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Ensemble processing and the global characterization of temporal and spatial fluctuations

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The fluctuation-spectrum theory aiming at the global characterization of temporal and spatial fluctuations, where the characteristic function λ_q plays the central role, is reconsidered from the ensemble-processing viewpoint. The order-q ensemble processing is constructed so as to extract the statistical-aspect characteristic of the variable q. The quantity $\alpha(q)$ $[\equiv d(q\lambda_q)/dq]$ is shown to be the average of the fluctuations in the processed ensemble. The susceptibility χ_q $[\equiv d\alpha(q)/dq]$ for the infinitesimal change of the degree of processing turns out to determine the asymptotic probability density of fluctuations in the processed ensemble.

I. INTRODUCTION

A new, remarkable trend in a recent study of a wide range of fluctuation phenomena is the global characterization. This is based on observing how the fluctuations of relevant quantities are enhanced or reduced as the scale over which the fluctuations are defined is varied. The multifractal theory[1–7] concerns the fluctuations in the probability exponents on strange sets by changing the cell size. On the other hand, the fluctuation-spectrum theory, proposed by the present authors,[8,9] aiming at the global characterization of temporal and spatial fluctuations, concerns how the local average over a finite time span or a space spread approaches the ensemble average as the scale is extended.

The remarkable feature of the above theories, where the Renyi exponent D_q (Ref. 10) and the characteristic function λ_q (Refs. 11–14) play the central role, is to contain the so-called *filtering parameter* q.[14] Owing to the introduction of this parameter, one can describe various global statistical aspects of fluctuations. However, the role of q is not so well understood. The fundamental purposes of this paper are to introduce the ensemble processing concept in order to clarify the role of the filtering parameter, and to discuss the interrelation between the ensemble processing and the fluctuation-spectrum theory.

Before going into the program, let us briefly review the fluctuation-spectrum theory for the temporal fluctuations.[8] The extension to the spatial fluctuations can be straightforwardly carried out (Sec. IV). Consider a steady time sequence

$$\{u_j\}_{j=1}^{nN}=u_1,u_2,u_3,\ldots,u_{nN}\ . \tag{1.1}$$

The time sequence is supposed to have the so-called natural measure obtained for almost any initial condition. We divide the above sequence into N subsequences each of which has a time length n, $\{u_j\}_{j=1}^{n},\{u_j\}_{j=n+1}^{2n},\ldots,\{u_j\}_{j=mn+1}^{(m+1)n},\ldots,\{u_j\}_{j=(N-1)n+1}^{nN}$. The N subsequences constitute the ensemble S_n. Let us define

$$\alpha_n(m)\equiv\frac{1}{n}\sum_{j=1}^{n}u_{(m-1)n+j} \tag{1.2}$$

$(m=1,2,3,\ldots,N)$, which is the average of u_j in the mth subsequence (ensemble member). The number density of subsequences in which α_n takes a value between α' and $\alpha'+d\alpha'$ is given by

$$\nu_n(\alpha')=\sum_{m=1}^{N}\delta(\alpha_n(m)-\alpha')\ , \tag{1.3}$$

where the normalization condition is written as

$$\int_{-\infty}^{\infty}\nu_n(\alpha')d\alpha'=N\ . \tag{1.4}$$

The probability density $\rho_n(\alpha')$ that α_n takes a value α' is therefore given by

$$\rho_n(\alpha')=\lim_{N\to\infty}\frac{\nu_n(\alpha')}{N}=\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}\delta(\alpha_n(m)-\alpha')\ . \tag{1.5}$$

For a sufficiently large n, $\rho_n(\alpha')$ is assumed to be expanded as

$$\rho_n(\alpha')\sim\sqrt{n}\,\exp[-\sigma(\alpha')n]\ , \tag{1.6}$$

where the function

$$\sigma(\alpha')=-\lim_{n\to\infty}\frac{1}{n}\ln\rho_n(\alpha') \tag{1.7}$$

describes the spectral structure of values of averages α_n's and is called the fluctuation spectrum.[8] By defining the characteristic function[11–14]

$$\lambda_q=\frac{1}{q}\lim_{n\to\infty}\frac{1}{n}\ln M_q(n)\ \text{ for }\ \frac{d\lambda_q}{dq}\geq 0\ , \tag{1.8}$$

where q is real and

$$\begin{aligned}M_q(n)&=\langle\exp(qn\alpha_n)\rangle\\&\equiv\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}\exp[qn\alpha_n(m)]\\&=\int_{-\infty}^{\infty}\rho_n(\alpha')e^{qn\alpha'}d\alpha'\ ,\end{aligned} \tag{1.9}$$

the steepest-descent method leads to[8]

$$\lambda_q=-\frac{1}{q}\min_{\alpha'}[-q\alpha'+\sigma(\alpha')]\ . \tag{1.10}$$

