Scaling theory for the localization length of the kicked rotor

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The relation $\xi=\frac{1}{2}D\hbar^{-2}$ between the localization length ξ and the diffusion coefficient D of the kicked rotor is derived in the framework of the scaling theory for localization. It is argued that this relation, first found by Shepelyansky [Phys. Rev. Lett. **56**, 677 (1986); Physica **28D**, 103 (1987)], reveals the special importance of the Lloyd model for the understanding of the quantal behavior of the kicked rotor and other dynamical systems. The finite-size-scaling form of the localization length and the conductance of the Lloyd model are derived.

I. INTRODUCTION

The kicked rotor[1-5] is a simple system that models the quantal dynamics of driven systems that are chaotic in the classical limit. It is defined by the Hamiltonian

$$H=H_0+V(\theta)\sum_m \delta(t-m)\ , \tag{1.1}$$

where

$$H_0=\frac{1}{2I}p^2 \tag{1.2}$$

with the angular momentum p and moment of inertia I. Classically the model exhibits diffusion in angular momentum for strong potentials $V(\theta)$, namely, for long time

$$\langle p^2\rangle=Dt\ , \tag{1.3}$$

where the average is over initial conditions and D is the diffusion coefficient. For the potential

$$V(\theta)=K\cos\theta\ , \tag{1.4}$$

the Hamiltonian (1.1) generates the well-known standard map where diffusion is found[6] for $K>K_c\approx 0.9716$.

In order to study the quantal behavior one has to investigate the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\theta,t)=H\psi(\theta,t) \tag{1.5}$$

with the Hamiltonian (1.1). Since this Hamiltonian is periodic in time the wave functions can be expanded in terms of the quasienergy states,

$$\chi_\alpha(\theta,t)=e^{-i\omega_\alpha t}u_\alpha(\theta,t)\ , \tag{1.6}$$

where $u_\alpha(\theta,t)=u_\alpha(\theta,t+1)$. The quasienergies ω_α are defined in the interval $(0,2\pi)$.

The dynamics are determined by the nature of the quasienergy states χ_α. It can be shown[5] that these satisfy the equation

$$T_n u_n+\sum_{r\neq 0} W_r u_{n+r}=Eu_n \tag{1.7}$$

with

$$u_r=\lim_{\delta t\to 0}\langle r|u_\alpha(\theta,m+\delta t)+u_\alpha(\theta,m-\delta t)\rangle\ , \tag{1.8}$$

where $|r\rangle$ are the eigenstates of the angular momentum operator $\hat{p}$ with the eigenvalues $r\hbar$, while

$$T_n=\tan\phi_n\ , \tag{1.9}$$

with

$$\phi_n=\tfrac{1}{2}(\tfrac{1}{2}\tau n^2-\omega),\quad \tau=\hbar/I \tag{1.10}$$

and

$$W_r=\langle 0|W(\theta)|r\rangle\ , \tag{1.11}$$

where

$$W(\theta)=\tan[\tfrac{1}{2}V(\theta)/\hbar] \tag{1.12}$$

and $W_0=-E$. But (1.7) is just a one-dimensional tight-binding model in solid-state physics. The correspondence between (1.1) and (1.7) implies that the quantum dynamics of a periodically driven rotor in momentum space resemble those of an electron in a one-dimensional solid.

It was argued[5] that if τ/π is a generic irrational number this sequence is pseudorandom. If the sequence $\{T_n\}$ is truly random and if the W_r fall off sufficiently fast with r, Eq. (1.7) is just the one-dimensional Anderson model for localization. It is well known[7,8] that for this model all the states are exponentially localized. On the basis of this analogy it was argued that for (generic) irrational τ/π all the quasienergy states are localized in momentum space.

