-1/2}$]. Using the large s result that

$$\sum_{n > n'} \frac{(-1)^{(n-n')}}{(n-n')^2} \approx -s(1 - \tfrac{1}{4} + \tfrac{1}{9} - \cdots) = -s\pi^2/12 , \tag{7.9}$$

we find that the phase variance vanishes as expected.

As a special example of a partial phase state, consider the "rectangular" state $|b,R\rangle$ for which the coefficients b_n equal a constant $(r^{-1/2})$ for $q \le n < q+r$ and are zero elsewhere. The photon number probability distribution is constant and nonzero between $|q\rangle$ and $|q+r-1\rangle$ and is zero outside this range—hence the name "rectangular" state. These states include as special cases both phase states (when $q=0$ and $r=s+1$) and the single number state $|q\rangle$ when $r=1$. The phase probability distribution can be obtained by summing the series in (7.2) to give

$$|\langle\theta_m|b,R\rangle|^2=\frac{\sin^2[r(\beta-\theta_m)/2]}{r(s+1)\sin^2[(\beta-\theta_m)/2]} . \quad (7.10)$$

When s tends to infinity this gives a continuous phase probability distribution,

$$P(\theta_m)=\frac{\sin^2[r(\beta-\theta_m)/2]}{2\pi r\sin^2[(\beta-\theta_m)/2]} , \quad (7.11)$$

which is peaked and symmetric about $\theta_m=\beta$. If we let r increase to give a broad number state distribution, then the phase probability distribution approaches a δ function. With the choice of reference phase (7.3), the expectation value of the phase is β (as it must be for all partial phase states). From (7.8) we find the phase variance in the rectangular state to be

$$\begin{aligned}\Delta\phi_\theta^2&=\frac{\pi^2}{3}-(4/r)[(r-1)-(r-2)/4+(r-3)/9\\&\qquad-\cdots+(-1)^r/(r-1)^2]\\&=\frac{\pi^2}{3}-4(1-\tfrac{1}{4}+\tfrac{1}{9}-\cdots)+(4/r)(1-\tfrac{1}{2}+\tfrac{1}{3}\ldots) .\end{aligned} \quad (7.12)$$

If r is reasonably large then we can replace the finite series by $\pi^2/12$ and ln2, respectively, giving the approximate result

$$\Delta\phi_\theta^2\approx(4/r)\ln2 . \quad (7.13)$$

To examine the validity of our approximation, we have calculated (7.12) explicitly for values $r=1$, 2, 3, and 4. We find for the ratio $\Delta\phi_\theta^2/[(4/r)\ln2]$ the results 1.19, 0.92, 1.03, and 0.98, respectively. It is clear that the approximation (7.13) is a good one for rectangular partial phase states with as low as three or perhaps even two different number state components.

The only rectangular partial phase state of interest to us which is not also a physical state is the phase state itself. We already know that the expectation value of the phase-number commutator vanishes for a phase state. We therefore restrict the following discussion of the minimum uncertainty properties to physical partial phase states. This restriction allows us to use expression (4.7) for the expectation value of the commutator in conjunction with (7.10) for a rectangular state. With our choice of θ_0 (7.3) and in the limit of large s we find

$$|\langle\theta_0|b,R\rangle|^2=\frac{1-(-1)^r}{2r(s+1)} , \quad (7.14)$$

with the result that the expectation value of the phase-number commutator is

$$\langle b,R|[\hat{\phi}_\theta,\hat{N}]|b,R\rangle=-i+\frac{i[1-(-1)^r]}{2r} . \quad (7.15)$$

For r even we regain the Dirac commutator expectation value. However, if r is odd then the situation is more complicated: the commutator expectation value vanishes if $r=1$ (single number state) and approaches the Dirac form for large r.

It is not difficult to calculate the photon number variance for the rectangular state $|b,R\rangle$:

$$\Delta N^2=(r^2-1)/12 . \quad (7.16)$$

This variance is zero for a single number state ($r=1$). In the limit of large r the number-phase uncertainty product becomes

$$\Delta N\Delta\phi_\theta\approx[(r\ln2)/3]^{1/2} . \quad (7.17)$$

Comparison of this expression with one half the modulus of the commutator which for large r is $\frac{1}{2}$ [see (7.15)], shows that the rectangular partial phase states will *not* in general be number-phase minimum uncertainty states. The exception among the physical rectangular phase states is the number state (with $r=1$) for which both the uncertainty product and the commutator expectation value vanish.