The above is equivalent to the Legendre transform

$$\alpha(q)=\partial_q(q\lambda_q) \text{ for } \partial_q\alpha(q)\geq 0 , \quad (1.11)$$

$$\sigma(\alpha(q))=-q\lambda_q+q\alpha(q) \quad (1.12)$$

$(\partial_q A\equiv\partial A/\partial q)$, i.e.,

$$\sigma(\alpha(q))=q^2\partial_q\lambda_q,\quad \sigma'(\alpha(q))=q . \quad (1.13)$$

The fluctuation spectrum $\sigma(\alpha)$ is a concave function $[\sigma''(\alpha)>0]$ and has a minimal value $\sigma(\alpha)=0$ at $\alpha=\lambda_0=\alpha(0)$, the ensemble average of u_n.

This paper is organized as follows. In Sec. II we shall reconsider the role of the filtering parameter, utilizing a simple chaotic dynamics. We shall be aware of how concretely the characteristic function succeeds in describing the global statistics of the temporal fluctuations. The ensemble processing concept to single out various statistical aspects of the temporal fluctuations will be introduced in Sec. III. This immediately leads to the introduction of the susceptibility. The similar approach to the statistically homogeneous spatial fluctuations will be developed in Sec. IV. A summary and concluding remarks are given in Sec. V. Simple examples are illustrated in Appendices A and B.

II. FILTERING-PARAMETER CONCEPT REVISITED

As is well known, the probability density $\rho_n(\alpha')$ near its peak position takes the Gaussian form,

$$\rho_n(\alpha')\sim\sqrt{n}\,\exp\left[-\frac{(\alpha'-\lambda_0)^2}{4D}n\right] \quad (2.1)$$

for a large n (the central limit theorem result). λ_0 is the ensemble average of u_n $(\lambda_0=\langle u_n\rangle)$, and D is given by

$$D=\partial_q\lambda_q|_{q=0}=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}C_n , \quad (2.2)$$

$C_n\equiv\langle u_n u_0\rangle-\lambda_0^2$ being the double time correlation function. This is equivalent to

$$\lambda_q=\lambda_0+Dq \quad (2.3)$$

because its Legendre transform leads to

$$\alpha(q)=\alpha(0)+2Dq,\quad \sigma(\alpha')=[\alpha'-\alpha(0)]^2/4D \quad (2.4)$$

$[\alpha(0)=\lambda_0]$. We note that the asymptotic law (2.3) holds usually for a small q.[8] If we consider only the peak position of $\rho_n(\alpha')$, the difference among various time sequences is distinguished through the difference among numerical values of λ_0's and D's. Hereafter we will show, utilizing a simple model, how the analysis only with λ_0 and D is far from the overall description of real temporal fluctuations.

The model is the temporal evolution of local expansion rates,

$$u_n=\ln|f'(x_n)| \quad (2.5)$$

for the Bernoulli map

$$x_{n+1}=f(x_n)\equiv\begin{cases}x_n/p, & 0\leq x_n<p\\ (x_n-p)/(1-p), & p\leq x_n\leq 1\end{cases} \quad (2.6)$$

