Boundary conditions of the diffusion equation and applications

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A particular expression of the mutual coherence function of a wave is first derived for a system of random layers with rough boundaries, starting from the unified Bethe-Salpeter equation that involves the random medium and boundaries on exactly the same footing. An effective scattering matrix of the medium is then introduced, which is the only medium-dependent quantity involved in the expression and is obtained as a boundary-value solution of the diffusion equation. The diffusion approximation is based on an eigenfunction expansion by using a set of eigenfunctions of the medium scattering cross section that can be even rotationally variant, and the first term is good enough in the diffusion region. Here each expansion coefficient is obtained as the solution of an integral equation with terms of the boundaries, and, for the first term, the equation can be converted to a diffusion equation with a source term, subjected to boundary conditions determined by the boundary scattering cross sections. Here the condition at each boundary is valid even when the averaged refractive indices of the two media differ from each other by a large amount, as contrasted to the condition of no reflection used so far when obtaining boundary-value solutions of diffusion equations. Specific expressions are obtained for both backscattered and transmitted waves through a random layer, along with numerical examples. The boundary condition is finally generalized to meet the case of a rough boundary between two random media of different kinds, consistent with power conservation.

I. INTRODUCTION

To investigate the wave coherence function in a system having a partial distribution of random media and random boundaries, it is possible to construct the Bethe-Salpeter (BS) equation of the entire system in such a way that the medium and the boundaries are involved in the equation on exactly the same footing, and this enables us to obtain several expressions of the solution to choose from, by introducing effective scattering matrices (of not only coherent but also incoherent characteristics) for the medium and/or the boundaries.[1,2] Here the transport equation can be conveniently utilized to obtain an elementary (incoherent) scattering matrix of the medium alone and thereby to construct the effective scattering matrices of the boundaries as affected by the medium fluctuation, for example (Sec. III). On the other hand, the transport equation has been solved mostly through numerical methods, while the diffusion equation has also been utilized as a simple alternative which enables us to obtain the solution analytically, because of the approximation involved which is often good enough to get practical answers. Also for the boundary-value problems, e.g., of a random layer with two free outside spaces (Fig. 1) the diffusion equation has been solved simply with the boundary condition of no reflection, regardless of whether the average refractive indices of the two media are nearly equal or differ from each other by a large amount, as in the case of a boundary between air and water with random scatterers in it.[3,4]

The diffusion approximation is based on an eigenfunction expansion of medium scattering matrices by using a set of eigenfunctions of the medium scattering cross section which can be arbitrarily anisotropic,[5] and the first term (to be referred to as the diffusion term hereafter) is good enough in the diffusion region (Sec. IV). Here in the case of a random layer, the diffusion term can be obtained as a boundary-value solution of a diffusion equation once the boundary condition is found in terms of a given scattering cross section of each boundary. To this end, the BS equation provides us with a powerful means by leading to an integral equation for the diffusion term, which has boundary terms so that the boundary condition can be found directly from the equation, and also showing the expression for both backscattered and transmitted waves through the layer, in terms of the diffusion term (Sec. V).

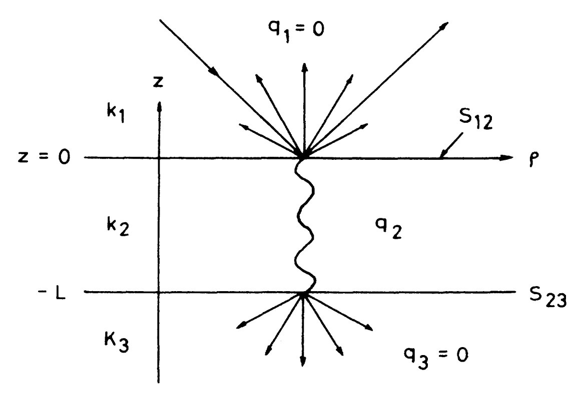

FIG. 1. Geometry of a random layer for Eqs. (2.23), (2.57), and (2.58).

The boundary condition can be generalized, consistently with power conservation, to meet the case of a rough boundary between two random media of different kinds [Sec. VI].

II. BS EQUATION FOR THE ENTIRE SYSTEM AND SCATTERING MATRICES

The coordinate vector in three-dimensional space is denoted by $\hat{\mathbf{x}}=(x_1,x_2,x_3)=(\boldsymbol{\rho},z)$ with $\boldsymbol{\rho}=(x_1,x_2)$ and $z=x_3$, where the z axis is taken in the direction normal to the average boundaries (Fig. 1). The scalar product of two space vectors $\hat{\mathbf{a}}=(\mathbf{a},a_z)$ and $\hat{\mathbf{b}}=(\mathbf{b},b_z)$ is denoted by $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=\mathbf{a}\cdot\mathbf{b}+a_zb_z$, where $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2$. We first consider two random layers separated by a rough boundary which is planar on average, as illustrated in Fig. 2. A scalar-wave function $\psi(\hat{\mathbf{x}})e^{i\omega t}$, where $\omega>0$ and t is time, is considered, and is denoted in each layer by $\psi_a(\hat{\mathbf{x}})$, $a=1,2$, whose wave equation is

$$[L_a-q_a(\hat{\mathbf{x}})]\psi_a(\hat{\mathbf{x}})=j_a(\hat{\mathbf{x}}) \;, \tag{2.1a}$$

$$L_a=-\left[\frac{\partial}{\partial\hat{\mathbf{x}}}\right]^2-k_a^2,\quad \mathrm{Im}(k_a)<0 \;. \tag{2.1b}$$

Here $q_a(\hat{\mathbf{x}})$ is the random part of the medium, and $j_a(\hat{\mathbf{x}})$ is a source term; k_a is the propagation constant when the medium is free from the random part, and the medium is assumed to be nondissipative for the time being. The boundary condition is assumed to be the continuity of ψ and its gradient normal to the (real) boundary surface, and consistently with this, the power vector $\hat{\mathbf{W}}(\hat{\mathbf{x}})$ is defined by

$$\hat{\mathbf{W}}(\hat{\mathbf{x}})=\frac{1}{2i}\psi^*\left[\frac{\overleftarrow{\partial}}{\partial\hat{\mathbf{x}}}-\frac{\overrightarrow{\partial}}{\partial\hat{\mathbf{x}}}\right]\psi(\hat{\mathbf{x}}) \;, \tag{2.2}$$

whose power equation is therefore

$$\frac{\partial}{\partial\hat{\mathbf{x}}}\cdot\hat{\mathbf{W}}(\hat{\mathbf{x}})=\frac{1}{2i}[\psi^*j(\hat{\mathbf{x}})-j^*\psi(\hat{\mathbf{x}})] \;. \tag{2.3}$$

The boundary condition can be transferred from the real boundary onto two reference boundary planes, say, S_1 and S_2 at $z=0$ and $z=-d_2$, respectively, chosen such that the change of the boundary height is ranged between S_1 and S_2 (Fig. 2); hence, with the notation $\partial_n^{(a)}=\hat{\mathbf{n}}^{(a)}\cdot\partial/\partial\hat{\mathbf{x}}$, where $\hat{\mathbf{n}}^{(a)}$ is the unit vector directed outward normally to S_a, the boundary equation can be written as[1]

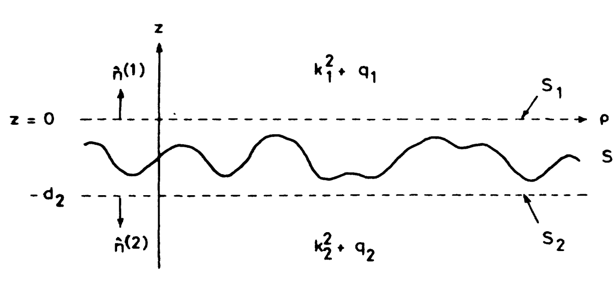

FIG. 2. Geometry of a rough boundary for Eq. (2.4). The real boundary S is distributed within the range $0>z>-d_2$.

$$-\partial_n^{(a)}\psi_a(\boldsymbol{\rho})=\sum_{b=1}^{2}\int d\boldsymbol{\rho}'B_{ab}^{(12)}(\boldsymbol{\rho}\,|\,\boldsymbol{\rho}')\psi_b(\boldsymbol{\rho}') \;. \tag{2.4}$$

Here $\psi_a(\boldsymbol{\rho})$ denotes $\psi_a(\hat{\mathbf{x}})$ bounded on S_a, and, when the boundary is nondissipative,

$$[B_{ab}^{(12)}(\boldsymbol{\rho}\,|\,\boldsymbol{\rho}')]^\dagger\equiv[B_{ba}^{(12)}(\boldsymbol{\rho}'\,|\,\boldsymbol{\rho})]^*=B_{ab}^{(12)}(\boldsymbol{\rho}\,|\,\boldsymbol{\rho}') \;, \tag{2.5}$$

i.e., the matrix defined by the elements $B_{ab}^{(12)}(\boldsymbol{\rho}\,|\,\boldsymbol{\rho}')$ is Hermitian with respect to both the coordinates and the indices. This means that $B^{(12)}$ is a real symmetrical matrix when the system is subject to the reciprocity relation with symmetrical elements of $B^{(12)}$. Hereafter, the boundary space enclosed by S_1 and S_2 will be neglected, on letting $d_2\to 0$, unless otherwise noted; so that $S_{12}=S_1+S_2$ at $z=0$ represents the two reference boundary planes, together.

The wave equations (2.1) and the boundary equation (2.4) can be unified to be written by one wave equation of the form

$$(L_a-q_a)\psi_a-\sum_{b=1}^{2}B_{ab}^{(12)}\psi_b=j_a \;. \tag{2.6}$$

Here both $B_{ab}^{(12)}$ and q_a are regarded as $\hat{\mathbf{x}}$ coordinate matrices, defined by the elements

$$B_{ab}^{(12)}(\hat{\mathbf{x}}\,|\,\hat{\mathbf{x}}')=\delta(z+d_a)B_{ab}^{(12)}(\boldsymbol{\rho}\,|\,\boldsymbol{\rho}')\delta(z'+d_b) \;,\quad d_1=0 \tag{2.7}$$

and $q_a(\hat{\mathbf{x}}\,|\,\hat{\mathbf{x}}')=q_a(\hat{\mathbf{x}})\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}')$, and the solution is subject to the new boundary condition that $\partial_n^{(a)}\psi_a=0$, $a=1,2$, inside the boundary space $0>z>-d_2$. The proof can be given by integrating Eq. (2.6) with respect to z over two infinitesimal regions enclosing S_1 and S_2, separately.

With a new matrix v_{ab}, defined by

$$v_{ab}=q_a\delta_{ab}+B_{ab}^{(12)} \;, \tag{2.8}$$

the equation of the deterministic Green function of the new wave equation (2.6), say, $g_{ab}(\hat{\mathbf{x}}\,|\,\hat{\mathbf{x}}')$, can be written as

$$\sum_c(L_a\delta_{ac}-v_{ac})g_{cb}(\hat{\mathbf{x}}\,|\,\hat{\mathbf{x}}')=\delta_{ab}\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}') \tag{2.9a}$$

or in matrix form as

$$(L-v)g=1,\quad v=q+B^{(12)} \;. \tag{2.9b}$$

Here v may be regarded as an effective medium representing both the medium and the boundary on an equal basis.

A. Statistical Green functions

Equation (2.9b) enables us to obtain the statistical Green functions in exactly the same form as those in an inhomogeneous random medium v, and the results are summarized as follows.[1,2] The averaged version of Eq. (2.9b) becomes written as

$$(L-M)G=1,\quad G=\langle g\rangle\ , \tag{2.10}$$

in terms of an effective medium M of v, defined by

$$MG=\langle vg\rangle,\quad M=M^{(q)}+M^{(12)}\ . \tag{2.11}$$

Here $M^{(q)}$ and $M^{(12)}$ are also defined in the same fashion, by

$$M^{(q)}G=\langle qg\rangle,\quad M^{(12)}G=\langle B^{(12)}g\rangle\ , \tag{2.12}$$

and are approximately equal to the independent contributions from the medium and the boundary, respectively, with the elements $M_a^{(q)}\delta_{ab}$ and $M_{ab}^{(12)}$.

For the statistical Green function of second order, defined by

$$I_{ab;cd}(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2\,|\,\hat{\mathbf{x}}_1';\hat{\mathbf{x}}_2')=\langle g_{ac}^*(\hat{\mathbf{x}}_1\,|\,\hat{\mathbf{x}}_1')g_{bd}(\hat{\mathbf{x}}_2\,|\,\hat{\mathbf{x}}_2')\rangle \tag{2.13a}$$

or in matrix form by

$$I(1;2)=\langle g^*(1)g(2)\rangle\ , \tag{2.13b}$$

(here and also hereafter, the subscript 1 is attached to the coordinates of quantities of the complex-conjugate wave function, and the subscript 2 is attached to those of the original wave function), we first introduce a matrix Δv, defined by

$$\Delta v=v-M=\Delta q+\Delta B^{(12)}\ , \tag{2.14a}$$

where

$$\Delta q=q-M^{(q)},\quad \Delta B^{(12)}=B^{(12)}-M^{(12)}\ , \tag{2.14b}$$

and employ the expression

$$g=G(1+\Delta vg),\quad \langle\Delta vg\rangle=0\ , \tag{2.15}$$

for both $g^*(1)$ and $g(2)$ in the right-hand side of Eq. (2.13b). Hence we obtain an expression

$$I(1;2)=G^*(1)G(2)[1+K(1;2)I(1;2)] \tag{2.16}$$

of the form of the Bethe-Salpeter equation, with a matrix $K(1;2)$, defined by

$$K(1;2)I(1;2)=\langle\Delta v^*(1)\Delta v(2)g^*(1)g(2)\rangle \tag{2.17}$$

in the same fashion as M by Eq. (2.11). Here $K(1;2)$ can also be approximated by the independent sum of $K^{(q)}$ from the medium and $K^{(12)}$ from the boundary, as

$$K(1;2)\simeq K^{(q)}(1;2)+K^{(12)}(1;2)\ . \tag{2.18}$$

Here $K^{(q)}$ is a diagonal matrix with respect to the subscripts, having only the elements $K_a^{(q)}\equiv K_{aa;aa}^{(q)}$, while the important elements of $K^{(12)}$ are limited to $K_{ab}^{(12)}\equiv K_{aa;bb}^{(12)}$. Hence, in terms of the notations $I_{ab}^{(q+12)}=I_{aa;bb}$ and

$$U_{ab}^{(C)}(1;2)=G_{ab}^*(1)G_{ab}(2)\ , \tag{2.19}$$

the BS equation (2.16) can be written, in 2×2 matrix form, as

$$I^{(q+12)}=U^{(C)}[1+(K^{(q)}+K^{(12)})I^{(q+12)}]\ . \tag{2.20}$$

The matrices M and K, as defined by Eqs. (2.11) and (2.17), respectively, are not quite independent of each other, subjected to a (optical) relation of the form

$$\delta(1;2)\{M^*(1)-M(2)-[G^*(1)-G(2)]K(1;2)\}=0\ . \tag{2.21}$$

Here the matrix $\delta(1;2)$ is defined by the elements $\delta(\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2)$, such that $\delta(1;2)A^*(1)B(2)$ represents

$$\int d\hat{\mathbf{x}}_1\int d\hat{\mathbf{x}}_2\delta(\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2)A^*(\hat{\mathbf{x}}_1\,|\,\hat{\mathbf{x}}_1')B(\hat{\mathbf{x}}_2\,|\,\hat{\mathbf{x}}_2')$$
$$=\int d\hat{\mathbf{x}}\,A^\dagger(\hat{\mathbf{x}}_1'\,|\,\hat{\mathbf{x}})B(\hat{\mathbf{x}}\,|\,\hat{\mathbf{x}}_2')$$
$$=A^\dagger B(\hat{\mathbf{x}}_1'\,|\,\hat{\mathbf{x}}_2')\ .$$

The relation (2.21) ensures conservation of integrated power of the entire system, whereas there exists also a local relation that ensures power conservation at every point in space.[2] These relations approximately hold true also for each $M^{(\ )}$ and $K^{(\ )}$ of the medium and the boundary, independently, leading to optical relations of their own.