In particular, this argument was tested for the driving potential

$$V(\theta)=2\hbar \arctan(\kappa \cos\theta) \ . \tag{1.13}$$

In the corresponding solid-state model only the hopping matrix elements W_r that connect nearest neighbors, namely, $W_{\pm 1}$ do not vanish. If one assumes that $\{\phi_n\}$ is truly random and uniformly distributed in $(0,2\pi)$, the $\{T_n\}$ follows a Lorentzian (or Cauchy) distribution. In this case (1.7) is the Lloyd model.[19] It was verified[5] that for the kicked rotor with the potential (1.13), the localization length is identical to the one of the (random) Lloyd model.[8,10]

For the standard kicked rotor with the driving potential (1.4) it was found that the localization length oscillates around the value that is found for the corresponding solid-state model, where ϕ_n is random.[11,12] For this potential $W(\theta)$ is singular for $K/\hbar > \pi$, and the behavior of the W_r is complicated. This singularity is, however, peculiar to the specific mapping that was used in the derivation of (1.7). If one[12] defines

$$\bar{u}_r = \lim_{\delta t \to 0} \langle r | e^{iV(\theta)/2\hbar} u(\theta, m+\delta t) \rangle \ , \tag{1.14}$$

it satisfies

$$\sum_r J_r(\tfrac{1}{2}K/\hbar)\sin(\phi_n + \pi r/2)\bar{u}_{n+r} = 0 \ , \tag{1.15}$$

where J_r is the ordinary Bessel function. The advantage of this model is that it is regular. It corresponds, however, to an Anderson model with correlated off-diagonal randomness that was not explored in solid-state physics.

A relation between the localization length ξ, and the classical diffusion coefficient D was found by Shepelyansky,[12] following arguments of Chirikov, Izrailev, and Shepelyansky.[3] It holds in the semiclassical limit $\hbar \to 0$ for strong kicking. It is argued that classical diffusion takes place until a time t^* that is determined by the localization length in momentum space, namely,

$$\hbar^2 \xi^2 \approx \langle p^2 \rangle = Dt^* \ . \tag{1.16}$$

But the time t^* is also the time when the discreteness of the local spectrum becomes important. If a typical separation between two quasienergies is $\Delta\omega$, then

$$\Delta\omega t^* \approx 1 \ . \tag{1.17}$$

Approximately ξ quasienergy states overlap, therefore $\Delta\omega = \xi^{-1}$. Consequently,

$$\xi = \alpha D/\hbar^2 \ , \tag{1.18}$$

where α is a numerical constant of order unity. This constant cannot be determined from such a heuristic argument. This argument is similar to the one that was introduced by Allen[13] in the localization theory for disordered solids. It assumes implicitly the existence of a scaling theory with one length scale, namely, the localization length. Such a theory is necessary for the existence of a relation between the localization length ξ, that is defined by the rate of exponential decay of wave functions on length scales that are larger than ξ, and suppression of classical diffusion on length scales of the order of ξ. Such a relation exists in the scaling theory for localization in a random solid.[7,14] For large values of ξ it was verified by model calculations in one dimension.[15–17] Shepelyansky found that $\alpha = \frac{1}{2}$ for the driving potentials (1.4) and (1.13) as well as for kicked rotors with other potentials. Some of these results were verified by other researchers.[11] The universality of this number calls for some theoretical reasoning

In Sec. II the relation (1.18) is verified in the framework of the scaling theory for localization with the help of the Landauer formula.[18] In this derivation the relation between the behavior on scales that are larger than ξ and the one on scales that are smaller than ξ is explicit. It is shown that if the distribution of the diagonal energies for the solid-state model (1.7) corresponding to (1.1) satisfied the conditions of the central limit theorem, i.e., had a finite second moment, one would expect $\alpha = 1$.

In Sec. III it is shown that for the Lloyd model one finds $\alpha = \frac{1}{2}$. From the derivation it is obvious that the Lorentzian distribution of the diagonal energies is essential for this result. For all the kicked rotors the diagonal energies, T_n of the analogous random solid-state model (1.7), exhibit a Lorentzian distribution. This is actually the case also in the formulation of Eq. (1.15) as is clear if one divides it by $\sin\phi_n$. Then all the pseudorandom terms are of the form $\mathrm{cotan}(\phi_n)$. If one assumes that ϕ_n can be considered random, the distribution of these terms is Lorentzian. In Sec. III the finite-size scalings of the conductivity and of the localization length are obtained for the Llyod model. To our knowledge, these results were not derived in the framework of the localization theory so far.

