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Spin-measurement retrodiction

Shahar Ben-Menahem*
School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Ramat Aviv, Tel-Aviv 69978, Israel
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We address the problem of choosing an initial state and final measurement that enable maximal retrodiction of the result of an unknown intermediate spin measurement of a spin-$\frac{1}{2}$ particle, along any one of a given set of directions. Details are given of the derivation of the most general possible solution of the problem, which we reported recently. The maximal retrodiction is shown to be the determination of the result of the intermediate spin measurement, along any of four different directions that span three-dimensional space and satisfy a specific linear relation.

I. INTRODUCTION

Due to the noncommutativity of different components of angular momentum in quantum mechanics, one cannot prepare a system that is a simultaneous eigenstate of more than one component. Thus, given a spin-$\frac{1}{2}$ particle, and given a set of distinct directions, one can at most predict the result of a future measurement of spin along *one* of these directions. However, the situation is quite different if we are asked to *retrodict* the result of the spin measurement; Vaidman, Aharonov, and Albert have recently demonstrated[1,2] that, provided the spinor is coupled to another "detector" spinor in a certain way, a suitable final measurement can retrodict the result of a past spin measurement (performed though after the initial coupling of the spinor to the detector) along any one of a (*a priori* given) set of three orthogonal directions.

In a recent work[3] I have briefly reported on the extension of their results to the maximal admissible number of candidate directions for the intermediate measurement (namely, four) and/or the most general directions allowed. In this paper I give the full derivation of my results.

Denote the set of *a priori* given candidate directions by $\{\mathbf{n}_l\}$, where l ranges from 1 to m and $\mathbf{n}_l$ are unit vectors. The possible intermediate measurements are then of the operators $\boldsymbol{\sigma}\cdot\mathbf{n}_l$, with $\boldsymbol{\sigma}$ the Pauli matrices of the spin-$\frac{1}{2}$ particle. If $m=1$ (the set $\{\mathbf{n}_l\}$ consisting of only one unit vector), the problem is trivially solved, either by choosing the state ϕ to be an eigenstate of $\boldsymbol{\sigma}\cdot\mathbf{n}$, or by measuring $\mathbf{Q}=\boldsymbol{\sigma}\cdot\mathbf{n}$ at the end. (The first alternative is really a *pre*diction rather than *retro*diction.) The case $m=2$ is also simple to solve—let ϕ be an eigenstate of the spin along one direction, say $\mathbf{n}_1$, and $\mathbf{Q}$ (measured at the end) be $\boldsymbol{\sigma}\cdot\mathbf{n}_2$. Both these cases do not require coupling the spin-$\frac{1}{2}$ particle to any other degrees of freedom. But what if we require $m\geq 3$, and m distinct unit vectors $\{\mathbf{n}_l\}$ to choose from? (Distinct here means also that no two are parallel, for opposite directions define the same spin measurement.) Here it is easy to see that no solution to the problem is possible if we restrict ourselves to just the spin-$\frac{1}{2}$ Hilbert space. As noted above, Refs. 1 and 2 have tackled the case where $m=3$ and $\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}$ is an orthonormal triple. Using a geometric method, they have found a particular solution to the retrodiction problem. We have chosen to treat the problem using a different formulation, which allows us to (a) find exactly for which m and for which sets of directions the problem is solvable, and (b) give a simple algebraic procedure for solving it when it is. Let us summarize our formulation of the problem and the results we obtained. The formulation involves the following.

(a) An initially prepared state ϕ, in a Hilbert space that is a direct product of the space of the spin-$\frac{1}{2}$ particle, and some spin J system; the dimension of the whole space is $d=2(2J+1)$, and must be even.[4]

(b) The "unknown measurement," of either of several spin components $\boldsymbol{\sigma}\cdot\mathbf{n}_l$, $1\leq l\leq m$. Here $\mathbf{n}_l$ are unit vectors, spanning three-space. (We refer to ordinary three-dimensional space as "three-space" in this paper.) [We make that last assumption because, as it turns out, the problem has no interesting (i.e., $m\geq 3$) solutions when all $\mathbf{n}_l$ are in a plane; this case is disposed of in the Appendix.] $\boldsymbol{\sigma}$ are the Pauli matrices on the space of the original spin-$\frac{1}{2}$ particle.

(c) A final measurement, of a Hermitian operator $\mathbf{Q}$. This operator has in general d distinct eigenvalues $\{\lambda_A\}$, and their corresponding orthonormal eigenstates φ_A.

(d) The final element of the formulation is the technical concept of *partition,* explained in Sec. II.

Our results are the following.

(1) Of the space of all possible operators $\mathbf{Q}$ for $J=\frac{1}{2}$ (a sixteen-dimensional real space) there is a twelve-parameter subset for which one can retrodict any one of σ_a, $1\leq a\leq 3$, i.e., solve the case when $m=3$ and $\{\mathbf{n}_l\}$ is an orthonormal triple.

(2) For $m=3$, $\mathbf{n}_l$ a nonorthonormal triple, J must be at least 1, and if $\mathbf{n}_l$ are to be unconstrained directions, one needs at least $J=\frac{3}{2}$ ($d=8$). For $d=8$, a construction is given of a possible pair (ϕ,$\mathbf{Q}$) for any triple $\{\mathbf{n}_l\}$ that spans the three-space.