It is beyond the scope of this paper to study further specific examples of partial phase states, but we conclude our discussion with a few general comments. Firstly, we emphasize the significance of the choice of reference phase θ_0. This phase can be assigned any value, but the choice $\theta_0=\beta-\pi$ is the natural (and most appealing) choice to make for partial phase states (7.1). With this choice of reference phase, the correction to the Dirac term in the expectation value of the phase-number commutator will be

$$\begin{aligned}(s+1)|\langle\theta_0|b\rangle|^2=[&(b_0+b_2+b_4+\cdots)\\&-(b_1+b_3+b_5+\cdots)]^2 .\end{aligned} \quad (7.18)$$

If the variation of b_n with n is sufficiently smooth and the distribution includes a large number of nonzero b_n, then it follows that this term will be small. In this case the Dirac term alone will be a good approximation to the expectation value of the phase-number commutator. For such states, the phase-number minimum uncertainty states will satisfy the relation

$$\Delta N\Delta\phi_\theta=\tfrac{1}{2} . \quad (7.19)$$

The parallel between this expression and that for position-momentum minimum uncertainty states suggests that a high-intensity phase-number minimum uncertainty state would have a Gaussian distribution of number states. A detailed calculation for a high-intensity coherent state (approximated by a Gaussian photon number distribution) is presented elsewhere[22] and shows that the minimum uncertainty relation (7.19) is satisfied by such states. The Hermitian phase operator provides a fully quantum-mechanical explanation for what hitherto has been discussed only in phenomenological and semiclassical terms.[9]

Finally, we note that when $(s+1)|\langle\theta_0|b\rangle|^2$ is small the probability density $P(\theta_0)$ (4.8) is also small. Therefore, the chosen range of phase eigenvalues completely encloses the peak of the phase distribution. The mean and variance of the phase will be reasonably insensitive to variations in the precise choice of θ_0, *provided* such variations do not take θ_0 too close to β, that is, if $|\beta-\theta_0|$ is still much greater than the width of the phase distribution.

VIII. PHASE DIFFERENCES

The phase difference between two independent *classical* field modes, or oscillators, is well defined and is independent of time if the oscillators have the same frequency. This phase difference is simply the difference between the individual phases $(\phi_1-\phi_2)$. The conventional Susskind-Glogower exponential operators are not operator functions of a Hermitian phase operator and such a simple notion of phase difference is not possible in their approach.[4] Instead, it is necessary to define phase difference operators separately rather than as the difference between phase operators. The definition chosen was that which *would* apply if the series were genuine exponential series:[25]

$$\widehat{\exp}_s[i(\phi_1-\phi_2)]\equiv\widehat{\exp}_s(i\phi_1)\widehat{\exp}_s(-i\phi_2)\ , \tag{8.1}$$

$$\widehat{\exp}_s[-i(\phi_1-\phi_2)]\equiv\widehat{\exp}_s(-i\phi_1)\widehat{\exp}_s(i\phi_2)\ , \tag{8.2}$$

where the subscripts 1 and 2 refer to modes 1 and 2. Hermitian cosine and sine operators were then constructed from these in the same manner as they would be constructed if these were genuine unitary operators. This definition leads to cosine and sine operators $[\widehat{\cos}_s(\phi_1-\phi_2)$ and $\widehat{\sin}_s(\phi_1-\phi_2)]$ that commute with the total number operator $(\hat{N}_1+\hat{N}_2)$ but *not* with each other. These operators have the expected behavior in the classical limit, but have quite peculiar properties in the quantum regime where the vacuum is an important component of either of the two field states. The lowest energy eigenstate of $\widehat{\cos}_s(\phi_1-\phi_2)$ is the double vacuum $|0,0\rangle$ with eigenvalue (labeled $\cos\theta$) of zero. This would correspond to a value of θ of $\pm\pi/2$. The next eigenstates are $2^{-1/2}\{|1,0\rangle\pm 0,1\rangle\}$ with eigenvalues $\cos\theta=\pm\frac{1}{2}$. The next set of eigenstates are three orthogonal linear combinations of $|2,0\rangle$, $|0,2\rangle$, and $|1,1\rangle$ with three eigenvalues, and so on. Only in the limit of large total excitation number does the spectrum of $\widehat{\cos}_s(\phi_1-\phi_2)$ become dense. These results would imply that a measurement of the phase difference between two modes, each in its vacuum state, *must* yield a phase difference of ±90°. If the state of the system is $|0,1\rangle$ (that is, one mode in its vacuum state and one containing a single photon), then a measurement of phase difference must yield either ±60° or ±120°. No other results are possible. The fundamental reason for these predictions is again the nonunitarity of $\widehat{\exp}_s(i\phi)$, which imparts nonrandom phase properties to the vacuum.