B. Case of three random layers

The situation is the same also for the case of random layers as illustrated in Fig. 1, and various equations formally remain unchanged with the setting

$$M=M^{(q)}+M^{(12)}+M^{(23)}\ , \tag{2.22a}$$

$$K\simeq K^{(q)}+K^{(12)}+K^{(23)}\ . \tag{2.22b}$$

Thus, using the notation $I_{ab}^{(q+12+23)}$, $a,b=1,2,3$, for the second-order Green function in this case, we obtain the BS equation in 3×3 matrix form, as

$$I^{(q+12+23)}=U^{(C)}[1+(K^{(q)}+K^{(12)}+K^{(23)})I^{(q+12+23)}]\ . \tag{2.23}$$

Here $K^{(q)}$ is a diagonal matrix with the elements $K_{ab}^{(q)}=K_a^{(q)}\delta_{ab}$, and $K^{(12)}$ and $K^{(23)}$ are the contributions purely from the boundaries S_{12} and S_{23}, with the nonvanishing elements $K_{ab}^{(12)}$, $a,b=1,2$, and $K_{ab}^{(23)}$, $a,b=2,3$.

C. Solutions and scattering matrices

To obtain the solution of the BS equation (2.20), we first introduce the solution in the special case $K^{(q)}=0$ (on keeping $M^{(q)}\neq0$, however), say, $I^{(12)}$; so that

$$I^{(12)}=U^{(C)}(1+K^{(12)}I^{(12)})\ . \tag{2.24}$$

Here the solution can be written as

$$I^{(12)}=U^{(C)}+U^{(C)}S^{(12)}U^{(C)}\ , \tag{2.25}$$

in terms of a (incoherent) scattering matrix of $K^{(12)}$, defined by

$$S^{(12)}=K^{(12)}(1+U^{(C)}S^{(12)}) \tag{2.26a}$$

$$=(1-K^{(12)}U^{(C)})^{-1}K^{(12)}\ . \tag{2.26b}$$

The Green function G_{ab} can also be written in the same form,

$$G_{ab}=G_a^{(0)}\delta_{ab}+G_a^{(0)}T_{ab}^{(12)}G_b^{(0)}\ , \tag{2.27}$$

in terms of the Green function $G_a^{(0)}$ in an unbounded medium of $M_a^{(q)}$, whose Fourier representation in the ρ space is therefore

$$G_a^{(0)}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')=(2\pi)^{-2}\int d\boldsymbol{\lambda}\exp[-i\boldsymbol{\lambda}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')]\times G_a^{(0)}(z-z') , \tag{2.28}$$

with

$$G_a^{(0)}(z-z')=[2i\tilde{h}_a(\boldsymbol{\lambda})]^{-1}\exp[-i\tilde{h}_a(\boldsymbol{\lambda})\,|\,z-z'\,|\,] . \tag{2.29}$$

Here

$$\tilde{h}_a(\boldsymbol{\lambda})=[(k_a^{(M)})^2-\boldsymbol{\lambda}^2]^{1/2} , \quad k_a^{(M)}=(k_a^2+\tilde{M}_a^{(q)})^{1/2}\simeq k_a , \tag{2.30}$$

where $\tilde{M}_a^{(q)}(\hat{\boldsymbol{\lambda}})$, $\hat{\boldsymbol{\lambda}}=(\boldsymbol{\lambda},\tilde{h}_a)$, is the Fourier transform of $M_a^{(q)}$, and $\mathrm{Im}(\tilde{h}_a)<0$. Hence $G_{ab}(\hat{\mathbf{x}}\,|\,\hat{\mathbf{x}}')$ also has the Fourier transform $G_{ab}(z\,|\,z')$ from Eq. (2.27), as

$$G_{ab}(z|z')=G_a^{(0)}(z-z')\delta_{ab}+G_a^{(0)}(z)\tilde{T}_{ab}^{(12)}G_b^{(0)}(-z') . \tag{2.31}$$

Here

$$\tilde{T}_{ab}^{(12)}=2i\tilde{h}_a\langle R_{ab}^{(12)}\rangle=\tilde{T}_{ba}^{(12)} , \tag{2.32}$$

where $\langle R_{ab}^{(12)}\rangle\neq\langle R_{ba}^{(12)}\rangle$ is the reflection-transmission coefficient of the boundary and, when it is perfectly smooth,

$$\langle R_{11}^{(12)}\rangle=\frac{\tilde{h}_1-\tilde{h}_2}{\tilde{h}_1+\tilde{h}_2},\quad \langle R_{21}^{(12)}\rangle=\frac{2\tilde{h}_1}{\tilde{h}_1+\tilde{h}_2} . \tag{2.33}$$

In the general case where $M_{ab}^{(12)}\neq 0$, $\langle R_{ab}^{(12)}\rangle$ is given in 2×2 matrix form by[1]

$$\langle R^{(12)}\rangle=(i\tilde{h}-\tilde{M}^{(12)})^{-1}(i\tilde{h}+\tilde{M}^{(12)}) \tag{2.34a}$$

or

$$\langle R^{(12)}\rangle+1=(i\tilde{h}-\tilde{M}^{(12)})^{-1}2i\tilde{h} , \tag{2.34b}$$

where $\tilde{h}$ is a diagonal matrix with the elements $\tilde{h}_{ab}=\tilde{h}_a\delta_{ab}$, and $\tilde{M}^{(12)}$ is the (two-dimensional) Fourier transform of $M^{(12)}$. Hence, setting $z=z'=0$ in Eq. (2.31), use of Eqs. (2.32) and (2.34b) leads to the expression

$$G(z=0\,|\,z'=0)=(i\tilde{h}-\tilde{M}^{(12)})^{-1} , \tag{2.34c}$$

which has a form similar to that given by Eq. (2.10) and is often referred to as the surface Green function.

Both Eqs. (2.27) and (2.31) can be written in matrix form by

$$G=G^{(0)}+G^{(0)}T^{(12)}G^{(0)} . \tag{2.35}$$

Therefore, by introducing a diagonal 2×2 matrix $U_{ab}=U_a\delta_{ab}$, defined by the elements

$$U_a(1;2)=[G_a^{(0)}(1)]^*G_a^{(0)}(2) , \tag{2.36}$$

$U^{(C)}(1;2)$ of Eq. (2.19) can also be written in the same form,

$$U^{(C)}(1;2)=U(1;2)+U(1;2)V^{(12)}(1;2)U(1;2) . \tag{2.37a}$$

Here

$$V^{(12)}(1;2)=[T^{(12)}(1)]^*T^{(12)}(2)+[T^{(12)}(1)]^*G^{-1}(2)+T^{(12)}(2)[G^*(1)]^{-1} , \tag{2.37b}$$

wherein the interference terms are negligible when the source and the observer are both separated enough from the boundary, whereas they are not negligible otherwise [see Eq. (2.39b)].

By using expression (2.37a), Eq. (2.25) is finally written as

$$I^{(12)}=U+U\sigma^{(12)}U . \tag{2.38}$$

Here $\sigma^{(12)}$ means a resultant scattering matrix of the boundary and is given by

$$\sigma^{(12)}=V^{(12)}+(1+V^{(12)}U)S^{(12)}(UV^{(12)}+1) , \tag{2.39a}$$

where, from Eqs. (2.37b) and (2.32),

$$F^{(C)}\equiv 1+UV^{(12)}=[1+\langle R^{(12)*}(1)\rangle][1+\langle R^{(12)}(2)\rangle] , \tag{2.39b}$$

which can be made more specific by using Eq. (2.34b); and $\bar{F}^{(C)}\equiv 1+V^{(12)}U$ is obtained from $F^{(C)}$ by the transposition.

The introduction of $I^{(12)}$ by Eq. (2.24) enables the BS equation (2.20) to be rewritten as

$$I^{(q+12)}=I^{(12)}(1+K^{(q)}I^{(q+12)}) \tag{2.40}$$

and hence the solution as

$$I^{(q+12)}=I^{(12)}+I^{(12)}S^{(q/12)}I^{(12)} , \tag{2.41}$$

in terms of a scattering matrix $S^{(q/12)}$ of $K^{(q)}$, defined by

$$S^{(q/12)}=K^{(q)}(1+I^{(12)}S^{(q/12)}) , \tag{2.42}$$

with the superscript $(q/12)$ to mean the dependence on $\sigma^{(12)}$ through $I^{(12)}$. Here the effect of $\sigma^{(12)}$ can be made explicit by introducing a solution of Eq. (2.42) in the special case $\sigma^{(12)}=0$, say, $S^{(0q)}$, governed by

$$S^{(0q)}=K^{(q)}(1+US^{(0q)}) , \tag{2.43}$$

so that Eq. (2.42) can be rewritten, on using Eq. (2.38), as

$$S^{(q/12)}=S^{(0q)}(1+U\sigma^{(12)}US^{(q/12)}) \tag{2.44a}$$

$$=(1-S^{(0q)}U\sigma^{(12)}U)^{-1}S^{(0q)} . \tag{2.44b}$$

Hence Eq. (2.41) becomes written finally in the form

$$I^{(q+12)}=I^{(12)}+(1+U\sigma^{(12)})I^{(s/12)}(\sigma^{(12)}U+1) . \tag{2.45}$$

Here the entire effect of the random medium appears only through a new matrix $I^{(s/12)}$, defined by

$$I^{(s/12)}=US^{(q/12)}U \tag{2.46}$$

and given as the solution of

$$I^{(s/12)}=I^{(0s)}(1+\sigma^{(12)}I^{(s/12)}) , \tag{2.47}$$

where

$$I^{(0s)}=US^{(0q)}U \quad (2.48a)$$
$$=UK^{(q)}(U+I^{(0s)}) , \quad (2.48b)$$

which is a diagonal matrix with respect to the subscripts, and tends to zero as $K^{(q)}\rightarrow 0$.

Example: Case of a semi-infinite random medium ($q_1=0$, $q_2\neq 0$)

We have $I_1^{(0s)}=I_{1b}^{(s/12)}=0$, and the only nonvanishing element of $I^{(s/12)}$ is $I_{22}^{(s/12)}$, which is the solution of

$$I_{22}^{(s/12)}=I_2^{(0s)}(1+\sigma_{22}^{(12)}I_{22}^{(s/12)}) . \quad (2.49)$$

Here $I_2^{(0s)}$ is the solution of

$$I_2^{(0s)}=U_2K_2^{(q)}(U_2+I_2^{(0s)}) , \quad (2.50)$$

and, therefore, is subject to the condition of no reflection at the boundary of $K_2^{(q)}$ that is distributed over $0>z>-\infty$ [Eqs. (3.11) and (3.22)].

Hence, when the wave source is located in the space k_1 [Fig. 1], $I^{(q+12)}$ in the same space is given according to Eq. (2.45), by

$$I_{11}^{(q+12)}=I_{11}^{(12)}+U_1\sigma_{12}^{(12)}I_{22}^{(s/12)}\sigma_{21}^{(12)}U_1 , \quad (2.51)$$

and the wave transmitted into the space k_2 is given by

$$I_{21}^{(q+12)}=I_{21}^{(12)}+(1+U_2\sigma_{22}^{(12)})I_{22}^{(s/12)}\sigma_{21}^{(12)}U_1 . \quad (2.52)$$

Also for the case of three random layers, as illustrated in Fig. 1, the situation becomes exactly the same by introducing a solution of when $K_a^{(q)}=0$, $a=1,2,3$, say, $I^{(12+23)}$, and letting $I^{(12+23)}$ do all the roles of $I^{(12)}$ in Eq. (2.45); that is, the basic equations (2.45)–(2.48) remain unchanged with the replacement of the superscript (12) by (12 + 23) and using the expression

$$I_{ab}^{(12+23)}=U_a\delta_{ab}+U_a\sigma_{ab}^{(12+23)}U_b . \quad (2.53)$$

Here, when the distance between the two boundaries, L, is sufficiently large compared with the wave coherence distance, say, γ_2^{-1}, so that $\gamma_2 L\gg 1$, $\sigma_{ab}^{(12+23)}$ can be approximated by

$$\sigma_{ab}^{(12+23)}\simeq\sigma_{ab}^{(12)}+\sigma_{ab}^{(23)}, \quad a,b=1,2,3 , \quad (2.54)$$

being the independent sum of the two boundary scattering matrices, $\sigma_{ab}^{(12)}$ of S_{12} and $\sigma_{ab}^{(23)}$ of S_{23}. Hence, for example, Eq. (2.45) is replaced by a 3×3 matrix equation, as

$$I^{(q+12+23)}=I^{(12+23)}+(1+U\sigma^{(12+23)})I^{(s/12+23)} \times(\sigma^{(12+23)}U+1) , \quad (2.55)$$

and, with (2.54), Eq. (2.47) is changed to

$$I^{(s/12+23)}=I^{(0s)}[1+(\sigma^{(12)}+\sigma^{(23)})I^{(s/12+23)}] . \quad (2.56)$$

Here $I^{(0s)}$ is still governed by Eq. (2.48b).