$(0<p<\tfrac{1}{2})$. The quantities λ_0 and D are obtained as

$$\lambda_0=p\ln p^{-1}+(1-p)\ln(1-p)^{-1} , \quad (2.7)$$

$$D=\frac{p(1-p)}{2}\left[\ln\left[\frac{1-p}{p}\right]\right]^2 . \quad (2.8)$$

If we define the variable $y_n=(u_n-\lambda_0)/\sqrt{2D}$, then the probability density for $n^{-1}\sum_{j=1}^{n}y_j$ asymptotically takes the Gaussian form with the mean value $\langle y_n\rangle=0$ and the variance $\langle(n^{-1}\sum_{j=1}^{n}y_j)^2\rangle\simeq 1/n$ for a large n. Therefore y_n's have a statistically similar structure for any p, if the Gaussianity is complete for describing $\{u_n\}$. Figure 1 shows the typical temporal evolutions of the normalized variable y_n for three values of p. Apparently their statistics depend on the parameter p, and one can easily distinguish them from each other. One remarkable feature is the development of the intermittency characteristic observed for $p\to 0$. This is due to the long duration in the region $x_n>p$ for a small p. This indicates that the Gaussian asymptotics characterized with two parameters λ_0 and D are insufficient for precisely describing even the long-time fluctuation characteristics. Since the law (2.3) is valid for a sufficiently small $|q|$, the above qualitative difference (Fig. 1) can be traced back to λ_q with a large $|q|$.

Figure 2 shows λ_q and $\alpha(q)$ for the variable (2.5), where their analytic expressions are[12]

$$\lambda_q=\frac{1}{q}\ln[p^{1-q}+(1-p)^{1-q}] , \quad (2.9)$$

$$\alpha(q)=\frac{p^{1-q}\ln p^{-1}+(1-p)^{1-q}\ln(1-p)^{-1}}{p^{1-q}+(1-p)^{1-q}} \quad (2.10)$$

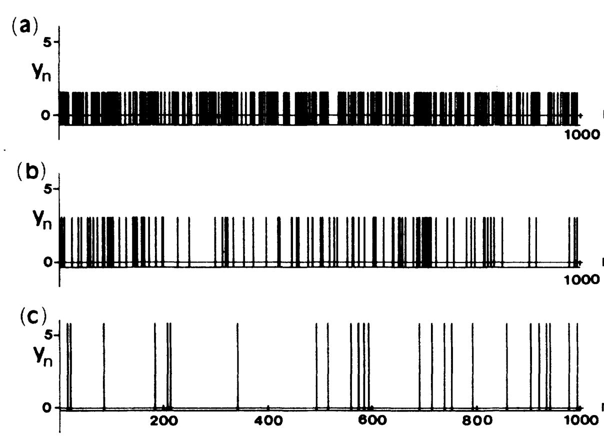

FIG. 1. Typical temporal evolutions of "normalized" local expansion rates $y_n=(u_n-\lambda_0)/\sqrt{2D}$ of the Bernoulli map (2.6) for three values of p: (a) $p=0.3$, (b) 0.1, and (c) 0.03. These temporal fluctuations have the mean value $\langle y_n\rangle=0$ and the variance $\langle(n^{-1}\sum_{j=1}^{n}y_j)^2\rangle\simeq 1/n$ for a large n, independently of p. As $p\to 0$, the intermittency characteristic is developed.

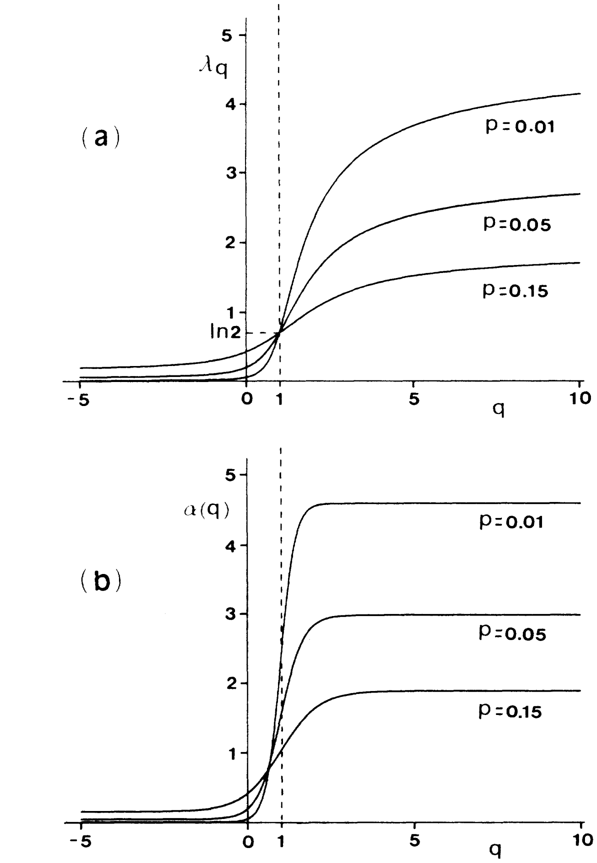

FIG. 2. Characteristic function λ_q and the quantity $\alpha(q)$ for three values of p. Critical behaviors of λ_q and $\alpha(q)$ as $p \to 0$ are related to the development of the intermittency characteristic observed in Fig. 1(c).