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(3) If $\{\mathbf{n}_l\}$ spans the three-space but $m=4$, J must be at least 1. Such a construction is given. It places only one restriction: that the signs of $\mathbf{n}_l$ can be chosen so that $\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4=0$. *Any* construction for $m=4$ will impose this constraint, not only the example we give—whatever J.

(4) Assuming $\{\mathbf{n}_l\}$ spans three-space, *m is at most* 4, no matter what J is.

The rest of this paper is organized as follows. In Sec. II the concept of partition is defined, and is used to formulate the retrodiction problem concisely. In Sec. III we apply our formalism to the case of three orthogonal directions, and in the process reproduce the solution of Ref. 1; indeed, we find for this case the most general solution which uses a $d=4$ dimensional Hilbert space. Section IV is devoted to the case of three nonorthogonal directions, Sec. V treats the $m>3$ case, while in Sec. VI we restate our conclusions briefly. The Appendix contains the proof that there are no solvable cases when $m\geq 3$ and $\{\mathbf{n}_l\}$ lie in a plane, thus justifying our assumption throughout that $\mathbf{n}_l$ span the three-space.

II. PARTITIONS

The initially prepared state is

$$\phi=\sum_{A=1}^{d} b_A\varphi_A\ , \tag{1}$$

where

$$\mathbf{Q}\varphi_A=\lambda_A\varphi_A\ , \tag{2}$$

where b_A are complex numbers; we choose the phases of φ_A so that b_A are real. φ_A are orthonormal. After $\boldsymbol{\sigma}\cdot\mathbf{n}_l$ is measured, giving a result $\eta_l=\pm1$, the state becomes the projected (no summation over l). (The X superscript signifies that this state is an unknown , to be "retrodicted"),

$$\phi^X_{\eta_l}(\mathbf{n}_l)=\tfrac{1}{2}(1+\eta_l\boldsymbol{\sigma}\cdot\mathbf{n}_l)\phi\ . \tag{3a}$$

Of course, the ϕ^X states as given by (3a) are not normalized. If the subsequent measurement of $\mathbf{Q}$ is to deterministically retrodict η_l, the expansions of the two states $\phi^X_\pm$ in the basis φ_A must consist of disjoint subsets of $\{A\}$; thus the value of λ that is measured will unambiguously tell us η_l.[5]

Let the disjoint subsets be $\{\lambda_A : A\in S_{\eta_l}(\mathbf{n}_l)\}$. Then since[5] $b_A\neq 0$, we have $S_+(\mathbf{n}_l)\cup S_-(\mathbf{n}_l)=\{A : 1\leq A\leq d\}$ for any $1\leq l\leq m$. Also

$$\phi^X_{\eta_l}(\mathbf{n}_l)=\sum_{A\in S_{\eta_l}(\mathbf{n}_l)} b_A\varphi_A\ , \tag{3b}$$

$$\phi=\phi^X_+(\mathbf{n}_l)+\phi^X_-(\mathbf{n}_l)\ . \tag{3c}$$

Thus for each $1\leq l\leq m$, there is a *partition* of $\{\lambda_A\}$ into the sets $S_\pm(\mathbf{n}_l)$. What's more, it is a nontrivial partition: $S_\pm$ are nonempty, for otherwise $\boldsymbol{\sigma}\cdot\mathbf{n}_l\phi=\pm\phi$ for some l, which contradicts Eq. (8a) below. This means that ϕ itself cannot be used to *predict* the outcome of the intermediate measurement along any of the directions, so we have true retrodiction— in the sense that the result of the final measurement is needed to deduce η_l, once l is given.

For each of these partitions, define the sign function $\epsilon^{(l)}$ (a mapping from $\{A\}$ onto Z_2) as follows:

$$\epsilon_A^{(l)}=\begin{cases}+1 \text{ if } A\in S_+(\mathbf{n}_l)\\ -1 \text{ if } A\in S_-(\mathbf{n}_l)\ .\end{cases} \tag{4}$$

In what follows, we will actually refer to the $\epsilon^{(l)}$ functions as the partitions.

Our problem is to construct pairs $(\phi,\mathbf{Q})$ for given $\{\mathbf{n}_l\}$. In $\mathbf{Q}$, λ_A are arbitrary as long as they remain distinct, so really only φ_A and b_A are to be found; φ_A should be expressed in terms of the Cartesian-product basis of the d-dimensional Hilbert space. But since Eqs. (3) are to be solved, it is more convenient to express instead the action of σ^a, $1\leq a\leq 3$, on the basis $\{\varphi_A\}$. We next express Eqs. (3) in terms of the partitions (4), and derive simple conditions that $b_A,\epsilon_A^{(l)}$ must satisfy. These conditions will be necessary and sufficient for the existence of σ^a in the φ_A basis, and once solutions $b_A,\epsilon_A^{(l)}$ are found, it is easy to construct σ^a. Only for point (1) (see Sec. I) will we explicitly build $\mathbf{Q}$; usually we will merely find $\{\epsilon_A^{(l)}\}$ and $\{b_A\}$, and that suffices to construct $\mathbf{Q}$ if so desired, as will be shown below.