It may be remarked that, besides expressions (2.45) and (2.55), several other expressions are possible, which are also useful depending on particular situations and information required.[1]

Example: Case of a random layer ($q_1=q_3=0$, $q_2\neq 0$)

The only nonvanishing element of $I^{(s/12+23)}$ in Eq. (2.55) is $I_{22}^{(s/12+23)}$ in this case. Hence, when the source is in the space k_1 and the layer width L is large enough such that $\gamma_2 L\gg 1$, $I^{(q+12+23)}$ within the same space is given by

$$I_{11}^{(q+12+23)}=I_{11}^{(12)}+U_1\sigma_{12}^{(12)}I_{22}^{(s/12+23)}\sigma_{21}^{(12)}U_1 , \quad (2.57)$$

and the transmitted wave into the space k_3 is given by

$$I_{31}^{(q+12+23)}=U_3\sigma_{32}^{(23)}I_{22}^{(s/12+23)}\sigma_{21}^{(12)}U_1 , \quad (2.58)$$

where the contribution from $I_{31}^{(12+23)}$ is presently negligible. Here the random medium is involved only through $I_{22}^{(s/12+23)}$, which is the solution of

$$I_{22}^{(s/12+23)}=I_2^{(0s)}[1+(\sigma_{22}^{(12)}+\sigma_{22}^{(23)})I_{22}^{(s/12+23)}] , \quad (2.59)$$

where $I_2^{(0s)}$ is the solution of Eq. (2.50) with the $K_2^{(q)}$ distributed over the range $0\geq z\geq -L$. Here, as for $\sigma^{(12)}$ and $\sigma^{(23)}$, we may utilize experimental values of the boundaries, instead of the theoreticals.

Thus the problem is reduced to finding the solution of Eq. (2.59), which will be obtained in this paper as a simple boundary-value solution of the diffusion equation, instead of solving the conventional transport equation depending actually on numerical methods.

III. $\hat{\Omega}$ REPRESENTATION AND SCATTERING CROSS SECTIONS

A specific expression of Eq. (2.59) is obtained in optical form by partially making the Fourier transformation. We first introduce relative coordinates $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\rho}}$, defined by

$$\hat{\mathbf{r}}=\hat{\mathbf{x}}_2-\hat{\mathbf{x}}_1, \quad \hat{\boldsymbol{\rho}}=\tfrac{1}{2}(\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_1) , \quad (3.1)$$

and the corresponding Fourier variables $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\lambda}}$, defined by

$$\hat{\mathbf{u}}=\tfrac{1}{2}(\hat{\boldsymbol{\lambda}}_2+\hat{\boldsymbol{\lambda}}_1), \quad \hat{\boldsymbol{\lambda}}=\hat{\boldsymbol{\lambda}}_2-\hat{\boldsymbol{\lambda}}_1 , \quad (3.2)$$

so that

$$-\hat{\mathbf{x}}_1\cdot\hat{\boldsymbol{\lambda}}_1+\hat{\mathbf{x}}_2\cdot\hat{\boldsymbol{\lambda}}_2=\hat{\mathbf{u}}\cdot\hat{\mathbf{r}}+\hat{\boldsymbol{\lambda}}\cdot\hat{\boldsymbol{\rho}} . \quad (3.3)$$

Then, the matrix elements of $K^{(q)}$ can be written in the form

$$K^{(q)}(\hat{\mathbf{x}}_1;\hat{\mathbf{x}}_2\,|\,\hat{\mathbf{x}}'_1;\hat{\mathbf{x}}'_2)=K^{(q)}(\hat{\mathbf{r}}\,|\,\boldsymbol{\rho}-\boldsymbol{\rho}'\,|\,\hat{\mathbf{r}}') , \quad (3.4)$$

in view of the translational invariance, approximately in the vertical direction, though; and its Fourier transform therefore has the form

$$\tilde{K}^{(q)}(\hat{\boldsymbol{\lambda}}_1;\hat{\boldsymbol{\lambda}}_2\,|\,\hat{\boldsymbol{\lambda}}'_1;\hat{\boldsymbol{\lambda}}'_2)=(2\pi)^3\delta(\hat{\boldsymbol{\lambda}}-\hat{\boldsymbol{\lambda}}')\tilde{K}^{(q)}(\hat{\mathbf{u}}\,|\,\hat{\boldsymbol{\lambda}}\,|\,\hat{\mathbf{u}}') . \quad (3.5)$$

On the other hand, the corresponding Fourier transforms of the wave quantities, e.g.,

$$\tilde{I}^{(s/12)}(\hat{\mathbf{u}},\hat{\boldsymbol{\lambda}}\,|\,\hat{\mathbf{u}}',\hat{\boldsymbol{\lambda}}')$$

of $I^{(s/12)}$ where $\hat{\boldsymbol{\lambda}}=(\boldsymbol{\lambda},\lambda_z)$, cannot be written in the same form, and, therefore, on suppressing $\boldsymbol{\lambda}$ and dropping the factor $(2\pi)^2\delta(\boldsymbol{\lambda}-\boldsymbol{\lambda}')$, we will hereafter use a composite expression, e.g.,

$$I^{(s/12)}(\hat{\mathbf{u}},z \mid \hat{\mathbf{u}}',z') ,$$

by making the Fourier inversion only with respect to λ_z.

As to the transform $\tilde{U}_a$ of U_a, we obtain

$$\tilde{U}_a(\hat{\mathbf{u}},\hat{\boldsymbol{\lambda}})=\tilde{G}_a^*(\hat{\mathbf{u}}-\tfrac{1}{2}\hat{\boldsymbol{\lambda}})\tilde{G}_a(\hat{\mathbf{u}}+\tfrac{1}{2}\hat{\boldsymbol{\lambda}}) \simeq \pi\delta(\hat{\mathbf{u}}^2-k_a^2)(k_a\gamma_a-i\hat{\mathbf{u}}\cdot\hat{\boldsymbol{\lambda}})^{-1} , \tag{3.6}$$

with the approximation $k_a^{(M)}\simeq k_a$ and the constant

$$\gamma_a=(2ik_a)^{-1}(\tilde{M}_a^{(q)*}-\tilde{M}_a^{(q)})(\hat{\mathbf{u}}) , \tag{3.7}$$

excluding the case of when the medium is intrinsically dispersive;[2] the expression (3.6) is a direct consequence of the identity

$$\tilde{G}_a^*(\hat{\boldsymbol{\lambda}}_1)-\tilde{G}_a(\hat{\boldsymbol{\lambda}}_2)=\{\tilde{G}_a^{-1}(\hat{\boldsymbol{\lambda}}_2)-[\tilde{G}_a^*(\hat{\boldsymbol{\lambda}}_1)]^{-1}\}\tilde{U}_a(\hat{\boldsymbol{\lambda}}_1;\hat{\boldsymbol{\lambda}}_2) ,$$

with the approximation of the left-hand side by $2\pi i\delta(\hat{\mathbf{u}}^2-k_a^2)$, and is valid under the condition $|\hat{\mathbf{u}}|\sim|k_a|\gg|\hat{\boldsymbol{\lambda}}|$.

Hence, on changing the variable $\hat{\mathbf{u}}$ by $\hat{\mathbf{u}}=u\hat{\boldsymbol{\Omega}}$, $d\hat{\mathbf{u}}=u^2du\,d\hat{\boldsymbol{\Omega}}$, where $u=|\hat{\mathbf{u}}|$ and $\hat{\boldsymbol{\Omega}}=(\boldsymbol{\Omega},\Omega_z)$, $\hat{\boldsymbol{\Omega}}^2=1$ is the unit vector, we obtain an important relation that, for any slowly changing function $f(\hat{\mathbf{u}})$,

$$(2\pi)^{-3}\int d\hat{\mathbf{u}}\,\tilde{U}_a(\hat{\mathbf{u}},\hat{\boldsymbol{\lambda}})f(\hat{\mathbf{u}})=\int_{4\pi}d\hat{\boldsymbol{\Omega}}\,\tilde{U}_a(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})f(\hat{\boldsymbol{\Omega}}) , \tag{3.8}$$

where

$$\tilde{U}_a(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})=(\gamma_a-i\hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{\lambda}})^{-1} , \tag{3.9}$$

$$f(\hat{\boldsymbol{\Omega}})=(4\pi)^{-2}f(\hat{\mathbf{u}}=k_a\hat{\boldsymbol{\Omega}}) . \tag{3.10}$$

Here the λ_z Fourier inversion of $\tilde{U}_a(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})$ is

$$U_a(\hat{\boldsymbol{\Omega}},z)\equiv\frac{1}{2\pi}\int d\lambda_z \exp(-i\lambda_z z)\tilde{U}_a(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}) = \begin{cases}\Omega_z^{-1}\exp[-\Omega_z^{-1}(\gamma_a-i\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})z], & \Omega_z z>0\\ 0, & \Omega_z z<0\end{cases} \tag{3.11}$$

while the three-dimensional inversion $U_a(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\rho}})$ is given by

$$U_a(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\rho}})=|\hat{\boldsymbol{\rho}}|^{-2}\exp(-\gamma_a|\hat{\boldsymbol{\rho}}|)\delta^2(\hat{\boldsymbol{\Omega}}-\hat{\boldsymbol{\rho}}/|\hat{\boldsymbol{\rho}}|) . \tag{3.12}$$

Hence, with the rule (3.8), Eq. (2.38), for example, leads to the expression

$$I_{ab}^{(12)}(\hat{\boldsymbol{\Omega}},z \mid \hat{\boldsymbol{\Omega}}',z')=U_a(\hat{\boldsymbol{\Omega}},z-z')\delta_{ab}\delta^2(\hat{\boldsymbol{\Omega}}-\hat{\boldsymbol{\Omega}}') + U_a(\hat{\boldsymbol{\Omega}},z)\sigma_{ab}^{(12)}(\hat{\boldsymbol{\Omega}}\mid\hat{\boldsymbol{\Omega}}') \times U_b(\hat{\boldsymbol{\Omega}}',-z') . \tag{3.13}$$

Here, from Eqs. (2.39) (see also Appendix B),

$$\sigma_{ab}^{(12)}(\hat{\boldsymbol{\Omega}}\mid\hat{\boldsymbol{\Omega}}')=\Omega_z^{(a)}|\langle R_{ab}^{(12)}(\hat{\boldsymbol{\Omega}})\rangle|^2\delta_S^2(\hat{\boldsymbol{\Omega}}^{(a)}-\hat{\boldsymbol{\Omega}}^{(a)\prime}) +\sigma_{ab}^{(I12)}(\hat{\boldsymbol{\Omega}}\mid\hat{\boldsymbol{\Omega}}') , \tag{3.14}$$

where $\hat{\boldsymbol{\Omega}}^{(a)}=[\boldsymbol{\Omega}^{(a)},\Omega_z^{(a)}]$ denotes the unit vector in space k_a, defined by $\mathbf{u}=k_a\boldsymbol{\Omega}^{(a)}$ and

$$\Omega_z^{(a)}=\pm[1-(\boldsymbol{\Omega}^{(a)})^2]^{1/2} ,$$

and δ_S^2 is a specular δ function of $\hat{\boldsymbol{\Omega}}$ which is not zero only when the scattering is made in the specular direction, regardless of the sign of $\Omega_z^{(a)}$; $\sigma_{ab}^{(I12)}$ is the incoherent cross section, given in terms of the Fourier transform $\tilde{S}^{(12)}$ by

$$\sigma_{ab}^{(I12)}(\hat{\boldsymbol{\Omega}}|\hat{\boldsymbol{\Omega}}')=\sum_{i,j,k,l}(4\pi)^{-2}\bar{F}_{aa;ij}(\mathbf{u})\tilde{S}_{ij;kl}^{(12)}(\mathbf{u}|\boldsymbol{\lambda}(=0)|\mathbf{u}')F_{kl;bb}(\mathbf{u}')\Big|_{\Omega} , \tag{3.15}$$

where the notation $|_{\Omega}$ means setting $\mathbf{u}=k_a\boldsymbol{\Omega}^{(a)}$ and $\mathbf{u}'=k_b\boldsymbol{\Omega}^{(b)}$, and neglecting the $\boldsymbol{\lambda}$ dependence.

The transform $\sigma_{ab}^{(12)}(\hat{\boldsymbol{\Omega}}\mid\hat{\boldsymbol{\Omega}}')$ provides the resultant (including both coherent and incoherent) cross section per unit area of the boundary for scattering of the wave from direction $\hat{\boldsymbol{\Omega}}'$ in space k_b to direction $\hat{\boldsymbol{\Omega}}$ in space k_a [Eq. (3.26)], and is subject to an optical relation resulting from relation (2.21) (with the details in Ref. 1) as

$$\sum_{a=1}^{2}\int_{2\pi}d\hat{\boldsymbol{\Omega}}'k_a\sigma_{ab}^{(12)}(\hat{\boldsymbol{\Omega}}'\mid\hat{\boldsymbol{\Omega}})=-k_b\Omega_n^{(b)}>0 . \tag{3.16a}$$

Here $\Omega_n^{(b)}=\hat{\mathbf{n}}^{(b)}\cdot\hat{\boldsymbol{\Omega}}$ and the reciprocity [similar to Eqs. (3.20)] holds:

$$\sigma_{ab}^{(12)}(\hat{\boldsymbol{\Omega}}\mid\hat{\boldsymbol{\Omega}}')=\sigma_{ba}^{(12)}(-\hat{\boldsymbol{\Omega}}'\mid-\hat{\boldsymbol{\Omega}}) . \tag{3.16b}$$

The expression (3.13) should be regarded as the matrix elements of $I^{(12)}$ with respect to $\hat{\boldsymbol{\Omega}}$ and z, to be multiplied with a $\hat{\boldsymbol{\Omega}}$-$z$ vector $f(\hat{\boldsymbol{\Omega}},z)$, defined by Eq. (3.10).