($\lambda_1 = \ln 2$). As is seen in Fig. 2, as $p \to 0$ the characteristic q regions can be divided into two regions, $q<1$ and $q \gtrsim 1$. The remarkable features for $p \to 0$ are observed in the enhancement of $\lambda_\infty = \alpha(\infty) = \ln p^{-1}$, and the diminishment of $\lambda_{-\infty} = \alpha(-\infty) = \ln(1-p)^{-1}$. The former describes bursts with the large amplitude ($u_n = \ln p^{-1} \to \infty$), and the latter explains laminar states (long-lived quiescent regions) with $u_n \simeq p \to 0$.

The above consideration implies that even a one-variable time sequence usually contains many kinds of statistical characteristics which cannot be described only with two quantities λ_0 and D. Each characteristic is supposed to be singled out by the scalar parameter q.

Let us expand the characteristic function in the cumulant expansion as $\lambda_q = \sum_{m=1}^{\infty} a_m q^{m-1}$. The relation (2.3) holds only for $|q| \ll \kappa$, κ being the convergence radius.[8,15] In the above simple model, $\kappa \sim 1/\ln p^{-1}$. For $|q| > \kappa$, the cumulant expansion does not converge. Instead, λ_q is usually expanded in the following form:[8]

$$\lambda_q \simeq \lambda_{\epsilon\infty} - \frac{1}{q}\left[\frac{1}{\tau_\epsilon} - c_\epsilon \exp(-\eta_\epsilon |q|)\right] \tag{2.11}$$

for $\epsilon q \gg \kappa$ ($\epsilon = \pm$), where τ_ϵ, c_ϵ, and η_ϵ are positive constants. The Legendre transformation gives

$$\alpha(q) \simeq \alpha(\epsilon\infty) - \epsilon c_\epsilon \eta_\epsilon \exp(-\eta_\epsilon |q|) , \tag{2.12}$$

$$\sigma(\alpha') \simeq \frac{1}{\tau_\epsilon} - \frac{1}{\eta_\epsilon}|\alpha' - \alpha(\epsilon\infty)| \ln\left[\frac{a_\epsilon}{|\alpha' - \alpha(\epsilon\infty)|}\right] , \tag{2.13}$$

where $\alpha(\epsilon\infty) = \lambda_{\epsilon\infty}$, $\eta_\epsilon \sim O(1/\kappa)$, and $a_\epsilon \equiv e c_\epsilon \eta_\epsilon$.

The existence of the convergence radius κ means that the statistical characteristics described with λ_q can be, roughly speaking, divided into three types, $q \ll -\kappa$, $|q| \ll \kappa$, and $q \gg \kappa$, which can be never perturbatively connected to each other.[8] In the sense that the parameter q selectively singles out the statistical characteristics relevant to it, it was called the *filtering parameter*. This concept was first introduced in Ref. 14 in connection with the thermodynamics formalism in fluctuation dynamics.

The probability density $\rho_n(\alpha')$ for $|\alpha' - \alpha(0)| \lesssim O(\sqrt{2D/n})$ is $O(\sqrt{n})$. For α' far from $\alpha(0)$, $\rho_n(\alpha') \sim \sqrt{n}\, e^{-\sigma(\alpha')n}$, where $\sigma(\alpha') = O(1)$. Therefore, almost all of the ensemble members in S_n with a large n give α' values in the Gaussian region. These correspond to $|q| \ll \kappa$. Other important characteristics, especially corresponding to $|q| \gg \kappa$, are contained in the minority of the ensemble members. They are in the tail regions of $\rho_n(\alpha')$.[16] In Sec. III we will discuss how we obtain precise information on overall statistical characteristics of α_n from the ensemble S_n.