From (1)–(4) it follows that (A always ranges $1\leq A\leq d$),

$$\begin{aligned}\boldsymbol{\sigma}\cdot\mathbf{n}_l\phi&=\sum_A\epsilon_A^{(l)}b_A\varphi_A,\\ \phi&=\sum_A b_A\varphi_A,\end{aligned}\qquad 1\leq l\leq m\ . \tag{5}$$

Now, assuming the σ matrices $(\sigma^a)_{AB}$ exist in the large Hilbert space, they must satisfy their usual algebra, which implies

$$(\boldsymbol{\sigma}\cdot\mathbf{n}_l)(\boldsymbol{\sigma}\cdot\mathbf{n}_j)=\mathbf{n}_l\cdot\mathbf{n}_j+i(\mathbf{n}_l\times\mathbf{n}_j)\cdot\boldsymbol{\sigma}\ . \tag{6}$$

Also, σ^a must be Hermitian.

By Eq. (5),

$$\langle\phi|\boldsymbol{\sigma}\cdot\mathbf{n}_l|\phi\rangle=\sum_A\epsilon_A^{(l)}d_A\ ,$$

where (7)

$$d_A=|b_A|^2\ .$$

The normalization of ϕ implies

$$\sum_A d_A = 1 \ . \tag{7a}$$

m is at least 3, and $\{\mathbf{n}_l\}$ is assumed to span the three-space (see the Introduction). Now take the expectation value of Eq. (6) in $|\phi\rangle$, making use of Eqs. (5) and (7) and the hermiticity of $\boldsymbol{\sigma}\cdot\mathbf{n}_l$; and use also the fact that $\{\mathbf{n}_l \times \mathbf{n}_j\}$, being essentially the dual basis to $\{\mathbf{n}_l\}$, also spans the three-space. Then the *imaginary* part of the resulting c-number equation is equivalent to

$$\sum_A \epsilon_A^{(l)} d_A = \langle \phi | \boldsymbol{\sigma}\cdot\mathbf{n}_l | \phi \rangle = 0 \ , \tag{8a}$$

whereas the *real* part gives

$$\mathbf{n}_l \cdot \mathbf{n}_j = \sum_A \epsilon_A^{(l)} \epsilon_A^{(j)} d_A \ . \tag{8b}$$

Note that $\epsilon_A^{(l)}\epsilon_A^{(j)} = \epsilon_A^{(lj)}$ is again a partition. It is useful to introduce some terminology for future convenience. A partition ϵ is "good" if $\sum_A \epsilon_A d_A = 0$; then

$$\sum_{A \in S_\eta} d_A = \tfrac{1}{2}, \quad \eta = \pm 1 \ (\text{good partition}) \ ; \tag{9}$$

otherwise ϵ is "bad." ϵ is "pointed" if any one of S_+, S_- that it induces has only a single element. Two partitions ϵ, ϵ' are "mutually nested" if $S_\eta \subset S'_{\eta'}$ for some η, $\eta' \in \{+,-\}$. From Eqs. (5) and (8) we easily find the following:

(1) ϵ^l is good.

(2) For $l \neq j$, $\epsilon^{(l)} \neq \pm\epsilon^{(j)}$; otherwise by (5), $\boldsymbol{\sigma}\cdot(\mathbf{n}_l + \eta\mathbf{n}_j)\phi = 0$ (for some sign η), which is impossible since $\mathbf{n}_l \neq \pm\mathbf{n}_j$ and the eigenvalues of $\boldsymbol{\sigma}\cdot(\mathbf{n}_l + \eta\mathbf{n}_j)$ are $\pm|\mathbf{n}_l + \eta\mathbf{n}_j|$.

(3) For $l \neq j$, $\epsilon^{(l)}$ and $\epsilon^{(j)}$ are not mutually nested. If they were, say, $S_+(\mathbf{n}_l) \subset S_+(\mathbf{n}_j)$, then $\sum_{A \in S_+(\mathbf{n}_l)} d_A < \sum_{A \in S_+(\mathbf{n}_j)} d_A$ (since all $b_A \neq 0$), contradicting Eq. (9), which states that both sides of the inequality are $\frac{1}{2}$.

(4) $\epsilon^{(l)}$ is not pointed; if it were for some l, it would (trivially) be mutually nested with any other $\epsilon^{(j)}$, $j \neq l$.

(5) $\epsilon^{(lj)}$ are true partitions for $l \neq j$ (i.e., they map $\{A\}$ *onto* Z_2). If two vectors $\mathbf{n}_l$, $\mathbf{n}_j$ are orthogonal, $\epsilon^{(lj)}$ is good, and otherwise it is bad.

In addition to (7a) and (8), there is yet another useful equation that follows from (5) if $m > 3$; there are $(m-3)$ linear relations between $\{\mathbf{n}_l\}$, say,

$$\mathbf{n}_{j+3} = \sum_{l=1}^{3} c_l^{(j)} \mathbf{n}_l, \quad 1 \leq j \leq m-3 \tag{10}$$

(where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are any three linearly independent $\mathbf{n}_l$'s). From (5) and (10) we get

$$\epsilon^{(j+3)} = \sum_{l=1}^{3} c_l^{(j)} \epsilon^{(l)} \tag{11}$$

(we sometimes drop the subscript A).

So far, (8) and (11) were derived as *necessary* conditions. But we now have a theorem.

Theorem. For given d and $\{\mathbf{n}_l\}$, a set of necessary and sufficient conditions for the existence of Hermitian matrices $(\sigma^a)_{AB}$ satisfying (5) and (6) is the following: (i) there exist $\epsilon_A^{(l)}$, $d_A = |b_A|^2$ that solve (7a) and (8), *and* (ii) if $m > 3$, there exist numbers $c_l^{(j)}$, $1 \leq j \leq m-3$, $1 \leq l \leq 3$ such that (11) holds.