In the same way, the $\hat{\boldsymbol{\Omega}}$-$z$ expression of Eq. (2.49) becomes

$$I_{22}^{(s/12)}(\hat{\boldsymbol{\Omega}},z\mid\hat{\boldsymbol{\Omega}}',z')=I_2^{(0s)}(\hat{\boldsymbol{\Omega}},z\mid\hat{\boldsymbol{\Omega}}',z') +\int d\hat{\boldsymbol{\Omega}}''d\hat{\boldsymbol{\Omega}}'''I_2^{(0s)}(\hat{\boldsymbol{\Omega}},z\mid\hat{\boldsymbol{\Omega}}'',z''(=0))\sigma_{22}^{(12)}(\hat{\boldsymbol{\Omega}}''\mid\hat{\boldsymbol{\Omega}}''')I_{22}^{(s/12)}(\hat{\boldsymbol{\Omega}}''',z'''(=0)\mid\hat{\boldsymbol{\Omega}}',z') . \tag{3.17}$$

Here the original $I_2^{(0s)}$ is the solution of Eq. (2.50), whose $\hat{\boldsymbol{\Omega}}$-$z$ version can be written most simply in the form of the transport equation, by multiplying the equation to the left with U_2^{-1} and using the relation

$$\left[\gamma_a + \Omega_z \frac{\partial}{\partial z}\right] U_a(\hat{\mathbf{\Omega}}, z) = \delta(z) \tag{3.18}$$

from (3.11); wherein the term $-i\boldsymbol{\lambda}\cdot\mathbf{\Omega}$ has been included in γ_a. Hence, with the notation $\partial_z = \partial/\partial_z$,

$$(\gamma_2 + \Omega_z \partial_z) I_2^{(0s)}(\hat{\mathbf{\Omega}}, z \mid \hat{\mathbf{\Omega}}', z') = K_2^{(q)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}') U_2(\hat{\mathbf{\Omega}}', z - z') + \int d\hat{\mathbf{\Omega}}'' K_2^{(q)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}'') I_2^{(0s)}(\hat{\mathbf{\Omega}}'', z \mid \hat{\mathbf{\Omega}}', z') . \tag{3.19}$$

Here $z, z' < 0$ in the present case, and, from Eq. (3.10),

$$K_2^{(q)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}') = (4\pi)^{-2} \tilde{K}_2^{(q)}(\hat{\mathbf{u}}(=k_2\hat{\mathbf{\Omega}}) \mid \hat{\boldsymbol{\lambda}}(=0) \mid \hat{\mathbf{u}}'(=k_2\hat{\mathbf{\Omega}}')) \tag{3.20a}$$

$$= K_2^{(q)}(-\hat{\mathbf{\Omega}}' \mid -\hat{\mathbf{\Omega}}) , \tag{3.20b}$$

where the last relation is from

$$\tilde{K}_2^{(q)}(\hat{\mathbf{u}} \mid \hat{\boldsymbol{\lambda}} \mid \hat{\mathbf{u}}') = \tilde{K}_2^{(q)}(-\hat{\mathbf{u}}' \mid -\hat{\boldsymbol{\lambda}} \mid -\hat{\mathbf{u}}) = \tilde{K}_2^{(q)*}(\hat{\mathbf{u}} \mid -\hat{\boldsymbol{\lambda}} \mid \hat{\mathbf{u}}') , \tag{3.20c}$$

which holds true whenever $K_2^{(q)}$ is a symmetrical matrix of the coordinates. By the same reason, Eq. (3.7) leads to

$$\gamma_2(\hat{\mathbf{\Omega}}) \equiv \gamma_2(\hat{\mathbf{u}}(=k_2\hat{\mathbf{\Omega}})) = \gamma_2(-\hat{\mathbf{\Omega}}) . \tag{3.20d}$$

Thus $K_2^{(q)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}')$ provides the scattering cross section per unit volume of the medium q_2 and is subject to the optical relation

$$\gamma_2(\hat{\mathbf{\Omega}}) = \int_{4\pi} d\hat{\mathbf{\Omega}}' K_2^{(q)}(\hat{\mathbf{\Omega}}' \mid \hat{\mathbf{\Omega}}) \tag{3.21}$$

from Eq. (2.21) with $M \to M_2^{(q)}$, $K \to K_2^{(q)}$, and $G \to G_2^{(0)}$.

Equation (3.19) is the same as the conventional transport equation with a source term in a random medium $K_2^{(q)}$, and the solution is subject to the condition of no reflection at the boundary, in view of Eq. (2.50) having the factor $U_2(\hat{\mathbf{\Omega}}, z)$ of Eq. (3.11); that is,

$$I_2^{(0s)}(\hat{\mathbf{\Omega}}, z(=0) \mid \hat{\mathbf{\Omega}}', z'(<0)) = 0, \quad \Omega_z < 0 . \tag{3.22}$$

With known $I_2^{(0s)}$, $I_{22}^{(s/12)}$ is obtained as the solution of integral equation (3.17), and thereby $I_{ab}^{(q+12)}$ is provided through Eqs. (2.51) and (2.52).

The same is true also in the case of a random layer, and the expressions of Eqs. (2.57) and (2.58) become

$$I_{11}^{(q+12+23)}(\hat{\mathbf{\Omega}}, z \mid \hat{\mathbf{\Omega}}', z') = I_{11}^{(12)}(\hat{\mathbf{\Omega}}, z \mid \hat{\mathbf{\Omega}}', z') + \int d\hat{\mathbf{\Omega}}'' d\hat{\mathbf{\Omega}}''' U_1(\hat{\mathbf{\Omega}}, z) \sigma_{12}^{(12)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}'')$$
$$\times I_{22}^{(s/12+23)}(\hat{\mathbf{\Omega}}'', z''(=0) \mid \hat{\mathbf{\Omega}}''', z'''(=0)) \sigma_{21}^{(12)}(\hat{\mathbf{\Omega}}''' \mid \hat{\mathbf{\Omega}}') U_1(\hat{\mathbf{\Omega}}', z') , \tag{3.23}$$

$$I_{31}^{(q+12+23)}(\hat{\mathbf{\Omega}}, z \mid \hat{\mathbf{\Omega}}', z') = \int d\hat{\mathbf{\Omega}}'' d\hat{\mathbf{\Omega}}''' U_3(\hat{\mathbf{\Omega}}, z+L) \sigma_{32}^{(23)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}'')$$
$$\times I_{22}^{(s/12+23)}(\hat{\mathbf{\Omega}}'', z''(=-L) \mid \hat{\mathbf{\Omega}}''', z'''(=0)) \sigma_{21}^{(12)}(\hat{\mathbf{\Omega}}''' \mid \hat{\mathbf{\Omega}}') U_1(\hat{\mathbf{\Omega}}', z') . \tag{3.24}$$

Here $I_{22}^{(s/12+23)}(\hat{\mathbf{\Omega}}, z \mid \hat{\mathbf{\Omega}}', z')$ is obtained as the solution of an integral equation similar to Eq. (3.17), with the additional boundary term of $\sigma_{22}^{(23)}$ at $z = -L$.

The three-dimensional expressions of Eqs. (3.23) and (3.24) are straightforward by making the Fourier inversion with respect to the variable $\boldsymbol{\lambda}$ that has been suppressed so far. For example, Eq. (3.23) is replaced by

$$I_{11}^{(q+12+23)}(\hat{\mathbf{\Omega}}, \hat{\boldsymbol{\rho}} | \hat{\mathbf{\Omega}}', \hat{\boldsymbol{\rho}}') = I_{11}^{(12)}(\hat{\mathbf{\Omega}}, \hat{\boldsymbol{\rho}} | \hat{\mathbf{\Omega}}', \hat{\boldsymbol{\rho}}')$$
$$+ \int d\boldsymbol{\rho}'' d\boldsymbol{\rho}''' \int d\hat{\mathbf{\Omega}}'' d\hat{\mathbf{\Omega}}''' U_1(\hat{\mathbf{\Omega}}, \hat{\boldsymbol{\rho}} - \boldsymbol{\rho}'') \sigma_{12}^{(12)}(\hat{\mathbf{\Omega}} | \hat{\mathbf{\Omega}}'')$$
$$\times I_{22}^{(s/12+23)}(\hat{\mathbf{\Omega}}'', z''(=0) | \boldsymbol{\rho}'' - \boldsymbol{\rho}''' | \hat{\mathbf{\Omega}}''', z'''(=0))$$
$$\times \sigma_{21}^{(12)}(\hat{\mathbf{\Omega}}''' | \hat{\mathbf{\Omega}}') U_1(\hat{\mathbf{\Omega}}', \boldsymbol{\rho}''' - \hat{\boldsymbol{\rho}}') . \tag{3.25}$$

Here $U_a(\hat{\mathbf{\Omega}}, \hat{\boldsymbol{\rho}})$ is given by Eq. (3.12), $I_{22}^{(s/12+23)}(\hat{\mathbf{\Omega}}, z \mid \boldsymbol{\rho} \mid \hat{\mathbf{\Omega}}', z')$ is the $\boldsymbol{\lambda}$ Fourier inversion of $I_{22}^{(s/12+23)}(\hat{\mathbf{\Omega}}, z \mid \hat{\mathbf{\Omega}}', z')$, and, from Eq. (3.13),

$$I_{11}^{(12)}(\hat{\mathbf{\Omega}}, \hat{\boldsymbol{\rho}} \mid \hat{\mathbf{\Omega}}', \hat{\boldsymbol{\rho}}') = U_1(\hat{\mathbf{\Omega}}, \hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\rho}}') \delta^2(\hat{\mathbf{\Omega}} - \hat{\mathbf{\Omega}}') + \int d\boldsymbol{\rho}'' U_1(\hat{\mathbf{\Omega}}, \hat{\boldsymbol{\rho}} - \boldsymbol{\rho}'') \sigma_{11}^{(12)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}') U_1(\hat{\mathbf{\Omega}}', \boldsymbol{\rho}'' - \hat{\boldsymbol{\rho}}') . \tag{3.26}$$

As to the physical quantities involved, the power flux is particularly important because of the continuity across the boundaries. The averaged power at a point $\hat{\rho}$ in space k_a for the wave from a point source at $\hat{\rho}'$ in space k_b, say, $\langle \hat{\mathbf{W}}_{ab}(\hat{\rho}\,|\,\hat{\rho}')\rangle$, is generally given according to the definition (2.2), by

$$\langle \hat{\mathbf{W}}_{ab}(\hat{\rho}\,|\,\hat{\rho}')\rangle = i\frac{\partial}{\partial\hat{\mathbf{r}}} I_{ab}(\hat{\mathbf{r}},\hat{\rho}\,|\,\hat{\mathbf{r}}',\hat{\rho}')\Big|_{\hat{\mathbf{r}}=\hat{\mathbf{r}}'=0} = (4\pi)^{-2}\int d\hat{\boldsymbol{\Omega}}\int d\hat{\boldsymbol{\Omega}}' k_a \hat{\Omega}^{(a)} I_{ab}(\hat{\boldsymbol{\Omega}},\hat{\rho}\,|\,\hat{\boldsymbol{\Omega}}',\hat{\rho}') ,$$

in terms of the relative coordinates of (3.1) and with the $\hat{\boldsymbol{\Omega}}$ representation by Eq. (3.10).

IV. EIGENFUNCTION EXPANSIONS AND DIFFUSION APPROXIMATION

The Green function $I^{(q+12)}$, as given by Eqs. (2.51) and (2.52) for a semi-infinite random layer, is dependent on the random medium only through the function $I_{22}^{(s/12)}$ that is the solution of the integral equation (3.17). Here, to a good approximation, this equation can be converted to a diffusion equation so that $I_{22}^{(s/12)}$ can be obtained as a simple boundary-value solution, including the cases of when the media's averaged refractive indices differ from each other by a large amount.

We first introduce a set of eigenfunctions $f_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})$ and $\overline{f_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})}$ of the cross section $K_2^{(q)}(\hat{\boldsymbol{\Omega}}\,|\,\hat{\boldsymbol{\Omega}}')$, defined by the eigenvalue equations[5]

$$\int d\hat{\boldsymbol{\Omega}}' K_2^{(q)}(\hat{\boldsymbol{\Omega}}|\hat{\boldsymbol{\Omega}}')\tilde{U}_2(\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}}) f_A(\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}}) = A(\hat{\boldsymbol{\lambda}}) f_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}) , \quad (4.1a)$$

$$\int d\hat{\boldsymbol{\Omega}}' \overline{f_A(\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}})}\tilde{U}_2(\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}}) K_2^{(q)}(\hat{\boldsymbol{\Omega}}'\,|\,\hat{\boldsymbol{\Omega}}) = A(\hat{\boldsymbol{\lambda}})\overline{f_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})} , \quad (4.1b)$$

with the normalization

$$\int d\hat{\boldsymbol{\Omega}}\, \overline{f_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})}\tilde{U}_2(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}) f_B(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}) = \delta_{AB} . \quad (4.2)$$

Here $\tilde{U}_2(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})$ is regarded as a weighting function when making the $\hat{\boldsymbol{\Omega}}$ integration, and $A(\hat{\boldsymbol{\lambda}})$ is the eigenvalue and tends to zero as $|\hat{\boldsymbol{\lambda}}|\to\infty$, in consequence of Eq. (3.9). In terms of the eigenfunctions and the eigenvalues, $K_2^{(q)}$ can be exhibited by the series

$$K_2^{(q)}(\hat{\boldsymbol{\Omega}}\,|\,\hat{\boldsymbol{\Omega}}') = \sum_A A(\hat{\boldsymbol{\lambda}}) f_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})\overline{f_A(\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}})} . \quad (4.3)$$

A similar expansion is possible also for the incoherent scattering matrix $S_2^{(0q)}$, i.e.,

$$\tilde{S}_2^{(0q)}(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}\,|\,\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}}') = \sum_A f_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})\tilde{S}_A(\hat{\boldsymbol{\lambda}}\,|\,\hat{\boldsymbol{\lambda}}')\overline{f_A(\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}}')} , \quad (4.4)$$

and, with Eq. (4.3), the substitution into Eq. (2.43), whose $\hat{\boldsymbol{\Omega}}$ version is

$$\tilde{S}^{(0q)}(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}\,|\,\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}}') = K^{(q)}(\hat{\boldsymbol{\Omega}}\,|\,\hat{\boldsymbol{\Omega}}')(2\pi)^3\delta(\hat{\boldsymbol{\lambda}}-\hat{\boldsymbol{\lambda}}') + \int d\hat{\boldsymbol{\Omega}}'' K^{(q)}(\hat{\boldsymbol{\Omega}}\,|\,\hat{\boldsymbol{\Omega}}'')\tilde{U}(\hat{\boldsymbol{\Omega}}'',\hat{\boldsymbol{\lambda}}) \times \tilde{S}^{(0q)}(\hat{\boldsymbol{\Omega}}'',\hat{\boldsymbol{\lambda}}\,|\,\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}}') , \quad (4.5)$$

leads to

$$[1-A(\hat{\boldsymbol{\lambda}})]\tilde{S}_A(\hat{\boldsymbol{\lambda}}\,|\,\hat{\boldsymbol{\lambda}}') = A(\hat{\boldsymbol{\lambda}})(2\pi)^3\delta(\hat{\boldsymbol{\lambda}}-\hat{\boldsymbol{\lambda}}') , \quad (4.6a)$$

or, by the Fourier inversion,

$$\left[1-A\left[i\frac{\partial}{\partial\hat{\rho}}\right]\right] S_A(\hat{\rho}\,|\,\hat{\rho}') = A\left[i\frac{\partial}{\partial\hat{\rho}}\right]\delta(\hat{\rho}-\hat{\rho}') . \quad (4.6b)$$

Here, from Eq. (4.4),

$$S_2^{(0q)}(\hat{\boldsymbol{\Omega}},\hat{\rho}\,|\,\hat{\boldsymbol{\Omega}}',\hat{\rho}') = \sum_A f_A\left[\hat{\boldsymbol{\Omega}}, i\frac{\partial}{\partial\hat{\rho}}\right] S_A(\hat{\rho}\,|\,\hat{\rho}')\overline{f_A\left[\hat{\boldsymbol{\Omega}}', -i\frac{\partial}{\partial\hat{\rho}'}\right]} . \quad (4.7)$$

To make a similar expansion of the wave quantities $I_2^{(0s)}$ and $I_{22}^{(s/12)}$, it is convenient to introduce another set of eigenfunctions $\phi_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})$ and $\overline{\phi_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})}$, defined by

$$\phi_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}) = \tilde{U}_2(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}) f_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}) , \quad (4.8a)$$

$$\overline{\phi_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})} = \overline{f_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})}\tilde{U}_2(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}) , \quad (4.8b)$$

which are subject to the normalization

$$\int d\hat{\boldsymbol{\Omega}}\, \overline{\phi_A}\tilde{U}_2^{-1}\phi_B(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}}) = \delta_{AB} , \quad (4.9)$$

and hence also

$$\tilde{U}_2(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})\delta(\hat{\boldsymbol{\Omega}}-\hat{\boldsymbol{\Omega}}') = \sum_A \phi_A(\hat{\boldsymbol{\Omega}},\hat{\boldsymbol{\lambda}})\overline{\phi_A(\hat{\boldsymbol{\Omega}}',\hat{\boldsymbol{\lambda}})} . \quad (4.10)$$

Hence, by definition (2.48a) and Eq. (4.7),

$$I_2^{(0s)}(\hat{\boldsymbol{\Omega}},\hat{\rho}|\hat{\boldsymbol{\Omega}}',\hat{\rho}') = \sum_A \phi_A\left[\hat{\boldsymbol{\Omega}}, i\frac{\vec{\partial}}{\partial\hat{\rho}}\right] S_A(\hat{\rho}|\hat{\rho}')\overline{\phi_A\left[\hat{\boldsymbol{\Omega}}', -i\frac{\partial}{\partial\hat{\rho}'}\right]} . \quad (4.11)$$

Here it may be remarked that the expansion coefficients $S_A(\hat{\rho}\,|\,\hat{\rho}')$ are the same as those in the series (4.7) for $S_2^{(0q)}$, and, therefore, that they are essentially medium waves defined only in the region $z<0$ where $K_2^{(q)}\neq 0$, being zero wherever $K_2^{(q)}=0$.