III. ENSEMBLE PROCESSING AND THE PROBABILITY DENSITY

The majority of the ensemble members in S_n take values $\alpha_n \simeq \langle u_n \rangle = \alpha(0)$. In order to amplify the relevant fluctuation characteristic in the minority, we consider the new ensemble $S_n(q)$ by suitably processing the ensemble S_n. This is done by increasing or decreasing the ensemble members with $\alpha_n = \alpha'$ according to

$$\nu_n(\alpha'; q) \propto \nu_n(\alpha') e^{qn\alpha'}, \quad -\infty < q < \infty . \tag{3.1}$$

The total number of the ensemble members in $S_n(q)$ is given by

$$N(q) = \int_{-\infty}^{\infty} \nu_n(\alpha'; q)\, d\alpha' . \tag{3.2}$$

For $q>0$ (<0), the number of ensemble members with large (small) α' values is increased. The ensemble $S_n(q=0)$ is identical to the original ensemble S_n, i.e., the processing with $q=0$ means that there is no ensemble processing. In the ensemble $S_n(q)$ there are $N(q)$ values of α_n. Renumbering these values as $\overline{m} = 1, 2, 3, \ldots, N(q)$, the corresponding α_n value is written as $\overline{\alpha}_n(\overline{m})$, where $\overline{\alpha}_n(\overline{m})$ is equal to, at least, one of $\alpha_n(m)$, $m = 1, 2, 3, \ldots, N$.

Hence the probability density $\rho_n(\alpha'; q)$ that α_n takes a value between α' and $\alpha' + d\alpha'$ in the ensemble $S_n(q)$ is given by

$$\rho_n(\alpha';q)=\lim_{N\to\infty}\frac{\nu_n(\alpha';q)}{N(q)}$$
$$=\lim_{N\to\infty}\frac{1}{N(q)}\sum_{\bar{m}=1}^{N(q)}\delta(\bar{\alpha}_n(\bar{m})-\alpha')$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}\delta(\alpha_n(m)-\alpha')g_n(\alpha_n(m);q)$$
$$=\rho_n(\alpha')g_n(\alpha';q)\ , \qquad (3.3)$$

with the processing factor

$$g_n(\alpha';q)=\frac{e^{qn\alpha'}}{M_q(n)} \qquad (3.4)$$

$[\langle g_n(\alpha';q)\rangle=1, g_n(\alpha';0)=1]$, where $M_q(n)$ is given in (1.9). The average of $F(\alpha_n(m))$ over the ensemble $S_n(q)$ is calculated as

$$\langle F(\alpha_n);q\rangle\equiv\lim_{n\to\infty}\frac{1}{N(q)}\sum_{\bar{m}=1}^{N(q)}F(\bar{\alpha}_n(\bar{m}))$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}F(\alpha_n(m))g_n(\alpha_n(m);q)$$
$$=\int_{-\infty}^{\infty}F(\alpha')\rho_n(\alpha';q)d\alpha' \qquad (3.5)$$

(Ref. 17). Especially, one obtains

$$\langle F(\alpha_n)\rangle=\langle F(\alpha_n);0\rangle\ , \qquad (3.6)$$

which is the conventional average over $\rho_n(\alpha')$. For example, the first moment is calculated as

$$\langle\alpha_n;q\rangle=\frac{1}{n}\partial_q\ln M_q(n)\ . \qquad (3.7)$$

The quantity

$$\chi_{q,n}\equiv\partial_q\langle\alpha_n;q\rangle \qquad (3.8)$$

has the meaning of the susceptibility because it is equal to the rate of the change of the average $\langle\alpha_n;q\rangle$ when q is infinitesimally changed.[18] Namely, $\chi_{q,n}$ indicates how $\langle\alpha_n;q\rangle$ is sensitive when the degree of the ensemble processing is changed.