Proof. We have already shown the necessity of (i) and (ii). If we assume (i) and (ii), we can simply *define* (for $l = 1,2,3$) the action of $\boldsymbol{\sigma}\cdot\mathbf{n}_l$ on ϕ according to (5). If $m > 3$, (10) and (11) then imply (5) also for $3 < l \leq m$. Since $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ span the three-space, we then know the four states ϕ, $\sigma^a\phi$, $1 \leq a \leq 3$. By (8a), (8b), and (5), we see that these four states are orthonormal. Through (6) we *define* the action of σ^a on these four states, and on the subspace they span. Equations (5)–(8) then imply that σ^a are Hermitian and satisfy the correct algebra on the subspace. Thus we have obtained a four-dimensional representation of σ^a. If $d > 4$ we simply extend the action of σ^a to the rest of Hilbert space by creating a direct sum of the four-dimensional representation with an arbitrary $d - 4 = 2(2J-1)$-dimensional representation of σ matrices. Thus we obtain the action of σ^a on the φ_A states. This procedure not only completes the proofs but is a way to construct φ_A, and thus ϕ and $\mathbf{Q}$, in terms of the usual angular momentum bases of the Hilbert space, or, if J is not an angular momentum, a Cartesian product of the usual spin-$\frac{1}{2}$ basis (say, $\sigma^3 = \pm 1$) and any basis of the spin-J system coupled to the spinor. [Such a Cartesian product basis is found by choosing any basis for the Hilbert subspace $\sigma^3 = 1$, and defining the corresponding basis of the $\sigma^3 = -1$ subspace by acting on the first basis with σ^1. Note that this pairing of the bases of the two subspaces $\sigma^3 = \pm 1$ guarantees that d is even; see parenthetical remark in point (a) of the Introduction.] In Sec. III we will demonstrate explicitly such a construction of $(\phi, \mathbf{Q})$.

III. THE $m = 3$, ORTHOGONAL CASE

We practice the above formalism on the case, $m = 3$, $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ an orthonormal basis of the three-space. In fact, let us choose Cartesian axes along these directions; we can render this a right-handed basis by proper choice of the signs of $\mathbf{n}_l$. We specify to $J = \frac{1}{2}$ ($d = 4$), which proves sufficient for this case. Since $\mathbf{n}_l \cdot \mathbf{n}_j = \delta_{lj}$, by (8b) $\epsilon^{(lj)}$ is a good partition if $l \neq j$. By the rules (1)–(4) above, each partition $\epsilon^{(l)}$ must divide $\{1,2,3,4\}$ into two sets, of two elements each. It is easy to see that the most general solution, up to a permutation of the labels A, is

$$S_+^{(1)} = \{1,3\}; \quad S_\eta^{(2)} = \{1,4\}; \quad S_+^{(3)} = \{1,2\} \ , \tag{12}$$

where $\eta = \pm 1$. Then we find $\epsilon^{(12)} = \eta\epsilon^{(3)}$, $\epsilon^{(23)} = \eta\epsilon^{(1)}$, $\epsilon^{(13)} = \eta\epsilon^{(2)}$, so $\epsilon^{(lj)}$ are good and (8b) is equivalent to (8a). Equation (8a) is equivalent to [see (9)]

$$\sum_{A \in S_+(\mathbf{n}_l)} d_A = \tfrac{1}{2} \ , \tag{9$'$}$$

so we find

$$d_1 + d_3 = d_2 + d_4 = d_1 + d_4 = d_1 + d_2 = \tfrac{1}{2}$$

and, hence,

$$d_1 = d_2 = d_3 = d_4 = \tfrac{1}{4} \ . \tag{13}$$

Thus, if the phase convention is that b_A are real and positive,

$$b_A = \tfrac{1}{2}, \quad 1 \le A \le 4 \; . \tag{14}$$

Then (5), (12), and (14) imply ($\eta = \pm 1$ is arbitrary)

$$\begin{aligned} 2\phi &= \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \; , \\ 2\sigma_x \phi &= \varphi_1 + \varphi_3 - \varphi_2 - \varphi_4 \; , \\ 2\sigma_y \phi &= \eta(\varphi_1 + \varphi_4 - \varphi_2 - \varphi_3) \; , \\ 2\sigma_z \phi &= \varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 \; . \end{aligned} \tag{15}$$

The usual σ^a algebra then tells us how σ^a operates on these four states, which form an orthonormal basis of Hilbert space. Thus, $(1/\sqrt{2})(\sigma_x + \zeta i \sigma_y)\phi$ and $(1/\sqrt{2})(1+\sigma_z \zeta)\phi$ ($\zeta = \pm 1$) are an orthonormal pair, and a basis of the sub-Hilbert space with $\sigma_z = \zeta$. Call the $J = \frac{1}{2}$ (auxiliary spinor) Pauli matrices τ^a; the states $|\zeta_1\rangle_\sigma |\zeta_2\rangle_\tau$, $\zeta_{1,2} = \pm 1$, are a basis for the full $d = 4$ Hilbert space. Since also $\{|\zeta\rangle_\sigma |+\rangle_\tau, |\zeta\rangle_\sigma |-\rangle_\tau\}$ is an orthonormal basis for the $\sigma_z = \zeta$ Hilbert subspace, a τ-spinor unitary matrix relates it to the above pair, namely,