II. THE RELATION BETWEEN THE LOCALIZATION LENGTH AND THE DIFFUSION COEFFICIENT

The evolution operator of the kicked rotor that is defined by the Hamiltonian (1.1) is

$$\hat{U} = e^{iH_0/\hbar} e^{-iV(\theta)/\hbar} \ . \tag{2.1}$$

It propagates the wave function one time step, namely,

$$\psi_-(\theta, m+1) = \hat{U}\psi_-(\theta, m) \ , \tag{2.2}$$

where $\psi_-(\theta,m)$ is the wave function just before the m-th kick. Since $\hat{U}$ is unitary, it can be expressed in the form

$$\hat{U} = e^{-i\tilde{H}/\hbar} \ , \tag{2.3}$$

where $\tilde{H}$ is Hermitian. The quasienergy operator $\tilde{H}/\hbar$ determines the quantal dynamics of the system. Its eigenstates are the quasienergy states χ_α of (1.6). They correspond to the energy states of a solid that is defined by the Hamiltonian $\tilde{H}$. Their projections on the angular momentum states satisfy (1.7).

Consider a chain consisting of N sites of this fictitious solid that is defined by $\tilde{H}$. Its conductance is analyzed in the framework of the scaling theory for localization, for various lengths N. The resistance of a one-dimensional chain is[16]

$$\rho_N = \tfrac{1}{2}[\cosh(2\gamma_N N) - 1] , \tag{2.4}$$

where $\gamma_N N$ is the logarithm of the largest eigenvalue of the product of N transfer matrices of the chain. It is found with the help of the Landauer formula. The work of Pichard[16] should be consulted for details. The localization length ξ of the infinite system is the limit $\xi^{-1} = \lim_{N\to\infty} \gamma_N$. Exponential localization is found for $N \gg \xi$, where[16,17]

$$\rho_N \sim \tfrac{1}{4} \exp(2N/\xi) . \tag{2.5}$$

For $N \ll \xi$ Ohmic behavior is expected, namely,[16,17]

$$\rho_N \sim \bar{\rho} N , \tag{2.6}$$

$\bar{\rho}$ is the resistivity that is independent of the length N. In order to introduce a scaling ansatz for γ_N, the averaged finite-size localization length, ξ_N is defined as

$$\xi_N^{-2} = \langle \gamma_N^2 \rangle , \tag{2.7}$$

where the average is over realizations of the random potential. This average is relevant for the average resistance, as can be seen from (2.4) for $\gamma_N N \ll 1$. Consistency of (2.4) with (2.6) for $N \ll \xi$ implies[16] that in this regime,

$$(\xi_N/\xi) \sim A\sqrt{N/\xi} , \tag{2.8}$$

where A is a constant. The dimensionless conductivity in this regime is

$$g = \bar{\rho}^{-1} = A^2 \xi . \tag{2.9}$$

Using a weak disorder expansion Pichard found[16] that $A^2 = \frac{1}{2}$, for all models with distributions of diagonal energies that satisfy the conditions of the central-limit theorem.

We assume that the chain of length N, of the fictitious solid that we consider, is embedded between two electron baths, as shown in Fig. 1. It is connected to the baths by segments of an ideal conductor, having identical site energies $T_n = 0$. This is a standard geometry[18–20] for the application of the Landauer formula. If a chemical potential difference is introduced between the baths, it will result in an electric current through the chain. The zero-temperature conductivity is

$$\sigma = g\frac{e^2}{2\pi\hbar} = (A^2\xi)\frac{e^2}{2\pi\hbar} , \tag{2.10}$$

where only one spin direction was taken into account. The conductivity is related to the diffusion coefficient by the Einstein relation,

$$\sigma = \tfrac{1}{2} D_0 e^2 (\delta n/\delta\tilde{E}) . \tag{2.11}$$

D_0 is the diffusion coefficient and $(\delta n/\delta\tilde{E})$ is the density of states per site for $\tilde{H}$. It is important to remember that the diffusion and conductivity that are considered in (2.11) and (2.10) are in the pure quantum regime, namely, between the points C_1 and C_2 of Fig. 1 and do not involve the baths. The energies $\tilde{E}$ of the Hamiltonian $\tilde{H}$ are distributed in the interval $(0, 2\pi\hbar)$, since the quasienergies are restricted to $(0, 2\pi)$. There is, on the average,