In the diffusion region where change of the wave intensity is sufficiently small within the wave coherence distance so that $|\hat{\boldsymbol{\lambda}}/\gamma_2| \ll 1$, the convergence of series (4.11) becomes good enough to be approximated by the first term and by the eigenfunction and the eigenvalue given by the first nonvanishing terms in their series expansions with respect to $\hat{\boldsymbol{\lambda}}/\gamma_2$.[5] Therefore we consider only the first term hereafter, and assume the form $K^{(q)}(\hat{\boldsymbol{\Omega}}\cdot\hat{\boldsymbol{\Omega}}')$ subject to the rotational invariance, when illustrated by specific expressions; its extension to the general case is straightforward according to the methods of Appendix A and Sec. V.

Hence, on suppressing λ and replacing $\hat{\lambda}$ by $i\partial_z$ whenever convenient, we obtain an expression of $A(\hat{\lambda})$ of the form (Appendix A)

$$1-A(i\partial_z)=\gamma_2^{-1}(\gamma^{(ab)}-D_2\partial_z^2)+O((\partial_z/\gamma_2)^4) . \quad (4.12)$$

Here $\gamma^{(ab)}$ is a newly added term to represent an intrinsic dissipation by the medium, for the latter's convenience; and

$$D_2=(3\gamma_2)^{-1}(1-a_1)^{-1} , \quad (4.13)$$

where a_1 is the average of the cosine of the scattering angle, defined by

$$a_1=\gamma_2^{-1}\int d\hat{\Omega}(\hat{\Omega}\cdot\hat{\Omega}')K_2^{(q)}(\hat{\Omega}\cdot\hat{\Omega}') . \quad (4.14)$$

Also, to the same approximation,

$$\phi_A(\hat{\Omega},i\partial_z)=\gamma_2^{-1}(1-3\Omega_z D_2\partial_z) , \quad (4.15a)$$

$$\overline{\phi_A(\hat{\Omega},i\partial_z)}=(4\pi)^{-1}(1-3\Omega_z D_2\partial_z) , \quad (4.15b)$$

which become both independent of $\hat{\Omega}$ when $\partial_z=0$.

Thus Eq. (4.6b) is reduced, on setting $A=1$ on the right-hand side, to

$$\gamma_2^{-1}(\gamma^{(ab)}-D_2\partial_z^2)S_A(z\mid z')=\delta(z-z') . \quad (4.16)$$

A. Boundary condition

The $\hat{\Omega}$-averaged power flux of the diffusion term, say, $\langle W_z\rangle$ in the z direction, can be written as a sum of the components $\langle W_z^{\pm}\rangle$ propagating in the positive and negative directions, respectively, by

$$\langle W_z\rangle=\langle W_z^{+}\rangle+\langle W_z^{-}\rangle . \quad (4.17)$$

Here, from Eqs. (4.11) and (3.27),

$$\langle W_z^{\pm}(z\mid z')\rangle =\frac{1}{4\pi}\int_{\Omega_z\gtrless 0}d\hat{\Omega}\, k_2\Omega_z\phi_A(\hat{\Omega},i\partial_z)S_A(z\mid z') , \quad (4.18)$$

where the additional factor $\overline{\phi_A}$ has been omitted for short. Hence, when using Eq. (4.15a),

$$\langle W_z^{\pm}\rangle=k_2(2\gamma_2)^{-1}(\pm\tfrac{1}{2}-D_2\partial_z)S_A(z\mid z') , \quad (4.19)$$

and therefore, from (4.17),

$$\langle W_z\rangle=-k_2\gamma_2^{-1}D_2\partial_z S_A(z\mid z') , \quad (4.20)$$

which shows that the left-hand side of the diffusion equation (4.16) is given, when $\gamma^{(ab)}=0$, by the space divergence of the power flux vector except the numerical factor k_2.

Here, on the boundary at $z=0$ (Fig. 1), we observe that $\langle W_z^{-}\rangle=0$ in consequence of Eq. (3.22) and $\langle W_z^{+}\rangle$ is continuous, so that, integrating both sides of Eq. (4.16) with respect to z over an infinitesimal range $0\geq z\geq -\epsilon$, $\epsilon=+0$, the result can be written, when $0>z'>-\epsilon$, as

$$\langle W_z(z\mid z')\rangle\,|_{z=-\epsilon}^{0}=-\langle W_z^{-}(z(=-\epsilon)\mid z')\rangle=k_2 , \quad (4.21)$$

whereas, when $z'<-\epsilon$, the right-hand side is zero. The results are therefore summarized, by using Eq. (4.19), as

$$(2\gamma_2)^{-1}(\tfrac{1}{2}+D_2\partial_z)S_A(z(=-\epsilon)\mid z') =\begin{cases}1, & 0>z'>-\epsilon \quad (4.22a)\\ 0, & z'<-\epsilon . \quad (4.22b)\end{cases}$$

Here Eq. (4.21) implies that the radiation from a source placed at the boundary is allowed only in the downward direction, i.e., toward the random-medium side; this is because $S_A(z\mid z')$ is a medium wave, as was emphasized below Eq. (4.11), and is never propagated into the upper region where $K_2^{(q)}=0$.

V. BOUNDARY-VALUE SOLUTIONS OF THE DIFFUSION EQUATION

An eigenfunction expansion similar to Eq. (4.11) is possible also for $I_{22}^{(s/12)}$ and, by the substitution into Eq. (3.17), the coefficient of the diffusion term, say, $S_A^{(12)}(z\mid z')$, is found to be the solution of the integral equation

$$S_A^{(12)}(z\mid z')=S_A(z\mid z')+S_A(z\mid z'(=0))\langle\sigma_{22}^{(12)}(-\bar{\partial}_z'\mid\vec{\partial}_z)\rangle \times S_A^{(12)}(z(=0)\mid z') , \quad (5.1)$$

in terms of the notation

$$\langle\sigma_{22}^{(12)}(-\bar{\partial}_z\mid\vec{\partial}_z)\rangle=\int d\hat{\Omega}\int d\hat{\Omega}'\overline{\phi_A(\hat{\Omega},-i\bar{\partial}_z)}\sigma_{22}^{(12)}(\hat{\Omega}\mid\hat{\Omega}') \times\phi_A(\hat{\Omega}',i\vec{\partial}_z) . \quad (5.2)$$

Hence the boundary condition for $S_A^{(12)}(z|z')$ is found, by applying Eqs. (4.22) to Eq. (5.1), to be

$$(2\gamma_2)^{-1}(\tfrac{1}{2}+D_2\partial_z)S_A^{(12)}(z(=0)\mid z'(<0)) =\langle\sigma_{22}^{(12)}(\vec{\partial}_z)\rangle S_A^{(12)}(z(=0)\mid z'(<0)) , \quad (5.3)$$

where

$$\langle\sigma_{22}^{(12)}(\vec{\partial}_z)\rangle=\langle\sigma_{22}^{(12)}(-\bar{\partial}_z(=0)\mid\vec{\partial}_z)\rangle , \quad (5.4)$$

showing that $S_A^{(12)}(z\mid z')$ can be obtained as a solution of the inhomogeneous diffusion equation (4.16) subjected to Eq. (5.3), instead of solving the integral equation (5.1) with known $S_A(z\mid z')$.

Here, when using Eqs. (4.15), Eq. (5.4) can be written in the form

$$\langle\sigma_{22}^{(12)}(\partial_z)\rangle=(2\gamma_2)^{-1}(\langle\sigma_{22}^{(12)}\rangle_0-\langle\sigma_{22}^{(12)}\rangle_1 D_2\partial_z) . \quad (5.5)$$

Here

$$\langle\sigma_{22}^{(12)}\rangle_0=\frac{1}{2\pi}\int_{2\pi}d\hat{\Omega}\int d\hat{\Omega}'\sigma_{22}^{(12)}(\hat{\Omega}\mid\hat{\Omega}') , \quad (5.6a)$$

$$\langle\sigma_{22}^{(12)}\rangle_1=\frac{3}{2\pi}\int_{2\pi}d\hat{\Omega}\int d\hat{\Omega}'\sigma_{22}^{(12)}(\hat{\Omega}\mid\hat{\Omega}')\Omega_z' . \quad (5.6b)$$

A. Case of a random layer (Fig. 1)

The situation is the same also for $I_{22}^{(s/12+23)}$ in Eqs. (2.57) and (2.58), as long as $\gamma_2 L \gg 1$, and its diffusion coefficient $S_A^{(12+23)}$, say, is the solution of the same diffusion equation subjected to the boundary condition (5.3) at $z=0$ and also that at $z=-L$, given similarly by

$$(2\gamma_2)^{-1}(\tfrac{1}{2}-D_2\partial_z)S_A^{(12+23)}(z(=-L)\,|\,z') = \langle \sigma_{22}^{(23)}(\partial_z)\rangle S_A^{(12+23)}(z(=-L)\,|\,z') , \quad 0>z'>-L , \tag{5.7}$$

with

$$\langle \sigma_{22}^{(23)}(\partial_z)\rangle = (2\gamma_2)^{-1}(\langle \sigma_{22}^{(23)}\rangle_0 + \langle \sigma_{22}^{(23)}\rangle_1 D_2\partial_z) . \tag{5.8}$$

Here the operators $\langle \sigma_{ab}^{(\,)}(\partial_z)\rangle$, as defined by Eqs. (5.4) and (5.2), are subject to a relation resulting from the optical relation (3.16a); that is,

$$\sum_{a=1}^{2} k_a \langle \sigma_{a2}^{(12)}(\partial_z)\rangle = \frac{k_2}{4\pi}\int_{2\pi} d\hat{\Omega}\,\Omega_z^{(2)}\phi_A(\hat{\Omega},i\partial_z) = k_2(2\gamma_2)^{-1}(\tfrac{1}{2}-D_2\partial_z) , \tag{5.9a}$$

and, similarly,

$$\sum_{a=2}^{3} k_a \langle \sigma_{a2}^{(23)}(\partial_z)\rangle = k_2(2\gamma_2)^{-1}(\tfrac{1}{2}+D_z\partial_z) . \tag{5.9b}$$

Also the boundary equations (5.3) and (5.7) can be rewritten, on using Eqs. (5.5) and (5.8), in the form

$$-D_2\partial_z S_A^{(12+23)}(z(=0)\,|\,z') = Z^{(12)}S_A^{(12+23)}(z(=0)\,|\,z') , \tag{5.10a}$$

$$+D_2\partial_z S_A^{(12+23)}(z(=-L)\,|\,z') = Z^{(23)}S_A^{(12+23)}(z(=-L)\,|\,z') . \tag{5.10b}$$

Here

$$Z^{(12)} = (\tfrac{1}{2}-\langle \sigma_{22}^{(12)}\rangle_0)/(1+\langle \sigma_{22}^{(12)}\rangle_1) , \tag{5.11a}$$

$$Z^{(23)} = (\tfrac{1}{2}-\langle \sigma_{22}^{(23)}\rangle_0)/(1+\langle \sigma_{22}^{(23)}\rangle_1) . \tag{5.11b}$$

A plane-wave solution of the diffusion equation (4.16) is given by

$$S_A^{(12+23)}(z\,|\,z') = C_A\varphi^{(12)}(z_>)\varphi^{(23)}(z_<) . \tag{5.12}$$

Here $z_>$ and $z_<$ designate the larger and the smaller of z and z', respectively, and $\varphi^{(12)}(z)$ and $\varphi^{(23)}(z)$ are solutions of the homogeneous diffusion equation subjected to the boundary conditions at $z=0$ and $-L$, respectively; C_A is a constant given by

$$C_A = \frac{\gamma_2}{D_2}\left[\varphi^{(12)}\frac{\partial}{\partial z}\varphi^{(23)} - \varphi^{(23)}\frac{\partial}{\partial z}\varphi^{(12)}\right]^{-1} . \tag{5.13}$$

Hence, with the notation $\kappa=(\gamma^{(ab)}/D_2)^{1/2}$, we can set

$$\varphi^{(12)}(z) = \varphi^{(12)}(0)\left[\cosh(\kappa z) - \frac{Z^{(12)}}{\kappa D_2}\sinh(\kappa z)\right] , \tag{5.14a}$$

$$\varphi^{(23)}(z) = \varphi^{(23)}(-L)\left[\cosh[\kappa(z+L)] + \frac{Z^{(23)}}{\kappa D_2}\sinh[\kappa(z+L)]\right] , \tag{5.14b}$$

and determine C_A at $z=0$; hence

$$S_A^{(12+23)}(z(=0)|z'(=-L)) = \gamma_2\left[Z^{(12)}\left[\cosh(\kappa L)+\frac{Z^{(23)}}{\kappa D_2}\sinh(\kappa L)\right] + \kappa D_2\sinh(\kappa L) + Z^{(23)}\cosh(\kappa L)\right]^{-1} , \tag{5.15}$$

$$S_A^{(12+23)}(z(=0)\,|\,z'(=0)) = S_A^{(12+23)}(z(=0)\,|\,z'(=-L))\left[\cosh(\kappa L)+\frac{Z^{(23)}}{\kappa D_2}\sinh(\kappa L)\right] . \tag{5.16}$$

B. Case of a nondissipative medium ($\gamma^{(ab)}=\kappa=0$)

In this case, Eq. (5.15) is reduced to

$$S_A^{(12+23)}(0\,|\,-L) = \gamma_2\left[Z^{(12)}\left[1+Z^{(23)}\frac{L}{D_2}\right] + Z^{(23)}\right]^{-1} , \tag{5.17}$$

which tends to zero as $L\to\infty$, being therefore a cutoff solution. Here holds the important relation

$$Z^{(12)}S_A^{(12+23)}(0\,|\,0) + Z^{(23)}S_A^{(12+23)}(-L\,|\,0) = \gamma_2 , \tag{5.18}$$

which can be shown more generally by integrating Eq. (4.16) over the range $0\geq z\geq -L$ with $z'=-0$ and followed using (5.10).