However, we have demonstrated the existence of the Hermitian phase operator and there is nothing to prevent us from adopting the natural definition of phase difference. Our phase difference operator is simply $\hat{\phi}_{\theta_1}-\hat{\phi}_{\theta_2}$, where again the subscripts 1 and 2 refer to the individual modes. The eigenstates of this operator are just the products of the individual phase eigenstates:

$$(\hat{\phi}_{\theta_1}-\hat{\phi}_{\theta_2})|\theta_{m1}\rangle|\theta_{m2}\rangle=(\theta_{m1}-\theta_{m2})|\theta_{m1}\rangle|\theta_{m2}\rangle\ . \tag{8.3}$$

The spectrum of this operator is dense (in the limit as s_1 and s_2 tend to infinity) *even for states with a low total excitation number.* A phase difference measurement can lead to a countably infinite number of results regardless of total excitation number. For example, the two-mode vacuum state is

$$|0,0\rangle=(s_1+1)^{-1/2}(s_2+1)^{-1/2}\sum_{m1=0}^{s_1}\sum_{m2=0}^{s_2}|\theta_{m1}\rangle|\theta_{m2}\rangle \tag{8.4}$$

and the system is equally likely to be found in any of the $(s_1+1)(s_2+1)$ phase difference eigenstates. In the limit, as s_1 and s_2 tend to infinity, there will be a countable infinity of possible values which, depending on the choice of $\theta_{01}-\theta_{02}$ would ensure that all phase differences are between -2π and 2π. Our phase difference operator is entirely consistent with two modes, in their respective vacuum states, having uncorrelated and random phases.

IX. UNITARY TRANSFORMATIONS AND MOMENTS

It follows from the definition of the number state (3.1) that the number operator is the generator of increments in the phase:

$$\exp(i\hat{N}\gamma)|\theta\rangle=|\theta+\gamma\rangle\ . \tag{9.1}$$

Integer multiples of 2π can be added to or subtracted from $\theta+\gamma$ without altering the state, in order to keep the value of $\theta+\gamma$ inside the chosen 2π window. The operator shifts the phase of the state by γ (mod 2π). If γ is an integer multiple of $2\pi/(s+1)$, then the action of $\exp(i\hat{N}\gamma)$ on an eigenstate of $\hat{\phi}_\theta$ produces another eigenstate of $\hat{\phi}_\theta$. When this is not so, we have a unitary operator which allows us to transform to a different phase state which is not an eigenstate of $\hat{\phi}_\theta$.

The existence of the Hermitian phase operator allows us to construct a general unitary operator $\exp(-i\lambda\hat{\phi}_\theta)$. When $\lambda=1$, this is just the "down-shift" operator introduced earlier (5.5). From expression (5.5) it follows that $\exp(ij\hat{\phi}_\theta)$ shifts the number state to a new number outside the "window" between $|0\rangle$ and $|s\rangle$, then the shift is by an amount j plus a suitable multiple of $(s+1)$, including the appropriate phase factor. For example, the unitary operator $\exp(i4\hat{\phi}_\theta)$ lowers the photon number by four so that:

$$\begin{aligned}&\exp(i4\hat{\phi}_\theta)|6\rangle=|2\rangle\ ,\\&\exp(i4\hat{\phi}_\theta)|1\rangle=\exp[i(s+1)\theta_0]|s-2\rangle\ .\end{aligned} \tag{9.2}$$

If λ is not an integer, then the state $\exp(-i\lambda\hat{\phi}_\theta)|n\rangle$ will *not* be an eigenstate of the number operator. Nevertheless, this state will be one of a complete orthonormal set of $(s+1)$ basis states which can be used to span the state space Ψ. These states are the noninteger number states which we label $|n+\lambda\rangle$. Access to these states by means of a unitary transformation generated by $\hat{\phi}_\theta$ may be useful in solving future problems.