By noting that

$$\lim_{n\to\infty}\langle\alpha_n;q\rangle=\partial_q(q\lambda_q)=\alpha(q)\ , \qquad (3.9)$$

(3.8) is written as[19]

$$\chi_q=\partial_q\alpha(q)\quad(>0)\ , \qquad (3.10)$$

where $\chi_q\equiv\lim_{n\to\infty}\chi_{q,n}$. Since $\sigma'(\alpha(q))=q$, the above is written as

$$\chi_q=\chi(\alpha(q))\ , \qquad (3.11)$$

with the function $\chi(\alpha)$ defined by

$$\chi(\alpha)\equiv[\sigma''(\alpha)]^{-1}\quad(>0)\ . \qquad (3.12)$$

We note that

$$\langle\alpha_n^2;q\rangle=\frac{1}{n^2}\frac{1}{M_q(n)}\partial_q^2M_q(n)\ , \qquad (3.13)$$

where $\partial_q^2=\partial_q\partial_q$. The variance is defined by

$$v_{q,n}\equiv\langle(\alpha_n-\langle\alpha_n;q\rangle)^2;q\rangle\ . \qquad (3.14)$$

From (3.7) and (3.13), one finds

$$\chi_{q,n}=nv_{q,n} \qquad (3.15)$$

(Ref. 20). This shows the explicit interrelation between the susceptibility and the variance of fluctuations.[21]

Let us go into the asymptotic form of $\rho_n(\alpha';q)$ for a large n. First we note that it asymptotically takes the form

$$\rho_n(\alpha';q)\sim\sqrt{n}\ \exp[-\sigma(\alpha';q)n]\ , \qquad (3.16)$$

where

$$\sigma(\alpha';q)=\sigma(\alpha')-q\alpha'+q\lambda_q$$
$$=\sigma(\alpha')-\sigma(\alpha(q))-\sigma'(\alpha(q))[\alpha'-\alpha(q)]\ , \qquad (3.17)$$

(Ref. 22). The function $\sigma(\alpha';q)$ is concave with respect to α' $[\partial_{\alpha'}^2\sigma(\alpha';q)=\sigma''(\alpha')>0]$ and will be hereafter called the order-q fluctuation spectrum. By expanding (3.17) around $\alpha'=\alpha(q)$, it is written, to the lowest order with respect to $\alpha'-\alpha(q)$, as

$$\sigma(\alpha';q)\simeq\frac{\sigma''(\alpha(q))}{2}[\alpha'-\alpha(q)]^2\ , \qquad (3.18)$$

and therefore

$$\rho_n(\alpha';q)\sim\sqrt{n}\ \exp\left[-\frac{[\alpha'-\alpha(q)]^2}{2\chi_q}n\right] \qquad (3.19)$$

for $|\alpha'-\alpha(q)|\lesssim O(\sqrt{\chi_q/n}\,)$ (Ref. 23). The probability density $\rho_n(\alpha';q)$ has asymptotically a Gaussian peak located at the average $\alpha'=\alpha(q)$, and its width is determined by the susceptibility χ_q. If χ_q's are large for certain q's, the fluctuations of α_n's corresponding to these q's are large.

The conventional central limit theorem result corresponds to $q=0$ [Eq. (2.1)], since $\alpha(0)=\lambda_0$ and $\chi_0=2D$. The expression (3.19) is the generalization to the set of processed ensembles $\{S_n(q)\}$.

Finally, let us add the asymptotic forms of χ_q and $\chi(\alpha)$. Making use of the expansions (2.3) and (2.11), we get

$$\chi_q\simeq2D\ , \qquad (3.20)$$

$$\chi(\alpha)\simeq2D \qquad (3.21)$$

for $|q|\ll\kappa$ and $|\alpha-\alpha(0)|\ll|\alpha(\pm\kappa)-\alpha(0)|$, and

$$\chi_q\simeq c_\epsilon\eta_\epsilon^2\exp(-\eta_\epsilon|q|)\ , \qquad (3.22)$$

$$\chi(\alpha)\simeq\eta_\epsilon|\alpha-\alpha(\epsilon\infty)| \qquad (3.23)$$

for $\epsilon q\gg\kappa$ and $|\alpha-\alpha(0)|\gg|\alpha(\pm\kappa)-\alpha(0)|$.

For illustrations, see examples in Appendices A and B.

IV. CHARACTERIZATION OF SPATIAL FLUCTUATIONS

The ensemble-processing concept introduced in Sec. III can be straightforwardly extended to the statistically homogeneous spatial fluctuations.[9] Let us consider a large d-dimensional system with the volume Ω. In each point indicated by the position vector $\mathbf{r}$, a scalar fluctuation variable $u(\mathbf{r})$ is defined.[24] The fluctuations $\{u(\mathbf{r})\}$ are homogeneous in the sense that its statistical property does not depend on $\mathbf{r}$.

