Electric fields in stationary ion flows in symmetric geometries

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A method is presented that yields exact closed-form solutions in curvilinear coordinates for the partial differential equations determining the electric field in a stationary flow of unipolar ions with constant mobility. The method is applied to the parabolic cylindrical, elliptic cylindrical, bipolar, parabolic, and spheroidal coordinates. The new solutions approach infinity roughly in the manner of the classical parallel-plate solution. They are expected to apply to analyses of non-self-sustained ion flows.

I. INTRODUCTION

The problems concerning the unipolar ion flow have been reviewed by Sigmond[1] with an emphasis on corona discharges. The main problem is the space-charge field. The differential equations determining the total electric field $\mathbf{E}$ are well known. The case usually considered is a stationary flow of ions of one species with constant mobility and negligible diffusion, when

$$\operatorname{div}(\mathbf{E}\operatorname{div}\mathbf{E})=0 , \tag{1}$$

$$\operatorname{curl}\mathbf{E}=0 . \tag{2}$$

Both industrial applications and scientific problems have called for investigation of these equations, but exact closed-form solutions have been found only for three general geometries. For the circular cylindrical geometry we have

$$E^2=A+B/r^2 , \tag{3}$$

where A and B are arbitrary constants and r is the cylindrical radius coordinate. The spherical solution is

$$E^2=A/R+B/R^4 , \tag{4}$$

where R is the spherical radius coordinate. For the parallel-plate geometry, the solution is

$$E^2=Ax+B , \tag{5}$$

where x denotes an axis along the middle plane of the plates.

Here we present a method by which we can find exact closed-form solutions for other usual[2] coordinate systems.

II. THE DIFFERENTIAL EQUATIONS IN CURVILINEAR COORDINATES

In orthogonal curvilinear coordinates (x_1,x_2,x_3), with the scale factors h_1,h_2,h_3, Eq. (1) becomes

$$\sum_i \partial_i \left[E_i \left[\sum_j \partial_j E_j h_1 h_2 h_3 / h_j \right] \Big/ h_i \right] = 0 , \tag{6}$$

where $\partial_i=\partial/\partial x_i$, and $\sum_i$ and $\sum_j$ mean sums over the indices 1,2,3. The usual assumption of rotational or translational symmetry eliminates the terms with $i=3$ and $j=3$ in (6) and reduces equation (2) to

$$\partial_1 E_2 h_2 = \partial_2 E_1 h_1 . \tag{7}$$

In solutions (3)–(5) we have $E_2=0$ and $E_1 h_1$ does not contain x_2, whence (7) is satisfied trivially. More general solutions of (7) can be given in terms of a function $U(x_1,x_2)$ which has the necessary derivatives and is defined as follows:

$$E_j h_j = U^m \partial_j U , \tag{8}$$

where m is a real number; in the present applications we need the value $m=\frac{1}{2}$. The function U is related to the potential V,

$$V=-\int E_j h_j dx_j = -U^{m+1}/(m+1)+c , \tag{9}$$

where c is a constant. By inserting (8) into (6), we obtain

$$\sum_i \partial_i \left[[(\partial_i U)/h_i^2] U^{2m} \times \sum_j [m(\partial_j U)^2 H_j/U + \partial_j(H_j \partial_j U)] \right] = 0 , \tag{10}$$

where $H_j=h_1h_2h_3/h_j^2$. A solution of (10) can be represented by two equations,

$$\sum_j \partial_j (H_j \partial_j U)=0 , \tag{11}$$

$$U^{2m-1}\sum_j (\partial_j U)^2/h_j^2 = C , \tag{12}$$

where C is a constant; Eq. (11) is used for both $\sum_j$ and $\sum_i$.

III. APPLICATIONS TO GEOMETRIES WITH TRANSLATIONAL SYMMETRY

We first consider translationally symmetric cases, for which $h_3=1$. We neglect the circular cylindrical geometry, which has been cleared up previously. The other usual coordinate systems with translational symmetry[2] fulfill the condition $h_1=h_2=h$, which we adopt. Then (11) and (12) become

$$\sum_j \partial_j \partial_j U = 0 \ , \tag{13}$$

$$U^{2m-1} \sum_j (\partial_j U)^2 = Ch^2 \ . \tag{14}$$

A simple solution of (13) is

$$U = px + k \ , \tag{15}$$

where $x = x_1$ and p and k are constants. In the Cartesian coordinates we have $h = 1$, where (15) also fulfills (14), and with $m = \frac{1}{2}$, Eq. (8) reproduces the classical solution (5).