C. Angle distributions of the scattered waves

From Eq. (3.23), a specific expression of $I_{11}^{(q+12+23)}$ to the diffusion approximation is obtained in the form

$$I_{11}^{(q+12+23)}(\hat{\Omega},z\,|\,\hat{\Omega}',z') = I_{11}^{(12)}(\hat{\Omega},z\,|\,\hat{\Omega}',z') + U_1(\hat{\Omega},z)\sigma_{12}^{(12)}(\hat{\Omega}\,|\,\vec{\partial}_z)S_A^{(12+23)}(z(=0)\,|\,z'(=0))\sigma_{21}^{(12)}(-\overleftarrow{\partial}_z'\,|\,\hat{\Omega}')U_1(\hat{\Omega}',z') . \tag{5.19}$$

Here the first term is given by Eq. (3.13), and

$$\sigma_{12}^{(12)}(\hat{\Omega}|\partial_z)=\int_{\Omega_z'>0} d\hat{\Omega}'\sigma_{12}^{(12)}(\hat{\Omega}|\hat{\Omega}')\phi_A(\hat{\Omega}',i\partial_z) , \quad (5.20a)$$

$$\sigma_{21}^{(12)}(-\partial_z\,|\,\hat{\Omega})=\int_{\Omega_z'<0} d\hat{\Omega}'\overline{\phi_A(\hat{\Omega}',-i\partial_z)}\sigma_{21}^{(12)}(\hat{\Omega}'\,|\,\hat{\Omega}) , \quad (5.20b)$$

which are related to $\langle\sigma_{ab}^{(\,)}(\partial_z)\rangle$ [Eq. (5.4)] by

$$\langle\sigma_{ab}^{(\,)}(\partial_z)\rangle=\frac{1}{4\pi}\int d\hat{\Omega}\,\sigma_{ab}^{(\,)}(\hat{\Omega}\,|\,\partial_z) , \quad (5.20c)$$

and, when using Eqs. (4.15),

$$4\pi\sigma_{21}^{(12)}(-\partial_z\,|\,-\hat{\Omega})=\gamma_2\sigma_{12}^{(12)}(+\hat{\Omega}\,|\,\partial_z)=\int d\hat{\Omega}'\sigma_{12}^{(12)}(\hat{\Omega}\,|\,\hat{\Omega}')(1-3\Omega_z'D_2\partial_z) . \quad (5.21)$$

Also, from (5.10a),

$$\vec{\partial}_z=-Z^{(12)}/D_2 , \quad (5.22)$$

while, from (5.14b),

$$-\bar{\partial}_z'=-\frac{\kappa\sinh(\kappa L)+(Z^{(23)}/D_2)\cosh(\kappa L)}{\cosh(\kappa L)+(Z^{(23)}/\kappa D_2)\sinh(\kappa L)} . \quad (5.23)$$

In the same way, for the transmitted waves of Eq. (3.24),

$$I_{31}^{(q+12+23)}(\hat{\Omega},z\,|\,\hat{\Omega}',z')=U_3(\hat{\Omega},z)\sigma_{32}^{(23)}(\hat{\Omega}\,|\,\vec{\partial}_z)S_A^{(12+23)}(z(=-L)\,|\,z'(=0))\sigma_{21}^{(12)}(-\bar{\partial}_z'\,|\,\hat{\Omega}')U_1(\hat{\Omega}',z') . \quad (5.24a)$$

Here $\sigma_{32}^{(23)}(\hat{\Omega}\,|\,\partial_z)$ differs from $\sigma_{12}^{(12)}(\hat{\Omega}\,|\,\partial_z)$ in Eq. (5.19), not only by k_3 and k_1, but also by the sign of the coefficients of ∂_z; and

$$\vec{\partial}_z=+Z^{(23)}/D_2,\quad -\bar{\partial}_z'=+Z^{(12)}/D_2 . \quad (5.24b)$$

When the medium is nondissipative, power conservation is strictly ensured by the boundary conditions; that is, the total scattered power away from both sides of the layer, say, $\langle W_z^{(s)}\rangle$, is given from Eqs. (5.19), (5.24a), and (3.13), by

$$\begin{aligned}\langle W_z^{(s)}\rangle&=\int_{2\pi} d\hat{\Omega}\,k_1\Omega_z^{(1)}I_{11}^{(q+12+23)}(\hat{\Omega},z(=0)\,|\,\hat{\Omega}',z')+\int_{2\pi} d\hat{\Omega}\,k_3\Omega_z^{(3)}I_{31}^{(q+12+23)}(\hat{\Omega},z(=-L)\,|\,\hat{\Omega}',z')-k_1\\&=k_1\int d\hat{\Omega}\,\sigma_{11}^{(12)}(\hat{\Omega}\,|\,\hat{\Omega}')U_1(\hat{\Omega}',z')\\&\quad+4\pi[k_1\langle\sigma_{12}^{(12)}(\partial_z)\rangle S_A^{(12+23)}(z(=0)\,|\,z'(=0))\\&\quad\quad+k_3\langle\sigma_{32}^{(23)}(\partial_z)\rangle S_A^{(12+23)}(z(=-L)\,|\,z'(=0))]\sigma_{21}^{(12)}(-\bar{\partial}_z'\,|\,\hat{\Omega}')U_1(\hat{\Omega}',z') ,\end{aligned} \quad (5.25)$$

in consequence of Eqs. (3.11) and (5.20c). Here, as it is proven below, the term in brackets [] in the last term is the constant k_2 independent of z', resulting in the fact that $-\bar{\partial}_z'$ involved in the following factor $\sigma_{21}^{(12)}(-\bar{\partial}_z'\,|\,\hat{\Omega}')$ becomes effectively zero. Hence the entire contribution from the last term is reduced to

$$\int d\hat{\Omega}\,k_2\sigma_{21}^{(12)}(\hat{\Omega}\,|\,\hat{\Omega}')U_1(\hat{\Omega}',z') , \quad (5.26)$$

being the incident power through S_{12}. To prove the above statement, we first eliminate the term of $\langle\partial_{22}^{(\,)}(\partial_z)\rangle$ from each of the boundary equations (5.3) and (5.7) by using relations (5.9), to get

$$[\gamma_2^{-1}k_2D_2\partial_z+k_1\langle\sigma_{12}^{(12)}(\partial_z)\rangle]S_A^{(12+23)}(z(=0)\,|\,z'(=0))=0 , \quad (5.27a)$$

$$[-\gamma_2^{-1}k_2D_2\partial_z+k_3\langle\sigma_{32}^{(23)}(\partial_p)\rangle]S_A^{(12+23)}(z(=-L)\,|\,z'(=0))=0 , \quad (5.27b)$$

and then sum up the above equations with the aid of Eqs. (5.10) and (5.18); hence the proof is given by obtaining

$$k_1\langle\sigma_{12}^{(12)}(\partial_z)\rangle S_A^{(12+23)}(z(=0)\,|\,0)+k_3\langle\sigma_{32}^{(23)}(\partial_z)\rangle S_A^{(12+23)}(z(=-L)\,|\,0)=k_2 . \quad (5.27c)$$

D. Numerical examples: Case of a nondissipative random layer with smooth boundaries

Shown in Fig. 3 is a curve of $Z^{(12)}$ versus k_1/k_2, given by Eqs. (5.11) when the boundary is perfectly smooth, with the details of the calculation in Appendix B. As k_1/k_2 changes from 0 to 1, $Z^{(12)}$ changes from 0 (perfectly conducting surface) to 0.5 for a boundary of no reflection [Eq. (4.22b)], as is expected. shown in Fig. 4 are the angle distributions of both backscattered (solid

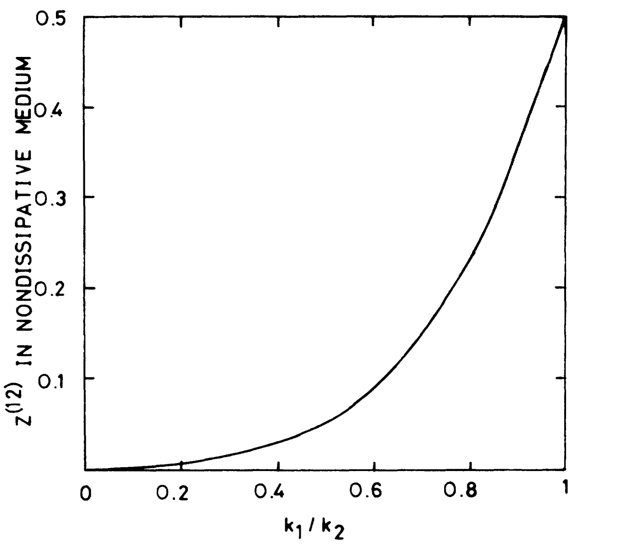

FIG. 3. $Z^{(12)}$ of Eqs. (5.11) is shown as a function of k_1/k_2 for a perfectly smooth boundary in a nondissipative medium, with the eigenfunction of Eq. (4.15a).

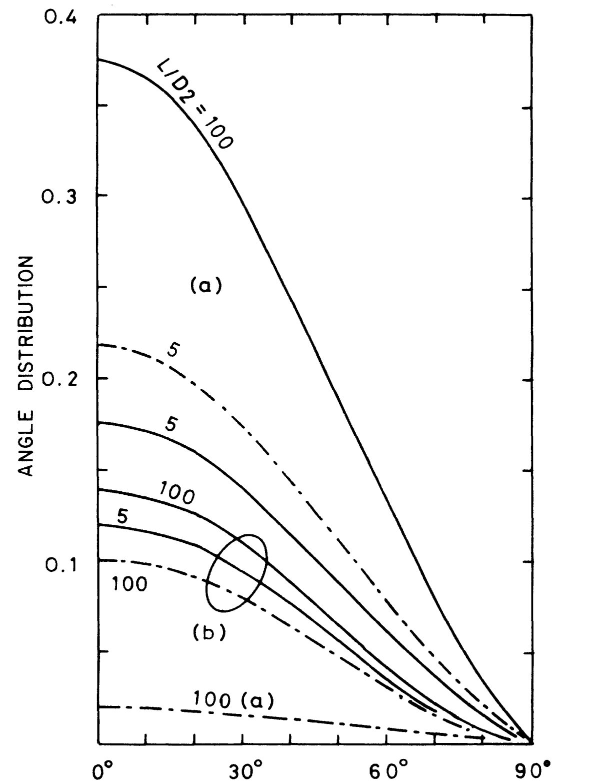

FIG. 4. Angle distributions of both backscattered and transmitted waves through a random layer with perfectly smooth boundaries and numerical width L/D_2. The wave is normally incident and the medium is nondissipative with (a) $k_1/k_2=0.99$; (b) $k_1/k_2=0.2$. The backscattered waves are shown by solid curves and the transmitted waves by dash-dot curves.

curves) and transmitted waves (dash-dot curves) through a random layer with diffusion constant D_2 [Eq. (4.13)], width L, averaged propagation constant k_2, and separated by two smooth boundaries from the outside free spaces with the equal propagation constants $k_1=k_3$ (Fig. 1). The basic equations used are Eqs. (5.19), (5.24), and (B21). The curves are shown for a normally incident wave and (a) $k_1/k_2=0.99$, $Z^{(12)}=0.4863$; (b) $k_1/k_2=0.2$, $Z^{(12)}=0.004\,122$, by using the numerical distance L/D_2 as a parameter. The sum of the backscattered and transmitted wave intensities are independent of L/D_2 in each case, reflecting the power conservation; here the sum in case (b) is smaller than that in case (a) by the different amount of specular reflection by the respective upper boundaries. The backscattered waves are generally stronger than the transmitted waves when $L/D_2=100$, whereas the situation is inversed when $L/D_2=5$ although the two waves are nearly of the same intensity in case (b); the smaller k_1/k_2 is, the more the waves are trapped inside the layer as a consequence of an enhanced multiple reflection between the two boundaries.

VI. GENERAL THEORY

The boundary equation can be written in a more general form so that the equation is applicable also to the case in which media are random on both sides of a boundary with the cross section of the general form $\sigma_{ab}(\hat{\Omega}\mid\hat{\Omega}')$. We designate the eigenfunctions of the diffusion term in space k_b by $\phi_A^{(b)}(\hat{\Omega},\hat{\lambda})$ and $\overline{\phi_A^{(b)}(\hat{\Omega},\hat{\lambda})}$ [Eqs. (4.8)], and first introduce two matrix operators, defined by

$$\sigma_{ab}(\hat{\Omega}\mid -\partial_n)=\int_{(-b)} d\hat{\Omega}'\sigma_{ab}(\hat{\Omega}\mid\hat{\Omega}')\phi_A^{(b)}(\hat{\Omega}',-i\hat{\partial}_n^{(b)})\ , \tag{6.1a}$$

$$\sigma_{ba}(+\partial_n|\hat{\Omega})=\int_{(+b)} d\hat{\Omega}'\overline{\phi_A^{(b)}(\hat{\Omega}',i\hat{\partial}_n^{(b)})}\sigma_{ba}(\hat{\Omega}'|\hat{\Omega})\ , \tag{6.1b}$$

similar to those defined by Eqs. (5.20). Here $\hat{\partial}_n^{(b)}=\hat{\mathbf{n}}^{(b)}\partial_n^{(b)}$, where $\partial_n^{(b)}=\hat{\mathbf{n}}^{(b)}\cdot\partial/\partial\hat{\rho}$, is the differential operator directed normally to the boundary toward the space k_b, and the integration ranges $(\pm b)$ designate the half solid angles of $\Omega_n^{(b)}=\hat{\mathbf{n}}^{(b)}\cdot\hat{\Omega}\gtrless 0$, respectively. Here, since $\tilde{U}_2(-\hat{\Omega},-\hat{\lambda})=\tilde{U}_2(\hat{\Omega},\hat{\lambda})$ in the eigenvalue equations (4.1) as a consequence of the reciprocity (3.20d), it follows that $\phi_A^{(b)}(\hat{\Omega},\hat{\lambda})$ and $\phi_A^{(b)}(-\hat{\Omega},-\hat{\lambda})$ are the same function except for a numerical factor, and, therefore, the same is true also for $\sigma_{ba}(\partial_n\mid\hat{\Omega})$ and $\sigma_{ab}(-\hat{\Omega}\mid -\partial_n)$, in view of Eq. (3.16b).