$$\begin{bmatrix} \frac{1}{\sqrt{2}}(\sigma_x + i\zeta\sigma_y)\phi \\ \frac{1}{\sqrt{2}}(1+\zeta\sigma_z)\phi \end{bmatrix} = \| e^{i\boldsymbol{\theta}_\zeta \cdot \boldsymbol{\tau} + i\chi_\zeta} \| \begin{bmatrix} |\zeta\rangle_\sigma |+\rangle_\tau \\ |\zeta\rangle_\sigma |-\rangle_\tau \end{bmatrix} , \tag{16}$$

where $\boldsymbol{\theta}_\pm$ are arbitrary real three-vectors, and $\chi_\pm$ are arbitrary scalars. Equations (15) and (16) easily yield φ_A as linear combinations of $|\zeta\rangle_\sigma |\zeta'\rangle_\tau$; then arbitrary (distinct) $\lambda_A, 1 \le A \le 4$, must be chosen, and then both $\mathbf{Q} = \sum_A \lambda_A |\varphi_A\rangle\langle\varphi_A|$ and the preprepared state $\phi = \sum_A \frac{1}{2}\varphi_A$ [by Eq. (15)] are known. Thus the general solution of our problem for $d = 4$ and $m = 3$, $\mathbf{n}_l \cdot \mathbf{n}_j = \delta_{lj}$ is characterized by 12 continuous real parameters $(\{\lambda_A\}, \boldsymbol{\theta}_\pm, \chi_\pm)$, as well as an arbitrary sign, η.

Upon consulting Ref. 1, it is a straightforward exercise to check that their solution to the retrodiction problem for the case in this section is a special case of our general solution, corresponding to the following parameters:

$$\begin{aligned} \boldsymbol{\theta}_+ &= \left[\frac{\pi}{2}, 0, 0 \right], \quad \boldsymbol{\theta}_- = \mathbf{0} \; , \\ \chi_+ &= -\frac{\pi}{2} \; , \\ \chi_- &= 0 \; , \\ \eta &= +1 \end{aligned}$$

(λ_A are four arbitrary distinct numbers).

IV. THE CASE $m = 3$, $\{\mathbf{n}_l\}$ NONORTHOGONAL

If $d = 4$, then the labels A can be permuted so any three partitions $\epsilon^{(l)}$, satisfying conditions (1)–(4) of Sec. II, are given by (12). Since then $\epsilon^{(12)} = \eta\epsilon^{(3)}$, etc. [see equations following Eq. (12)], $\epsilon^{(lj)}$ are good partitions for $l \ne j$, so by (5), $\mathbf{n}_l \cdot \mathbf{n}_j = \delta_{lj}$, which is not the case in this section. This proves that we need $d \ge 6$ ($J \ge 1$) for the nonorthogonal case. Note that for $d = 4$ there are no further distinct partitions which satisfy (1)–(4) beyond $\epsilon^{(1,2,3)}$, so $m > 3$ cannot be handled with $d = 4$; this fact will be used in Sec. V. Going back to $m = 3$, we consider $d = 6$ ($J = 1$). There are a number of different ways to choose $\{\epsilon_A^{(l)}\}$; we find that several lead to solutions, but in all cases $\mathbf{n}_l \cdot \mathbf{n}_j$ are not the most general cosines between three directions. This means that they are not three independent real numbers between 0 and 1.[6] In fact, there cannot be a solution with three free parameters, because there are only six d_A, and the "goodness" conditions (8a) (of which there are three) together with (7a) are four equations, so there remain only two real parameters, on which $\mathbf{n}_l \cdot \mathbf{n}_j$ depend through (8b). Nor could (7a) and (8a) be linearly dependent, for that would imply a linear dependence $\sum_{l=1}^{3} c_l \epsilon_A^{(l)} = 0$, implying via (5) that $\sum_{l=1}^{3} c_l \mathbf{n}_l = 0$, contradicting the assumption that $\mathbf{n}_l$ span the three-space.

Despite the fact that $d = 6$ cannot give the most general nonorthogonal solution, it *can* give a two-parameter subset, as the following example demonstrates. Let the partitions be

$$\begin{aligned} S_+(\mathbf{n}_1) &= \{2,4,6\} \; , \\ S_+(\mathbf{n}_2) &= \{1,2\} \; , \\ S_+(\mathbf{n}_3) &= \{2,3,6\} \; . \end{aligned} \tag{17}$$

They clearly satisfy conditions (1)–(4) of Sec. **II**. The goodness equations become

$$\begin{aligned} d_1 + d_2 &= d_2 + d_4 + d_6 = d_2 + d_3 + d_6 = \tfrac{1}{2} \; , \\ d_1 + d_2 &+ d_3 + d_4 + d_5 + d_6 = 1 \; . \end{aligned} \tag{18}$$

If d_1, d_3 are taken as independent, the solution of (18) is

$$\begin{aligned} d_2 &= \tfrac{1}{2} - d_1 \; , \\ d_4 &= d_3 \; , \\ d_6 &= d_1 - d_3 \; , \\ d_5 &= \tfrac{1}{2} - d_1 - d_3 \end{aligned} \tag{19}$$

and after a little algebra, (8b) gives for the angles between the measurement axes,

$$\begin{aligned} \mathbf{n}_1 \cdot \mathbf{n}_2 &= 1 - 4d_1 \; , \\ \mathbf{n}_2 \cdot \mathbf{n}_3 &= 1 - 4d_1 \; , \\ \mathbf{n}_3 \cdot \mathbf{n}_1 &= 1 - 2d_1 - 4d_3 \; , \end{aligned}$$

so the angles are determined by only two parameters. Note that $\mathbf{n}_2 \cdot \mathbf{n}_3 = \mathbf{n}_1 \cdot \mathbf{n}_2$ in this construction.