Another polynomial solution of (13) is

$$U = b(\xi^2 - \eta^2 + k) \ , \tag{16}$$

where b is a constant. This form applies to the parabolic cylindrical coordinate system[2] (ξ, η, z), where $h^2 = \xi^2 + \eta^2$. From (16) and (9) we obtain

$$V = q(\xi^2 - \eta^2 + k)^{3/2} + c \ , \tag{17}$$

where q is a constant.

A nonpolynomial solution of (13) is

$$U = B \sinh(Pu) \sin(Pv) + D \sinh(Qu) \cos(Qv) + F \cosh(Ru) \sin(Rv) + G \cosh(Su) \cos(Sv) \ , \tag{18}$$

where $u = x_1$, $v = x_2$, and B, D, F, G, P, Q, R, and S are constants. For the special case $U = B \sinh u \sin v$, Eq. (14) gives

$$B^2(\sinh^2 u + \sin^2 v) = Ch^2 \ . \tag{19}$$

This agrees with the expression of h^2 in the elliptic cylindrical coordinate system (u, v, z).[2] The corresponding field is given by

$$\begin{aligned} E_1 &= [B \sinh u \sin v / (1 + \tanh u \cot v)]^{1/2} \ , \\ E_2 &= [B \sinh u \sin v / (1 + \coth u \tan v)]^{1/2} \ , \\ V &= -(\tfrac{2}{3})(B \sinh u \sin v)^{3/2} + c \ . \end{aligned} \tag{20}$$

Other coordinate systems corresponding to (18) are not elaborated here.

A more complicated solution of (13) is

$$U = B(\sinh\eta + \sin\xi)/(\cosh\eta - \cos\xi) \ , \tag{21}$$

for which

$$\sum_j (\partial_j U)^2 = 2B/(\cosh\eta - \cos\xi)^2 \ . \tag{22}$$

This gives the expression of h^2 in the bipolar coordinates (ξ, η, z).[2] The corresponding potential is

$$V = -\tfrac{2}{3}[B(\sinh\eta + \sin\xi)/(\cos\eta - \cos\xi)]^{3/2} + c \ . \tag{23}$$

IV. APPLICATIONS TO GEOMETRIES WITH ROTATIONAL SYMMETRY

Here we neglect the spherical geometry, which has been cleared up previously. The other usual geometries with rotational symmetry[2] comply with the condition $h_1 = h_2 = h$. Since $h_3 \neq 1$, Eq. (11) becomes

$$\sum_j \partial_j h_3 \partial_j U = 0 \ . \tag{24}$$

In the parabolic coordinates[2] (ξ, η, φ), the scale factor h is the same as in the parabolic cylindrical coordinates, where (14) would be satisfied with the previous solution (16). Although we now have $h_3 = \xi\eta$, solution (16) also fulfills condition (24). Thus, the expression of V is the same as (17), but it means here a rotationally symmetric potential.

In the prolate spheroidal coordinates[2] (u, v, φ) for which $h = a(\sinh^2 u + \sin^2 v)^{1/2}$ and $h_3 = a \sinh u \sin v$, the solution of (24) and (14) is

$$U = a \cosh u \cos v \ . \tag{25}$$

The corresponding potential is

$$V = b(\cosh u \cos v)^{3/2} + c \ . \tag{26}$$

The case of the oblate spheroidal coordinates is similar, except that $h_3 = a \cosh u \cos v$, where

$$U = a \sinh u \sin v \ . \tag{27}$$

V. CONCLUSIONS

The new solutions are functions of two variables, which is a novel feature in comparison with solutions (3)–(5). The present solutions do not vanish at infinity; they approach infinity roughly in the same way as the parallel-plate solution (5). On the basis of this analogy, the new solutions are expected to apply to analyses of non-self-sustained ion flows. Even in the simple parallel-plate geometry, such analyses have produced important general conclusions about unipolar ion flow.[1] The present method does not solve the question of possible closed-form solutions applying to self-sustained coronas.

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[1]R. S. Sigmond, J. Electrostat. **18**, 249 (1986).
[2]G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1970).