The unitary operators described here have an application in that they allow us to construct phase- and number-moment generating functions. Given the density matrix for the field ρ, we can define a phase-moment generating function or characteristic function:

$$\chi_\theta(\lambda) \equiv \mathrm{Tr}[\rho \exp(i\lambda\hat{\phi}_\theta)] \ . \tag{9.3}$$

The moments of $\hat{\phi}_\theta$ are given by differentiation with respect to λ:

$$\langle \hat{\phi}_\theta^k \rangle = \left[\frac{-i\partial}{\partial\lambda} \right]^k \chi_\theta(\lambda) \bigg|_{\lambda=0} \ . \tag{9.4}$$

The moment generating function is related by Fourier transform to the phase probability density introduced in Sec. IV:

$$\begin{aligned} P(\theta) &= \int_{-\infty}^{\infty} \frac{d\lambda}{(s+1)} e^{-i\lambda\theta} \chi_\theta(\lambda) \\ &= \frac{2\pi}{(s+1)} \mathrm{Tr}[\rho\delta(\hat{\phi}_\theta - \theta)] \ . \end{aligned} \tag{9.5}$$

Evaluating the trace shows that this is indeed the phase distribution

$$P(\theta) = \frac{2\pi}{(s+1)} \sum_m \langle \theta_m | \rho | \theta_m \rangle \delta(\theta_m - \theta) \tag{9.6}$$

and that it is correctly normalized

$$\begin{aligned} \int_{\theta_0}^{\theta_0+2\pi} \frac{(s+1)}{2\pi} P(\theta) d\theta &= \mathrm{Tr}\rho \\ &= 1 \ . \end{aligned} \tag{9.7}$$

A parallel analysis involving the operator $\exp(i\hat{N}\gamma)$ will generate a number-moment generating function.

X. CONCLUSION

We have presented a mathematical model of the single-mode electromagnetic field which involves a finite but arbitrarily large state space Ψ. The dimensionality of Ψ is allowed to tend to infinity only *after* calculation of physical results, such as expectation values, are made. Our model and the usual harmonic-oscillator model are equally valid and are physically indistinguishable. The advantage of our approach is that it permits the existence of an Hermitian phase operator, thus removing phase from its hitherto rather unique position as a classical observable without a quantum Hermitian operator counterpart.

We have described how the existence and form of the phase operator follow directly and uniquely from the states of well-defined phase. The physical state expectation value of the resulting phase-number commutator corresponds precisely with the classical Poisson bracket of the single-valued phase with the energy. The commutator contains Dirac's term and an additional contribution which resolves the anomalies associated with Dirac's commutator. However, the *exact form* of the phase-number commutator (which must be used for unphysical states such as phase states) is such as to preclude a direct extrapolation from the Poisson bracket.

A vital term in the unitary operator function $\exp(i\hat{\phi}_\theta)$ vanishes when combined with $\hat{N}^{1/2}$ in order to form the annihilation operator. This is the source of the difficulty in attempting to define a phase operator from the annihilation operator. The genuine unitarity of $\exp(i\hat{\phi}_\theta)$ gives it very different properties than the conventional Susskind-Glogower operator $\widehat{\exp}_s(i\phi)$. This is particularly evident when operating on field states with a significant vacuum component. Our unitary operator is in no sense an approximation to the Susskind-Glogower operator. Indeed, the phase properties of the vacuum state, and particularly the phase difference between two fields, are dramatic illustrations of the difference between our formulation and that of Susskind-Glogower.

The derived phase properties of the partial phase states, which include the coherent states, are consistent with those obtained by phenomenological methods[9] where the latter are valid. The phase operator allows us to construct a continuous unitary transformation between the number states. This transformation also allows us to access new basis sets of noninteger number states which are *not* eigenstates of $\hat{N}$. The continuous unitary transformation is of utility in constructing moment generating functions for the phase operator.

We conclude that our model of an electromagnetic field mode is not only physically indistinguishable from the conventional mathematical model involving the infinite Hilbert space harmonic operator, but is also more useful in that it allows us to define a well-behaved phase operator. Optical phase can at last be treated within the framework of quantum electrodynamics.

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[25]We follow the notation of our earlier work (Ref. 7) and denote the Susskind-Glogower operators as $\hat{\exp}_s(\pm i\phi)$.