We also introduce an operator $\langle\sigma_{ab}(-\partial_n)\rangle$, defined by

$$\langle \sigma_{ab}(-\partial_n)\rangle = \int_{(+a)} d\hat{\Omega} \int_{(-b)} d\hat{\Omega}' \overline{\phi_A^{(a)}(\hat{\Omega}, \hat{\lambda}(=0))} \sigma_{ab}(\hat{\Omega} \mid \hat{\Omega}') \phi_A^{(b)}(\hat{\Omega}', -i\hat{\partial}_n^{(b)}) \tag{6.2a}$$

$$= \frac{1}{4\pi} \int_{(+a)} d\hat{\Omega}\, \sigma_{ab}(\hat{\Omega} \mid -\partial_n) , \tag{6.2b}$$

where use has been made of Eq. (A7); and similarly $p_b^{\pm}(\partial_n)$, defined by

$$p_b^{\pm}(\partial_n) = \frac{1}{4\pi} \int_{(\pm b)} d\hat{\Omega}\, \Omega_n^{(b)} \phi_A^{(b)}(\hat{\Omega}, i\hat{\partial}_n^{(b)}) , \tag{6.3}$$

which is reduced, when using Eqs. (4.15) with $\Omega_z \partial_z \to \Omega_n^{(b)} \partial_n^{(b)}$, $\gamma_2 \to \gamma_b$, and $D_2 \to D_b$, to

$$p_b^{\pm}(\partial_n) = (2\gamma_b)^{-1}(\pm\tfrac{1}{2} - D_b \partial_n^{(b)}) . \tag{6.4}$$

Now, the generalized version of boundary equation (5.3) is written as

$$p_b^{+}(\partial_n) S_A^{(a)} = \sum_b \langle \sigma_{ab}(-\partial_n)\rangle S_A^{(b)} . \tag{6.5}$$

Here $S_A^{(a)}$ denotes the boundary value (and its normal derivative) on the side of space k_a, and the left-hand side means the power scattered toward the space k_a by the boundary. From Eqs. (6.2) and (6.3), we can write $p_a^{\pm}(\partial_n)$ and $\langle \sigma_{ab}(-\partial_n)\rangle$ in the form

$$p_a^{\pm}(\partial_n) = p_{a,0}^{\pm} - p_{a,1}^{\pm} \partial_n^{(a)} , \tag{6.6}$$

$$\langle \sigma_{ab}(-\partial_n)\rangle = \langle \sigma_{ab}\rangle_0 + \langle \sigma_{ab}\rangle_1 \partial_n^{(b)} . \tag{6.7}$$

In 2×2 matrix form, Eqs. (6.5)–(6.7) can be written simply by

$$p^{+}(\partial_n) S_A = \langle \sigma(-\partial_n)\rangle S_A , \tag{6.8}$$

$$p^{\pm}(\partial_n) = p_0^{\pm} - p_1^{\pm} \partial_n , \tag{6.9}$$

$$\langle \sigma(-\partial_n)\rangle = \langle \sigma\rangle_0 + \langle \sigma\rangle_1 \partial_n , \tag{6.10}$$

where $p^{\pm}(\partial_n)$ is regarded as a diagonal matrix with the elements $p_a^{\pm}(\partial_n)$. Hence Eq. (6.8), upon substitution of Eqs. (6.9) and (6.10), becomes written finally in the form

$$\partial_n S_A = Z S_A \tag{6.11a}$$

or, more explicitly,

$$\partial_n^{(a)} S_A^{(a)} = \sum_b Z_{ab} S_A^{(b)} . \tag{6.11b}$$

Here Z is a 2×2 matrix, defined by

$$Z = (p_1^{+} + \langle \sigma\rangle_1)^{-1}(p_0^{+} - \langle \sigma\rangle_0) . \tag{6.12}$$

Equations (6.11) and (6.12) obviously correspond to Eqs. (5.10) and (5.11), respectively, and they are in fact equivalent to each other in the case of a semi-infinite random layer.

To confirm the consistency of the boundary equation (6.5) with power conservation, we observe that the $\langle \sigma_{ab}(-\partial_n)\rangle$'s are subject to a constraint resulting from optical relation (3.16a), as

$$\sum_a k_a \langle \sigma_{ab}(-\partial_n)\rangle = -k_b p_b^{-}(\partial_n) . \tag{6.13}$$

Hence, using Eq. (6.5),

$$\sum_a k_a p_a^{+}(\partial_n) S_A^{(a)} = \sum_{a,b} k_a \langle \sigma_{ab}(-\partial_n)\rangle S_A^{(b)}$$
$$= -\sum_b k_b p_b^{-}(\partial_n) S_A^{(b)} , \tag{6.14}$$

which can be written, in terms of the notation

$$p_a(\partial_n) = p_a^{+}(\partial_n) + p_a^{-}(\partial_n) , \tag{6.15}$$

by

$$\sum_a k_a p_a(\partial_n) S_A^{(a)} = 0 . \tag{6.16}$$

Here the sum means the total (angle-averaged) power away from both sides of the boundary, as is clear from the definition (6.3).

The function $S_A^{(a)}$ in each space is governed by the diffusion equation from Eq. (A31), i.e., with $\hat{\mathbf{p}}_A \to \hat{\mathbf{p}}_A^{(a)}$, by

$$\frac{\partial}{\partial \hat{\boldsymbol{\rho}}} \cdot \hat{\mathbf{p}}_A^{(a)}(i\partial/\partial\hat{\boldsymbol{\rho}}) S_A^{(a)}(\hat{\boldsymbol{\rho}} | \hat{\boldsymbol{\rho}}') = \delta(\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\rho}}') , \tag{6.17}$$

where $K_a^{(q)}(\hat{\Omega} | \hat{\Omega}')$ can be even rotationally variant. Here $p_a(\partial_n)$ of Eq. (6.15) is the same as the component of $\hat{\mathbf{p}}_A^{(a)}$ in the direction $\hat{\mathbf{n}}^{(a)}$, and can also be written in the form

$$\hat{\mathbf{n}}^{(a)} \cdot \hat{\mathbf{p}}_A^{(a)}(\partial_n) = p_a(\partial_n) = p_{a,0} - p_{a,1} \partial_n^{(a)} . \tag{6.18}$$

Here $p_{a,0}$ is generally dependent on the horizontal $\boldsymbol{\lambda} = i\partial/\partial\boldsymbol{\rho}$ although it is identically zero when the scattering cross section is rotationally invariant.

VII. SUMMARY AND DISCUSSION

For a system of random layers with rough boundaries, it is generally possible to construct the BS equation in a unified form so that the random medium and boundaries are involved on exactly the same footing, as given by Eq. (2.23) for a random layer with two rough boundaries (Fig. 1). There are several expressions of the solution to choose from, depending on the situation and information required. One of the expressions is Eq. (2.55) with specific expressions (2.57) and (2.58) for the backscattered and transmitted waves, respectively; which are particularly convenient in the present case because the only medium-dependent function is $I_{22}^{(s/12+23)}$ so that the diffusion approximation can be limitedly applied only to this function, without affecting the other terms and factors at all.

The diffusion approximation is based on the series expansion (4.7) for the scattering matrix of the medium alone (free from the boundary characteristics) by using the eigenfunctions of the medium cross section [Eqs.

(4.1)]. Here the expansion coefficients are given as solutions of Eqs. (4.6), and the first (diffusion) term is generally good enough in the diffusion region, the second- and the higher-order terms being short-range functions decreasing rapidly with the distance from the source. It should be mentioned, however, that, when the forward scattering is absolutely dominant as in the case of light wave propagation through a turbulent medium, the first several terms of the series become equally important and the eigenfunction expansion is therefore not useful any longer.[5] The present method of the eigenfunction expansion is quite general in the sense of the applicability for a wide class of medium scattering cross sections including those rotationally invariant in space (Appendix A).

The boundary-dependent diffusion term is governed by Eq. (5.1), wherein the second term provides the entire boundary effect due to S_{12} and the first term is subject to the condition of no reflection that has been the only condition used so far whenever solving diffusion equations, regardless of whether the average refractive indices of the two media are nearly equal or differ from each other by a large amount.[3,4] In the case of a random layer, the boundary conditions are written in the form (5.10) with Eqs. (5.11), which are simple enough to obtain the boundary-value solution of diffusion equation (4.16); and the resulting backscattered and transmitted waves through the layer are given by Eqs. (5.19) and (5.24a). Here power is strictly conserved in spite of the diffusion approximation that discards all the terms other than the first in the series expansion, primarily because the second- and the higher-order terms are not directly concerned with the propagation of power, provided that the two boundaries are separated enough so that their multiple reflection between the boundaries is negligible.

In Sec. VI, the boundary equation was generalized to the case of a rough boundary between two random media of different kinds. The equation is given by Eq. (6.5), and ensures power conservation in virtue of relation (6.13) resulting from the optical relation for the boundary cross section.

APPENDIX A: EIGENFUNCTION EXPANSIONS ASSOCIATED WITH DIFFUSION APPROXIMATION

The eigenvalue equations for ϕ_A and $\overline{\phi_A}$ become, on using Eqs. (4.1) and (4.8), as

$$\int d\hat{\mathbf{\Omega}}' K_2^{(q)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}')\phi_A(\hat{\mathbf{\Omega}}',\hat{\boldsymbol{\lambda}}) = A(\hat{\boldsymbol{\lambda}})\tilde{U}_2^{-1}\phi_A(\hat{\mathbf{\Omega}},\hat{\boldsymbol{\lambda}}) , \tag{A1}$$

$$\int d\hat{\mathbf{\Omega}}' \overline{\phi_A(\hat{\mathbf{\Omega}}',\hat{\boldsymbol{\lambda}})} K_2^{(q)}(\hat{\mathbf{\Omega}}' \mid \hat{\mathbf{\Omega}}) = A(\hat{\boldsymbol{\lambda}})\overline{\phi_A}\tilde{U}_2^{-1}(\hat{\mathbf{\Omega}},\hat{\boldsymbol{\lambda}}) , \tag{A2}$$

where

$$\tilde{U}_2^{-1}(\hat{\mathbf{\Omega}},\hat{\boldsymbol{\lambda}}) = \gamma_2 - i\hat{\mathbf{\Omega}}\cdot\hat{\boldsymbol{\lambda}} . \tag{A3}$$

Hence, when $\hat{\boldsymbol{\lambda}}=0$, ϕ_A and $\overline{\phi_A}$ are reduced to $\phi_a(\hat{\mathbf{\Omega}})$ and $\phi_a(\hat{\mathbf{\Omega}})$ with the eigenvalues a, respectively, defined by

$$\int d\hat{\mathbf{\Omega}}' K_2^{(q)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}')\phi_a(\hat{\mathbf{\Omega}}') = a\gamma_2\phi_a(\hat{\mathbf{\Omega}}) , \tag{A4}$$

$$\int d\hat{\mathbf{\Omega}}' \overline{\phi_a(\hat{\mathbf{\Omega}}')} K_2^{(q)}(\hat{\mathbf{\Omega}}' \mid \hat{\mathbf{\Omega}}) = a\gamma_2\overline{\phi_a(\hat{\mathbf{\Omega}}')} , \tag{A5}$$

subjected to the normalization

$$\int d\hat{\mathbf{\Omega}}\, \overline{\phi_a}\gamma_2\phi_b(\hat{\mathbf{\Omega}}) = \delta_{ab} . \tag{A6}$$

Here we directly see that $a=1$ is one of the eigenvalues with the eigenfunction of a uniform distribution, $\overline{\phi_1}$. Hence, according to (A6), we can set

$$\overline{\phi_1(\hat{\mathbf{\Omega}})} = (4\pi)^{-1}, \quad \phi_1(\hat{\mathbf{\Omega}}) = \overline{\gamma_2}^{-1} , \tag{A7}$$

where $\overline{\gamma_2}$ is the angle-averaged value of $\gamma_2(\hat{\mathbf{\Omega}})$, and use has been made of the reciprocities (3.20). Also $|a| \le 1$ is proven.[5] The $\hat{\mathbf{\Omega}}$ integration of (A4) on both sides and followed by use of Eq. (3.21) lead to the important relation

$$(1-a)\int d\hat{\mathbf{\Omega}}\, \gamma_2\phi_a(\hat{\mathbf{\Omega}}) = 0 , \tag{A8}$$

which shows that

$$a = 1 \quad \text{if} \int d\hat{\mathbf{\Omega}}\, \gamma_2\phi_a(\hat{\mathbf{\Omega}}) \neq 0 , \tag{A9}$$

whereas

$$\int d\hat{\mathbf{\Omega}}\, \gamma_2\phi_a(\hat{\mathbf{\Omega}}) = 0 \quad \text{if } a \neq 1 . \tag{A10}$$

To obtain ϕ_A in a series of the ϕ_a's, we set

$$\phi_A = \sum_b \phi_b C_{bA}, \quad C_{aA} = 1 , \tag{A11}$$

which, substituting into Eq. (A1) and using (A3)–(A6), lead to

$$(A-b)C_{bA} = iA\sum_c \langle \hat{\mathbf{\Omega}}\cdot\hat{\boldsymbol{\lambda}} \rangle_{bc} C_{cA} , \tag{A12}$$

in terms of the notation

$$\langle \hat{\mathbf{\Omega}}\cdot\hat{\boldsymbol{\lambda}} \rangle_{bc} = \int d\hat{\mathbf{\Omega}}\, \overline{\phi_b}\hat{\mathbf{\Omega}}\cdot\hat{\boldsymbol{\lambda}}\phi_c . \tag{A13}$$

Equation (A12) gives, to the first order of $\hat{\boldsymbol{\lambda}}$,

$$C_{bA} = i(a-b)^{-1}a\langle \hat{\mathbf{\Omega}}\cdot\hat{\boldsymbol{\lambda}} \rangle_{ba}, \quad b \neq a . \tag{A14}$$

Hence, particularly when $a=1$ (diffusion term),

$$\phi_A = \phi_1 + i\sum_{b(\neq 1)} \phi_b(1-b)^{-1}\langle \hat{\mathbf{\Omega}}\cdot\hat{\boldsymbol{\lambda}} \rangle_{b1} , \tag{A15}$$

and the eigenvalue A is given from (A12), by

$$1 - A(\hat{\boldsymbol{\lambda}}) = \sum_{b(\neq 1)} (1-b)^{-1}\langle \hat{\mathbf{\Omega}}\cdot\hat{\boldsymbol{\lambda}} \rangle_{1b}\langle \hat{\mathbf{\Omega}}\cdot\hat{\boldsymbol{\lambda}} \rangle_{b1} \tag{A16}$$

$$= -i\frac{1}{4\pi}\int d\hat{\mathbf{\Omega}}(\hat{\boldsymbol{\lambda}}\cdot\hat{\mathbf{\Omega}})\phi_A(\hat{\mathbf{\Omega}},\hat{\boldsymbol{\lambda}}) \tag{A17}$$

(where $\langle \hat{\mathbf{\Omega}}\cdot\hat{\boldsymbol{\lambda}} \rangle_{11}=0$), which provides the nonvanishing $\hat{\boldsymbol{\lambda}}$ term of $A(\hat{\boldsymbol{\lambda}})$ to the lowest order.