Even for $d > 6$, not all choices of partitions give three independent directions. We will simply record a particular solution that does give independent directions. Let $d>6$ be arbitrary (even ; see the Introduction). Let

$$\sum_{A>6} d_A = B \ . \tag{20a}$$

Also, we decree that

$$\epsilon_A^{(l)} = +1 \quad (1 \le l \le 3, \ A > 6) \tag{20b}$$

(d could be chosen 8, i.e., $J=\frac{3}{2}$). Consider the following partitions [each $S_+(\mathbf{n}_l)$ set contains every $A>6$, which is indicated by dots]:

$$\begin{aligned} &S_+(\mathbf{n}_1)=\{5,6,\ldots\} \ , \\ &S_+(\mathbf{n}_2)=\{1,2,5,\ldots\} \ , \\ &S_+(\mathbf{n}_3)=\{2,4,6,\ldots\} \ . \end{aligned} \tag{21}$$

Taking into account (20), the goodness conditions become

$$\begin{aligned} d_2+d_4+d_6 &= d_1+d_3+d_5-B \\ &= d_5+d_6=d_1+d_2+d_5 \\ &= \tfrac{1}{2}-B \ . \end{aligned} \tag{22}$$

Taking B, d_4, and d_6 as independent, we get the solution

$$\begin{aligned} &d_1=-\tfrac{1}{2}+B+d_4+2d_6, \quad d_2=\tfrac{1}{2}-B-d_4-d_6 \ , \\ &d_3=\tfrac{1}{2}-d_4-d_6, \quad d_5=\tfrac{1}{2}-B-d_6 \ . \end{aligned} \tag{23}$$

So from Eqs. (8b), (21), and (23) we get

$$\begin{aligned} &\mathbf{n}_1\cdot\mathbf{n}_2=1-4d_6 \ , \\ &\mathbf{n}_2\cdot\mathbf{n}_3=1-4d_4-4d_6 \ , \\ &\mathbf{n}_3\cdot\mathbf{n}_1=4B-1+4d_6 \ . \end{aligned} \tag{24}$$

Thus, the three angles are indeed free. In fact, the only constraints on B, d_4, d_6 are (since $d_A=|b_A|^2>0$) $B>0$, $d_4>0$, $d_6>0$, and $d_1>0$, $d_2>0$, etc., in (23). It is tedious but straightforward to prove that, if relabeling of $\mathbf{n}_l$ is allowed, no inequalities are imposed on $\mathbf{n}_l\cdot\mathbf{n}_j$ beyond the geometric ones.[6] This completes our construction for $m=3$ and $\{\mathbf{n}_l\}$ nonorthogonal. As $d_4\to 0$, $d_6\to\frac{1}{4}$, $B\to 0$, $d_1\to 0$ our solution tends to the case $\mathbf{n}_l\cdot\mathbf{n}_j=\delta_{lj}$, and the partitions (21), with $A=1,4$ and all $A>6$ removed, become (modulo a relabeling of A values) the $d=4$ solution of Sec. III.

V. THE CASE $m>3$

As noted at the beginning of Sec. IV, $m>3$ is impossible for $d=4$. The current section consists of two parts: first we show that $m=4$ is possible for $d=6$ by an explicit solution (for which $\{\mathbf{n}_l\}$ satisfies the linear relation recorded in the Introduction; this relation precludes any three among them from being orthogonal). Then, we prove that $m>4$ is impossible, and that any $m=4$ solution must obey that linear relation among the four unit vectors.

Consider the partitions ($d=6$),

$$\begin{aligned} &S_+(\mathbf{n}_1)=\{1,2,3\} \ , \\ &S_+(\mathbf{n}_2)=\{1,5,6\} \ , \\ &S_+(\mathbf{n}_3)=\{3,4,6\} \ . \end{aligned} \tag{25}$$

They satisfy the conditions (1)–(4) of Sec. II, and by (8a), (d_5, d_6 taken as independent),

$$\begin{aligned} d_4+d_5+d_6 &= d_1+d_2+d_3=d_3+d_4+d_6 \\ &= d_1+d_5+d_6=\tfrac{1}{2} \ , \end{aligned} \tag{26a}$$

so

$$d_1=d_4=\tfrac{1}{2}-d_5-d_6, \quad d_2=d_6, \quad d_3=d_5 \tag{26b}$$

and (8b) gives

$$\begin{aligned} &\mathbf{n}_1\cdot\mathbf{n}_2=1-4d_5-4d_6 \ , \\ &\mathbf{n}_2\cdot\mathbf{n}_3=4d_6-1 \ , \\ &\mathbf{n}_3\cdot\mathbf{n}_1=4d_5-1 \end{aligned} \tag{26c}$$