The local average of $u(\mathbf{r})$ taken over a d-dimensional sphere with the volume V whose center is located at $\mathbf{r}_0$ is given by

$$\alpha_V(\mathbf{r}_0) \equiv \frac{1}{V}\int_{|\mathbf{r}'|<l} u(\mathbf{r}'+\mathbf{r}_0)d\mathbf{r}' , \tag{4.1}$$

where l is the radius of the sphere, $V=\pi^{d/2}l^d/\Gamma(1+d/2)$, $\Gamma(z)$ being the γ function. Let us divide the whole system into the set of subsystems each of which has the volume V. These subsystems constitute the ensemble S_V, the number of the ensemble members being N. The number density of subsystems for which α_V takes a value between α' and $\alpha'+d\alpha'$ is given as

$$\nu_V(\alpha') = \sum_{m=1}^{N} \delta(\alpha_V(\mathbf{r}_{0m})-\alpha') , \tag{4.2}$$

where $\mathbf{r}_{0m}$ is the center of the mth subsystem (mth ensemble member of S_V). The probability density $\rho_V(\alpha')$ for which α_V takes a value α' is thus calculated as

$$\rho_V(\alpha') = \lim_{\Omega\to\infty} \frac{\nu_V(\alpha')}{N} . \tag{4.3}$$

The order-q moment is introduced by

$$M_q(V) \equiv \langle \exp(qV\alpha_V)\rangle \equiv \int_{-\infty}^{\infty} \rho_V(\alpha')e^{qV\alpha'}d\alpha' . \tag{4.4}$$

The characteristic function λ_q and the fluctuation spectrum $\sigma(\alpha')$ are defined as

$$\lambda_q = \frac{1}{q}\lim_{V\to\infty}\frac{1}{V}\ln M_q(V) , \tag{4.5}$$

$$\rho_V(\alpha') \sim \sqrt{V}e^{-\sigma(\alpha')V} , \tag{4.6}$$

where the limit $\lim_{V\to\infty}$ should be read as $\lim_{V\to\infty}\lim_{\Omega\to\infty}$. The $\sigma(\alpha')$ is derived from λ_q with the Legendre transformation as in Sec. I.

Now we introduce the ensemble processing as in Sec. III. $\rho_V(\alpha')$ is the probability density in the original ensemble S_V. We construct a new ensemble $S_V(q)$ so that $\rho_V(\alpha';q)$, the probability density that α_V takes a value α' in $S_V(q)$, is given by

$$\rho_V(\alpha';q) \equiv \rho_V(\alpha')g_V(\alpha';q) , \tag{4.7}$$

with the processing factor $g_V(\alpha';q)=e^{qV\alpha'}/M_q(V)$. The average of $F(\alpha_V)$ with the ensemble $S_V(q)$ is calculated by

$$\langle F(\alpha_V);q\rangle = \int_{-\infty}^{\infty} F(\alpha')\rho_V(\alpha';q)d\alpha' . \tag{4.8}$$

The first moment is written as

$$\langle \alpha_V;q\rangle = \frac{1}{V}\partial_q \ln M_q(V) . \tag{4.9}$$

The susceptibility

$$\chi_{q,V} \equiv \partial_q \langle \alpha_V;q\rangle \tag{4.10}$$

is determined by the fluctuation of α_V in $S_V(q)$ as

$$\chi_{q,V} \equiv V\langle(\alpha_V-\langle\alpha_V;q\rangle)^2;q\rangle . \tag{4.11}$$

Since $\lim_{V\to\infty}\langle\alpha_V;q\rangle=\alpha(q)$ $[=d(q\lambda_q)/dq]$ is finite, $\chi_q=\lim_{V\to\infty}\chi_{q,V}$ is also a definite function of q. We obtain

$$\chi_q \equiv \partial_q\alpha(q) = 1/\sigma''(\alpha(q)) . \tag{4.12}$$

The probability density $\rho_V(\alpha';q)$ asymptotically takes the form

$$\rho_V(\alpha';q) \sim \sqrt{V}e^{-\sigma(\alpha';q)V} , \tag{4.13}$$

where $\sigma(\alpha';q)$ is given in (3.17). The above density has a single peak at $\alpha'=\alpha(q)$, near which (4.13) is written as

$$\rho_V(\alpha';q) \sim \sqrt{V}\exp\left[-\frac{[\alpha'-\alpha(q)]^2}{2\chi_q}V\right] . \tag{4.14}$$

This is again the generalization of the conventional central limit theorem result for a processed ensemble.