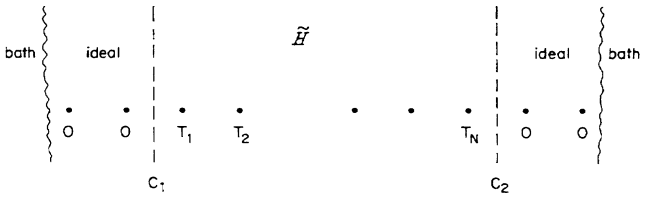

FIG. 1. The geometry where the quantum conductance and diffusion between C_1 and C_2 are defined.

one quasienergy state on each angular momentum state of (1.1), leading to the density of states per site

$$\delta n/\delta\tilde{E} = 1/2\pi\hbar . \tag{2.12}$$

Therefore (2.10) and (2.11) lead to

$$\xi = D_0/2A^2 . \tag{2.13}$$

The sites of the solid that is studied in this section correspond to angular momentum states of the kicked rotor. Their separation is $\hbar$; therefore the corresponding diffusion coefficient in momentum space is $D = D_0/\hbar^2$. In particular, for small values of $\hbar$ one obtains (1.18) where $\alpha = A^{-2}/2$.

For solid-state models with distributions of diagonal energies that have a finite second moment one finds[16] $A^2 = \frac{1}{2}$ leading to $\alpha = 1$. For the model corresponding to the kicked rotor this distribution is Lorentzian and consequently a different value of α may be found. In Sec. III, it will be shown that for the Lloyd model $A^2 = 1$ implying $\alpha = \frac{1}{2}$, in agreement with the numerical results.

III. FINITE-SIZE SCALING FOR THE LLOYD MODEL

In this section we show that (2.8) holds for the Lloyd model in the band center, $E = 0$, and calculate the constant A. We consider the tight-binding model

$$\epsilon_n u_n + u_{n+1} + u_{n-1} = 0 \tag{3.1}$$

in the geometry of Fig. 2. For $n \le 0$, the system is ideal and is modeled by $\epsilon_n = 0$, while for $n > 0$ it is disordered, with a Lorentzian (Cauchy) distribution of diagonal energies

$$P_{\eta+i\delta}(\epsilon_n) = \frac{1}{\pi}\frac{\delta}{(\epsilon_n - \eta)^2 + \delta^2} , \tag{3.2}$$

where $\eta = 0$.

The relation between the "localization length" ξ_N of the system of size N and the Lyapunov exponent of the transfer matrix is used in what follows. For this purpose that rate of growth of u_n is calculated using arbitrary initial conditions.[11,12,16,21]

First note that on the pure side $n < 0$, where $\epsilon_n = 0$ (see Fig. 2), the eigenstates satisfy

$$u_{n+1} = \pm i u_n . \tag{3.3}$$

Starting from an arbitrary real initial condition on the pure side, (3.3) implies that for all $n < 0$ the norm of the

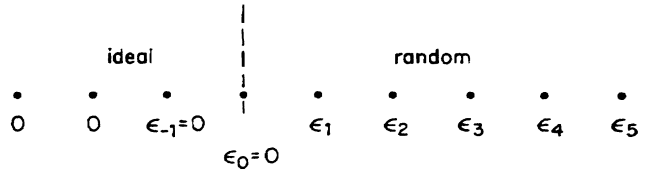

FIG. 2. The geometry where the finite-size scaling of the localization length is defined.

vector (u_{n+1},u_n) is constant. It will be chosen to be unity. The various initial conditions on the pure side will be determined by a parameter θ on the interface, namely,

$$\mathbf{v}_0=\begin{bmatrix} u_0 \\ u_{-1} \end{bmatrix}=\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} . \tag{3.4}$$