(these three angles depend on only two parameters, which is always the case for $d=6$, as explained in Sec. IV). But if $m=4$, there is one more condition on the partitions: Eq. (11), which reads

$$\epsilon_A^{(4)}=\sum_{l=1}^{3} c_l\epsilon_A^{(l)} \tag{11'}$$

for some three real numbers c_l, of which at least two are not equal to 0. Taking A to range from 1 to 6 and reading off $\epsilon_A^{(l)}$ from (25), we find

$$\begin{aligned} &c_1+c_2-c_3=\text{sign} \ , \\ &c_1+c_3-c_2=\text{sign} \ , \\ &c_2+c_3-c_1=\text{sign} \ . \end{aligned} \tag{27}$$

The signs on the right-hand side are unspecified, and form part of the fourth partition $\epsilon^{(4)}$. Note that solving (27) is sufficient (recalling that at least two c's are nonzero); properties (1)–(4) of Sec. II, and Eq. (8b) for all $1\le l\le 4$ then follow from the $1\le l\le 3$ case. Thus $\epsilon^{(4)}$ will automatically be good, etc. Equation (27) is a necessary *and* sufficient condition when augmented with the $1\le l\le 3$ conditions.

Equation (27) may be simply solved: with the convention $c_1<0$ we find $c_1=c_2=c_3=-1$. Thus

$$\sum_{l=1}^{4}\mathbf{n}_l=0 \ . \tag{28}$$

This requires that $\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3$ be a unit vector, a fact already guaranteed by the parametrization (26c). Later we shall see, as a corollary to our proof that $m\le 4$, that (28) is *always* true if $m=4$ (of course, up to arbitrary signs in the definitions of the unit vectors $\mathbf{n}_l$). Thus the above $m=4$ construction, albeit a particular one, gives the most general retrodiction possible for the $m=4$ case.

We conclude with the proof that $m>4$ is impossible and that (28) is the most general $m=4$ case. We proceed by assuming numbers c_l, $1\le l\le 3$ such that

$\sum_{l=1}^{3} c_l \epsilon_A^{(l)} = \text{sign}$, and showing that if there is a solution, it is unique. Thus in (11), j may take at most one value; this will prove that $m \leq 4$.

Proof. Consider any three partitions, $\epsilon_A^{(l)}$, $1 \leq l \leq 3$, that satisfy conditions (1)–(4) of Sec. II. A ranges from 1 to an arbitrary d; the dimension d does not enter at all in the proof. Let (ijk) be some permutation of (123). Let $T = \{A\}$, the set of all A labels.

Lemma. For any (ijk) if ($\varnothing$ the empty set)

$$\begin{aligned} \varnothing &= S_+(\mathbf{n}_i) \cap S_-(\mathbf{n}_j) \cap S_-(\mathbf{n}_k) \\ &= S_-(\mathbf{n}_i) \cap S_+(\mathbf{n}_j) \cap S_+(\mathbf{n}_k) , \end{aligned} \tag{29}$$

then $c_i = \zeta$, $c_j = c_k = -\zeta$, where $\zeta = \pm 1$.

Proof of Lemma. If (29) holds, then recalling $S_+ \cup S_- = T$ (for any l), we easily see that

$$\begin{aligned} &S_+(\mathbf{n}_j) \cup S_+(\mathbf{n}_k) \supset S_+(\mathbf{n}_i) \supset S_+(\mathbf{n}_j) \cap S_+(\mathbf{n}_k) \neq \varnothing , \\ &S_+(\mathbf{n}_i) \cap S_-(\mathbf{n}_j) \neq \varnothing , \\ &S_+(\mathbf{n}_i) \cap S_-(\mathbf{n}_k) \neq \varnothing , \\ &S_+(\mathbf{n}_j) \cap S_-(\mathbf{n}_i) \neq \varnothing , \\ &S_+(\mathbf{n}_k) \cap S_-(\mathbf{n}_i) \neq \varnothing . \end{aligned} \tag{30}$$

By taking $A \in S_+(\mathbf{n}_j) \cap S_+(\mathbf{n}_k)$, the condition $\sum_{l=1}^{3} c_l \epsilon_A^{(l)} = \text{sign}$ gives us

$$c_i + c_j + c_k = \text{sign} . \tag{31a}$$

Next, take $A \in S_+(\mathbf{n}_j) \cap S_-(\mathbf{n}_i)$; there the condition gives

$$c_j - c_k - c_i = \text{sign} , \tag{31b}$$

whereas $A \in S_+(\mathbf{n}_k) \cap S_-(\mathbf{n}_i)$ gives

$$c_k - c_j - c_i = \text{sign} . \tag{31c}$$

In addition, $A \in S_+(\mathbf{n}_i) \cap S_-(\mathbf{n}_k)$ and $A' \in S_+(\mathbf{n}_i) \cap S_-(\mathbf{n}_j)$ give, respectively (all right-hand sides are independent signs),

$$c_i + c_j - c_k = \text{sign} , \tag{31d}$$

$$c_i + c_k - c_j = \text{sign} . \tag{31e}$$

Equation (31d) is redundant with (31c), (31e) with (31b), and the general solution of the remaining independent equations is just as stated in the lemma.