Here, when assuming the conventional form $K_2^{(q)}(\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{\Omega}}')$ for $K_2^{(q)}$, we can choose ϕ_b, $b \neq 1$, as

$$\phi_b(\hat{\mathbf{\Omega}}) = \Omega_j, \quad \overline{\phi_b(\hat{\mathbf{\Omega}})} = (3/4\pi\gamma_2)\Omega_j, \quad j=1,2,3 , \tag{A18}$$

which are consistent with (A6) and also have the orthogonality with respect to the subscript j. Hence

$$\langle \hat{\Omega}\cdot\hat{\lambda}\rangle_{1b}=\tfrac{1}{3}\lambda_j,\quad \langle \hat{\Omega}\cdot\hat{\lambda}\rangle_{b1}=\gamma_2^{-2}\lambda_j\ , \tag{A19}$$

and the eigenvalue b is found, by substitution of (A18) into (A4), to be

$$b=a_1\equiv\gamma_2^{-1}\int d\hat{\Omega}\,K_2^{(q)}(\hat{\Omega}\cdot\hat{\Omega}')\hat{\Omega}\cdot\hat{\Omega}'\ . \tag{A20}$$

Thus, in terms of the notation

$$D_2=(3\gamma_2)^{-1}(1-a_1)^{-1}\ , \tag{A21}$$

Eq. (A16) can be written finally as

$$1-A(\hat{\lambda})=\gamma_2^{-1}D_2\hat{\lambda}^2\ , \tag{A22}$$

and, from (A15) and (A7),

$$\phi_A(\hat{\Omega},\hat{\lambda})=\gamma_2^{-1}(1+i3D_2\hat{\Omega}\cdot\hat{\lambda})\ , \tag{A23}$$

which leads to Eq. (4.15a); Eq. (4.15b) for $\overline{\phi_A}$ is also obtained in the same manner.

The eigenvalues $A(\hat{\lambda})$ are directly connected with power fuxes carried by the respective eigenfunction terms, as follows: The $\hat{\Omega}$ integration on both sides of Eq. (4.1a) and use of (3.21) yield

$$\int d\hat{\Omega}\,\gamma_2\tilde{U}_2 f_A(\hat{\Omega},\hat{\lambda})=A(\hat{\lambda})\int d\hat{\Omega}\,f_A(\hat{\Omega},\hat{\lambda})\ , \tag{A24}$$

which, upon substitution of

$$\gamma_2\tilde{U}_2=i\hat{\Omega}\cdot\hat{\lambda}\tilde{U}_2+1 \tag{A25}$$

in the left-hand side, can be rewritten as

$$1-A(\hat{\lambda})=-i\int d\hat{\Omega}\,\hat{\lambda}\cdot\hat{\Omega}\phi_A(\hat{\Omega},\hat{\lambda})\Big/\int d\hat{\Omega}\,f_A(\hat{\Omega},\hat{\lambda}) \tag{A26}$$

$$\equiv -i\hat{\lambda}\cdot\hat{\mathbf{p}}_A(\hat{\lambda})\ . \tag{A27}$$

Hence the substitution into Eq. (4.6a) shows that

$$-i\int d\hat{\Omega}\,\hat{\lambda}\cdot\hat{\Omega}\phi_A(\hat{\Omega},\hat{\lambda})\tilde{S}_A(\hat{\lambda}\,|\,\hat{\lambda}')=A(\hat{\lambda})(2\pi)^3\delta(\hat{\lambda}-\hat{\lambda}')\int d\hat{\Omega}\,f_A(\hat{\Omega},\hat{\lambda})\ , \tag{A28}$$

which represents the power equation for each wave of $S_A(\hat{\rho}\,|\,\hat{\rho}')$; in fact, by multiplying Eq. (A28) with $\phi_A(\hat{\Omega}',\hat{\lambda}')$ and summing up over all the A's, we obtain the entire power equation for $I_2^{(0s)}$ from Eq. (4.11), as

$$-i\int d\hat{\Omega}\,\hat{\lambda}\cdot\hat{\Omega}\tilde{I}_2^{(0s)}(\hat{\Omega},\hat{\lambda}\,|\,\hat{\Omega}',\hat{\lambda}')=\int d\hat{\Omega}\,K_2^{(q)}(\hat{\Omega}\,|\,\hat{\Omega}')\tilde{U}_2(\hat{\Omega}',\hat{\lambda}')(2\pi)^3\delta(\hat{\lambda}-\hat{\lambda}')\ , \tag{A29}$$

reproducing the Fourier transform of what is given by the $\hat{\Omega}$ integration of the transport equation (3.19). Note that the source term on the right-hand side (rhs) of Eq. (A28) tends to zero as $\hat{\lambda}\rightarrow 0$ for all the terms except the diffusion, in view of Eq. (A10).

Here, for the diffusion term, definition (A27) gives, to the first order of $\hat{\lambda}$,

$$\hat{\mathbf{p}}_A(\hat{\lambda})=\frac{1}{4\pi}\int d\hat{\Omega}\,\hat{\Omega}\phi_A(\hat{\Omega},\hat{\lambda})\ , \tag{A30}$$

which is consistent with Eq. (A17), and Eq. (A28) [Eq. (4.6a)] becomes written as

$$-i\hat{\lambda}\cdot\hat{\mathbf{p}}_A(\hat{\lambda})\tilde{S}_A(\hat{\lambda}\,|\,\hat{\lambda}')=(2\pi)^3\delta(\hat{\lambda}-\hat{\lambda}')\ , \tag{A31}$$

with the approximation $A(\hat{\lambda})\simeq 1$ on the right-hand side. Here, using Eq. (A23) in (A30),

$$\hat{\mathbf{p}}_A(\hat{\lambda})=i\gamma_2^{-1}D_2\hat{\lambda}\ , \tag{A32}$$

and Eq. (A22) is reproduced by Eq. (A27). It may be remarked that the form (A32) remains unchanged even when the scattering cross section $K_2^{(q)}(\hat{\Omega}\,|\,\hat{\Omega}')$ is rotationally variant in space, with the replacement of D_2 by a 3×3 matrix to be multiplied by $\hat{\lambda}$, as given from Eq. (A16) with (A27).

APPENDIX B: COHERENT PART OF BOUNDARY SCATTERING CROSS SECTION

The coherent part of $\sigma_{ab}^{(12)}(\hat{\Omega}|\hat{\Omega}')$ in (3.14) is given by the Fourier transform of matrix $V^{(12)}$ [Eq. (2.37b)] which can be written in the form [see also Eq. (3.5)]

$$\tilde{V}_{ab}^{(12)}(\mathbf{u}\,|\,\lambda(=0)\,|\,\mathbf{u}')=(2\pi)^2\delta(\mathbf{u}-\mathbf{u}')\tilde{V}_{ab}^{(12)}(\mathbf{u})\ . \tag{B1}$$

Here, neglecting the interference terms and using Eq. (2.32),

$$\tilde{V}_{ab}^{(12)}(\mathbf{u})=4\,|\,\tilde{h}_a(\mathbf{u})\langle R_{ab}^{(12)}(\mathbf{u})\rangle\,|^2=\tilde{V}_{ba}^{(12)}(-\mathbf{u})\ . \tag{B2}$$

Hence, according to (3.10), we obtain

$$V_{ab}^{(12)}(\hat{\Omega}\,|\,\hat{\Omega}')=(4\pi)^{-2}\tilde{V}_{ab}^{(12)}(\mathbf{u}(=k_a\mathbf{\Omega})\,|\,\lambda(=0)\,|\,\mathbf{u}'(=k_b\mathbf{\Omega}')) \tag{B3}$$

$$=\Omega_z^{(a)}\,|\,\langle R_{ab}^{(12)}(\mathbf{u}(=k_a\mathbf{\Omega}^{(a)}))\rangle\,|^2\delta_S^2(\hat{\Omega}^{(a)}-\hat{\Omega}^{(a)\prime}) \tag{B4}$$

$$=\Omega_z^{(b)}\,|\,\langle R_{ba}^{(12)}(\mathbf{u}(=k_b\mathbf{\Omega}^{(b)}))\rangle\,|^2\delta_S^2(\hat{\Omega}^{(b)}-\hat{\Omega}^{(b)\prime})\ , \tag{B5}$$

where use has been made of

$$\tilde{h}_a=k_a\Omega_z^{(a)},\quad d\mathbf{u}=k_a^2\Omega_z^{(a)}d\hat{\Omega}^{(a)}\ , \tag{B6}$$

$$\delta(\mathbf{u}-\mathbf{u}')=(k_a^2\Omega_z^{(a)})^{-1}\delta_S^2(\hat{\Omega}^{(a)}-\hat{\Omega}^{(a)\prime})\ . \tag{B7}$$

Here we have the reciprocity

$$V_{ab}^{(12)}(\hat{\Omega}\,|\,\hat{\Omega}')=V_{ba}^{(12)}(-\hat{\Omega}'\,|\,-\hat{\Omega})\ , \tag{B8}$$

although this is not quite clear in the expressions (B4) and (B5) wherein $\langle R_{ab}\rangle\neq\langle R_{ba}\rangle$.

To evaluate $\langle\sigma_{22}^{(12)}\rangle$ of Eqs. (5.6), it is more convenient to rewrite the equations in terms of $\langle \sigma_{12}^{(12)} \rangle$, on using relation (5.9a) in Eq. (5.5), as

$$\langle \sigma_{22}^{(12)} \rangle_0 = \frac{1}{2} - \frac{k_1}{k_2} \langle \sigma_{12}^{(12)} \rangle_0 \;, \tag{B9}$$

$$\langle \sigma_{22}^{(12)} \rangle_1 = 1 - \frac{k_1}{k_2} \langle \sigma_{12}^{(12)} \rangle_1 \;, \tag{B10}$$

so that, from Eq. (5.11a),

$$Z^{(12)} = (k_1/k_2) \langle \sigma_{12}^{(12)} \rangle_0 / [2 - (k_1/k_2) \langle \sigma_{12}^{(12)} \rangle_1] \;. \tag{B11}$$

Here, when the boundary is perfectly smooth, use of expression (B4) yields

$$\sigma_{12}^{(12)}(\hat{\mathbf{\Omega}} \mid \hat{\mathbf{\Omega}}') = \Omega_z^{(1)} \mid \langle R_{12}^{(12)}(\hat{\mathbf{\Omega}}) \rangle \mid^2 \delta_S^2(\hat{\mathbf{\Omega}}^{(1)} - \hat{\mathbf{\Omega}}^{(1)\prime}) \;, \tag{B12}$$

where

$$\langle R_{12}^{(12)}(\hat{\mathbf{\Omega}}) \rangle = 2\Omega_z^{(2)} / [\Omega_z^{(2)} + (k_1/k_2)\Omega_z^{(1)}] \;, \tag{B13}$$

and, when $k_1/k_2 < 1$,

$$(k_1/k_2)\Omega_z^{(1)} = [(\Omega_z^{(2)})^2 - c^2]^{1/2} \;, \quad c = [1 - (k_1/k_2)^2]^{1/2} \;. \tag{B14}$$

Hence, when the expression (B12) for a smooth boundary is used in Eq. (B9), change of the variable of integration by

$$\Omega_z^{(2)} = c \cosh x, \quad (k_1/k_2)\Omega_z^{(1)} = c \sinh x \tag{B15}$$

leads, according to definition (5.6a), to

$$\frac{k_1}{k_2} \langle \sigma_{12}^{(12)} \rangle_0 = \frac{k_1}{k_2} \int_0^1 d\Omega_z^{(2)} \Omega_z^{(1)} \mid \langle R_{21}^{(12)}(\hat{\mathbf{\Omega}}) \rangle \mid^2 \tag{B16}$$

$$= 4c^2 \int_0^{x_c} dx \frac{\cosh^2 x \sinh^2 x}{(\sinh x + \cosh x)^2} \;, \tag{B17}$$

where

$$x_c = \operatorname{arctanh}(k_1/k_2) \;. \tag{B18}$$

In the same way, for Eq. (B10), we obtain

$$\frac{k_1}{k_2} \langle \sigma_{12}^{(12)} \rangle_1 = 12c^3 \int_0^{x_c} dx \frac{\cosh^3 x \sinh^2 x}{(\sinh x + \cosh x)^2} \;. \tag{B19}$$

Hence $Z^{(12)}$ is given by Eq. (B11) with (B17) and (B19), and when $x_c \sim k_1/k_2 \ll 1$, is reduced to

$$Z^{(12)} \sim \tfrac{2}{3}(k_1/k_2)^3, \quad |k_1/k_2| \ll 1 \;, \tag{B20}$$

while, when $k_1/k_2 \sim 1$ so that $x_c \sim +\infty$, $Z^{(12)} = 0.5$, reproducing the condition (4.22b) at a free boundary.

To evaluate the associated $\sigma_{12}^{(12)}(\hat{\mathbf{\Omega}} \mid \partial_z)$ in (5.19) in the present case, it is convenient to use Eq. (5.21) with (B5), because of the $\hat{\mathbf{\Omega}}^{(2)}$ integration involved. Hence

$$\begin{aligned} \gamma_2 \sigma_{12}^{(12)}(\hat{\mathbf{\Omega}} \mid \partial_z) &= 4\pi \sigma_{21}^{(12)}(-\partial_z \mid \hat{\mathbf{\Omega}}) \\ &= (1 - 3 \mid \Omega_z^{(2)} \mid D_2 \partial_z) \mid \Omega_z^{(2)} \mid \\ &\quad \times \mid \langle R_{21}^{(12)}(\hat{\mathbf{\Omega}}) \rangle \mid^2 \;, \end{aligned} \tag{B21}$$

where $\Omega_z^{(2)} < 0$, $\Omega_z^{(1)} < 0$, and

$$\langle R_{21}^{(12)}(\hat{\mathbf{\Omega}}) \rangle = 2\Omega_z^{(1)} / [\Omega_z^{(2)} + (k_1/k_2)\Omega_z^{(1)}] \;. \tag{B22}$$

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