Starting from this initial condition, the wave function on the disordered side can be found by the transfer matrix method.[11,12,16,21] For this purpose, (3.1) is written in the form

$$\mathbf{v}_{n+1}=M_n\mathbf{v}_n , \tag{3.5}$$

with

$$\mathbf{v}_n=\begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} \tag{3.6}$$

and

$$M_n=\begin{bmatrix} -\epsilon_n & -1 \\ 1 & 0 \end{bmatrix} . \tag{3.7}$$

It is useful to define the product matrix

$$R_N=M_NM_{N-1}\cdots M_0 . \tag{3.8}$$

The wave function at an arbitrary site is obtained by the application of R_N to the initial conditions, namely,

$$\mathbf{v}_N=R_N\mathbf{v}_0 . \tag{3.9}$$

For random matrices M_n, Furstenberg's theorem[22] assures that the norm of $\mathbf{v}_n$ grows exponentially in the limit $N\rightarrow\infty$.

In this work we are interested mainly in the behavior of $\mathbf{v}_N$ for $N\ll\xi$. Let us denote by

$$\mathbf{v}_\pm^N=(x_\pm^N,y_\pm^N) \tag{3.10}$$

the eigenvectors of R_N with eigenvalues $\exp(\pm\gamma_NN)$, respectively. Their norm is taken to be unity. The initial vector $\mathbf{v}_0$ can be expanded in terms of these nonorthogonal vectors in the form

$$\mathbf{v}_0=a_+^N\mathbf{v}_+^N+a_-^N\mathbf{v}_-^N , \tag{3.11}$$

with

$$\begin{aligned} a_+^N&=\frac{1}{\Delta_N}(y_-^N\cos\theta-x_-^N\sin\theta) , \\ a_-^N&=\frac{1}{\Delta_N}(x_+^N\sin\theta-y_+^N\cos\theta) , \end{aligned} \tag{3.12}$$

where

$$\Delta_N=x_+^Ny_-^N-y_+^Nx_-^N . \tag{3.13}$$

The vector $\mathbf{v}_N$ is expressed in terms of these vectors as

$$\mathbf{v}_n=R_N\mathbf{v}_0=a_+^Ne^{\gamma_NN}\mathbf{v}_+^N+a_-^Ne^{-\gamma_NN}\mathbf{v}_-^N \tag{3.14}$$

and the square of its length is

$$(\mathbf{v}_N\cdot\mathbf{v}_N)=1+|a_+^N|^2(e^{2\gamma_NN}-1)+|a_-^N|^2(e^{-2\gamma_NN}-1) . \tag{3.15}$$

The normalization of $\mathbf{v}_0$ and $\mathbf{v}_\pm^N$ was used. In order to simplify (3.15) it is averaged over the initial conditions θ. From (3.12), one finds

$$\langle|a_+^N|^2\rangle_\theta=\langle|a_-^N|^2\rangle_\theta\frac{1}{2|\Delta_N|^2} , \tag{3.16}$$

where $\langle\ \rangle_\theta$ denotes the average over θ, with a uniform distribution. For $\gamma_NN\ll1$,

$$\langle(\mathbf{v}_N\cdot\mathbf{v}_N)\rangle_\theta=1+2(\gamma_NN)^2/|\Delta_N|^2+O((\gamma_NN)^2) . \tag{3.17}$$

Since $\lim_{N\rightarrow0}|\Delta_N|^2=1$, the elongation of $\mathbf{v}_N$, to the leading order, is

$$d\equiv\langle(\mathbf{v}_N\cdot\mathbf{v}_N)\rangle_\theta-1=2\gamma_N^2N^2 . \tag{3.18}$$

Its average over the realizations of randomness is

$$\langle d\rangle=2\langle\gamma_N^2\rangle N^2 . \tag{3.19}$$

Actually the quantity that should be averaged is $\ln(1+d)$ but for $N\ll\xi$ it reduces to $\langle d\rangle$ of (3.19) (see Ref. 16). In this regime the localization length ξ_N is defined by (2.7).