Using the lemma, the theorem now follows easily. For if the assumption (29) of the lemma is false for all permutations (ijk), we can pick

$$\begin{aligned} A_1 \in &[S_+(\mathbf{n}_1) \cap S_-(\mathbf{n}_2) \cap S_-(\mathbf{n}_3)] \\ &\cup [S_-(\mathbf{n}_1) \cap S_+(\mathbf{n}_2) \cap S_+(\mathbf{n}_3)] , \end{aligned}$$

and similarly A_2, A_3 in a cyclic manner; then the condition (11′) at $A = A_1, A_2, A_3$ yields

$$\begin{aligned} c_1 + c_2 - c_3 &= \text{sign} , \\ c_2 + c_3 - c_1 &= \text{sign} , \\ c_1 + c_3 - c_2 &= \text{sign} . \end{aligned} \tag{32}$$

This, together with the fact (noted above) that at least two c_l's must be nonzero, implies $c_1 = c_2 = c_3 = \zeta$, ζ a sign. Thus there can be only one $c^{(j)}$ in Eqs. (10) and (11), since two $\mathbf{n}_j$ vectors differing by a minus sign are not allowed (they define the same measurement). Hence, the only case where it remains to prove the theorem is when there is a permutation such that (29) holds. But in that case the lemma assures us that, once again, the ordered set of c_l's is unique up to a sign; so the proof is complete.

VI. CONCLUSIONS

We summarize the results obtained in this paper. Given a spin-$\frac{1}{2}$ particle and a set of directions in space, we wish to interact with (measure) the particle (with some auxiliary quantum system added) at two instances in time, in such a way that when told along which of those directions the spin was measured (at an intermediate time), the result of that measurement will be known (retrodicted). The first "measurement" consists in preparing a pure state. Our aim here was to find how large and general the set of directions can be while still allowing retrodiction, and to find a simple way to generate solutions to the problem when they exist. We also asked, what is the minimal number of degrees of freedom the auxiliary system must have for each given set of directions. We found that, beyond the trivial $m \leq 2$ cases, the directions must span three-space in order that there be a solution. We also found that m is at most 4, that when $m = 4$ there are solutions but only if the four unit vectors satisfy a certain linear relation, and that the case $m = 3$ can be solved for any three distinct directions. We gave the minimal dimensions of Hilbert space for which a solution exists for each of the above cases. We have shown a general way to construct solutions $(\phi, \mathbf{Q})$, and gave several particular solutions, including the general one for the case $d = 4$, $m = 3$ and an orthogonal triple of directions $\mathbf{n}_l$; this last has as a special case the solution of Ref. 1. Finally, we remark that our procedure is in fact capable of yielding the most general solution for *any* d and $\{\mathbf{n}_l\}$, provided all inequivalent sets of partitions $\{\epsilon^{(l)}, 1 \leq l \leq m\}$ that solve Eqs. (7), (8), and (11) are tried; searching through all such sets is a straightforward, if tedious, combinatorical task.

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APPENDIX

Let us prove that there are no solutions to the retrodiction problem when $m \geq 3$ unless the directions $\mathbf{n}_l$ span the three-space. If they do not, they lie in a plane, say, XY. If $m \geq 3$, there are at least three pairs of projected states $\phi^X_{\eta_l}(\mathbf{n}_l)$, and each of these pairs defines a partition [see Eqs. (3a)–(5)]. However, the remark after Eq. (3c) is no longer valid, since it depends on Eq. (8a), which depends on the fact that $\mathbf{n}_l \times \mathbf{n}_j$ span the three-space, which they don't now. Thus, there is no reason why $\epsilon^{(l)}$ have to be true partitions. That is, it is possible that $\epsilon_A^{(l)} = +1$ for all A and some l value; then η_l would be "predicted." But whether this happens or not does not matter to our proof. Let $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ be any three of the directions. Since they are linearly dependent, so are there corresponding partitions; thus there are two nonzero numbers c_1, c_2 such that

$$c_1 \epsilon_A^{(1)} + c_2 \epsilon_A^{(2)} = \text{signs} , \tag{A1}$$

where the signs on the right-hand side are $\epsilon_A^{(3)}$. Now, the partitions $\epsilon^{(1,2)}$ are distinct [e.g., by point (2) of Sec. II], so one can find an A value where they are equal, and another where they are opposite; thus,

$$c_1 + c_2 = \text{sign} , \tag{A2}$$

$$c_1 - c_2 = \text{sign} . \tag{A3}$$

Since $c_{1,2}$ must be different from 0, these two equations have no solution. This completes the proof.

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*Present address: Lyman Lab of Physics, Harvard University, Cambridge, MA 02138.

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[3] S. Ben-Menahem (unpublished).

[4] J need not be an angular momentum; we refer to it as "spin" for convenience only. In general, the only operators in our formalism that must obey the angular momentum Lie algebra are the Pauli matrices of the original spin-$\frac{1}{2}$ particle. Still, J should be a half-integer, since the full product Hilbert space has as one of its factors the two-dimensional space of the spin-$\frac{1}{2}$ particle; thus d is even.

[5] In what follows we assume $b_A \neq 0$ and all λ_A distinct; relaxing these assumptions means merely that the effective d is lowered. In physical terms, this means that only a subset of the auxiliary degrees of freedom coupled to the spinor are involved in the preparation of ϕ and subsequent measurements. (This fact follows easily from the procedure outlined in this section, and will not be proved here.) Also, all $\mathbf{n}_l$ are mutually nonparallel, for parallel $\mathbf{n}$'s correspond to the same spin measurement.

[6] Actually there are three inequalities satisfied by these three numbers, for any three directions spanning the three-space. One of these expresses that $\mathbf{n}_l$ are linearly independent—it is that their box product is not equal to 0. The other two express the limited range of possible angles between the three directions, e.g., $\mathbf{n}_l \cdot \mathbf{n}_j = -1$ for all $l \neq j$ is ruled out by geometrical considerations.