In order to calculate ξ_N a relation between $\langle d\rangle$ and ξ will be found, using a method that was introduced by Ishii.[8] For this purpose we write Eq. (3.1) in the form

$$z_{n+1}=-\epsilon_n-1/z_n , \tag{3.20}$$

where $z_n=u_n/u_{n-1}$. He found that for a Lorentzian distribution, $P_{i\delta}(\epsilon_n)$, of ϵ_n, the distribution of the ratios z_n is Lorentzian, as well. It is the distribution $P_{s_n}(z_n)$ of (3.2), where the s_n satisfies the recursion relation

$$s_{n+1}=i\delta-1/s_n . \tag{3.21}$$

Ishii used this recursion relation, in order to find the stationary distribution of z_n and the localization length. In particular, for $E=0$, he found the result

$$\cosh\gamma=[1+(\delta/2)^2]^{1/2} , \tag{3.22}$$

where $\gamma=1/\xi$. For $\xi\gg1$ ($\gamma\ll1$, $\delta\ll1$), that is of interest in the present work

$$\gamma\approx\delta/2 . \tag{3.23}$$

We use (3.21), in order to find the distribution of ratios z_n for $N\ll\xi$ iterations. It is different from the stationary distribution.[8] In the pure region, where $\delta=0$, one finds $s_n=i$. The initial condition at the interface between the pure and random regions is that the distribution of z_0 is $P_i(z_0)$, i.e., $s_0=i$. It corresponds to a distribution of ini-

tial conditions, where θ of (3.4) is uniformly distributed. With this initial condition, one finds, for $n \ll \xi$ and $\delta \ll 1$,

$$s_n = \begin{cases} i + O(\delta^2), & n \text{ even} \\ i(1+\delta) + O(\delta^2), & n \text{ odd} . \end{cases} \tag{3.24}$$

For $1 \ll \xi$, the typical growth rate at each site is small, i.e., $|z_n| - 1 \ll 1$ so that $|z_n| \approx 1 + \ln|z_n|$. Since the distribution of z_n is Lorentzian, its average does not exist. For the scaling theory the behavior of the typical quantities should be considered.[15-17] These can be very different from the averages, if these are dominated by the tail of the distribution, as is the case in particular for Lorentzians. The average of $\ln|z_n|$ exists and for $N \ll \xi_N$ the average of the typical elongation of the vector $\mathbf{v}_0$, corresponding to (3.19) is

$$\langle d \rangle = \sum_{n=1}^{N} \langle \ln|z_n| \rangle . \tag{3.25}$$

With the help of the integral,

$$\int_{-\infty}^{\infty} dz P_{i\sigma}(z) \ln|z| = \ln\sigma \tag{3.26}$$

for real σ, one finds, to the first order in δ

$$\langle \ln|z_n| \rangle = \begin{cases} 0, & n \text{ even} \\ \delta, & n \text{ odd} . \end{cases} \tag{3.27}$$

Consequently for $1 \ll N \ll \xi$, to the leading order,

$$\langle d \rangle = \delta N . \tag{3.28}$$

Comparison with (3.19) leads to

$$\xi_N / \xi = \sqrt{N/\xi} . \tag{3.29}$$

This is the result (2.8) with $A = 1$. Consequently one expects that (1.18) is satisfied with $\alpha = \frac{1}{2}$ for a kicked rotor model, where the corresponding solid state model (1.7) is a Lloyd model.

IV. SUMMARY

The relation (1.18) between the localization length and the classical diffusion coefficient was derived in the framework of the scaling theory for localization. This provides additional support for the detailed correspondence between quantal behavior of driven chaotic systems and Anderson localization in disordered solids. The fact that the solid state systems that correspond to the kicked rotor exhibit a Lorentzian distribution of diagonal energies, manifests itself by the value $\alpha = \frac{1}{2}$. Therefore this value of α reveals the special importance of the Lloyd model for the understanding of the quantal behavior of the kicked rotor and other dynamical systems. If there are dynamical systems corresponding to solid-state systems with distributions of diagonal energies satisfying the central limit theorem, then we expect to find $\alpha = 1$. We are not aware of any such system, however. All calculations in Secs. II and III were performed for models with hopping to nearest neighbors only. We conjecture that these are representative and the results should hold for models with hopping that is of short range but not restricted to nearest neighbors.

It was shown that the finite-size scaling of the localization length and of the conductance of the Lloyd model is similar to the one that is found for generic models for localization, namely, models with distributions of diagonal energies, that satisfy the central limit theorem. These scaling forms differ however, by a numerical constant [see (2.8) and (3.29)].

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