V. SUMMARY AND CONCLUDING REMARKS

In the present paper we have reconsidered the fluctuation spectrum theory of temporal and spatial fluctuations from the ensemble-processing viewpoint. First, the order-q ensemble $S_n(q)$ $[S_V(q)]$ was constructed by appropriately changing the ensemble members of the original ensemble [Eqs. (3.1) and (4.7)]. The quantity $\alpha(q)$ $[=d(q\lambda_q)/dq]$ turned out to have the meaning of the average of α_n (α_V) over the processed ensemble $S_n(q)$ $[S_V(q)]$. Thus the filtering parameter q has also the meaning of the parameter measuring the *degree of processing*.

The ensemble processing immediately leads to the introduction of the susceptibility concept. Namely, the quantity χ_q $[=d\alpha(q)/dq]$ evaluates the change of the average $\alpha(q)$ when q is infinitesimally changed. Then we find that the probability density for $S_n(q)$ $[S_V(q)]$ asymptotically takes the Gaussian form whose variance is determined by the susceptibility χ_q. Depending on q, it has quite different q dependences [Eqs. (3.20) and (3.22)].

Next we will discuss the scaling relations near critical points. Near a certain chaotic transition point, the characteristic function λ_q often obeys a scaling law[25–27]

$$\lambda_q = \kappa^\mu L(q/\kappa) , \tag{5.1}$$

where μ is a constant, the characteristic value κ evaluates the convergence radius, and $L(x)$ is a scaling function. Therefore the relevant thermodynamics variables also satisfy the scaling relations

$$\alpha(q)=\kappa^{\mu}A(q/\kappa)\ , \quad (5.2)$$

$$\sigma(\alpha)=\kappa^{\mu+1}S(\alpha/\kappa^{\mu})\ , \quad (5.3)$$

$$\chi_q=\kappa^{\mu-1}A'(q/\kappa),\ \ \chi(\alpha)=\kappa^{\mu-1}[S''(\alpha/\kappa^{\mu})]^{-1}\ , \quad (5.4)$$

$$\sigma(\alpha;q)=\kappa^{\mu+1}B(\alpha/\kappa^{\mu},q/\kappa)\ , \quad (5.5)$$

where scaling functions are

$$A(x)=\frac{d}{dx}[xL(x)]\ , \quad (5.6)$$

$$S(y)=[x^2L'(x)]_{x=A^{-1}(y)}\ , \quad (5.7)$$

$$B(z,x)=S(z)-xz+xL(x) \quad (5.8)$$

$[S'(A(x))=x,\ L(x)=A(x)-S(A(x))/x],\ x=A^{-1}(y)$ being the inverse function of $A(x)=y$. Expanding $B(z,x)$ around $z=A(x)$, we get

$$B(z,x)\simeq\frac{S''(A(x))}{2}[z-A(x)]^2 \quad (5.9)$$

to the lowest order in $z-A(x)$. A simple example of the scaling relations is given in Appendix B 2.

Finally, let us give a comment on a temporal correlation.[28,29] If $M_q(n)$ is written as

$$M_q(n)=Q_q(n)\exp(q\lambda_q n)\ , \quad (5.10)$$

where $\lim_{n\to\infty}n^{-1}\ln Q_q(n)=0$, then the explicit temporal correlation is contained in $Q_q(n)$. In Ref. 28 we reported models having temporal correlation, where $Q_q(n)$'s are given by the superposition of the exponential decaying terms. Let us take a time series whose $Q_q(n)$ is given as[28]

$$Q_q^{(n)}=J_q^{(0)}+J_q^{(1)}\exp(-\gamma_q n)\ , \quad (5.11)$$

where $J_q^{(0)}$ and $J_q^{(1)}$ are constants and $\gamma_q>0$. This is the simplest case having a temporal correlation. The above correlation function can be described with one damping rate. If one tries to discuss the temporal correlation by observing the temporal evolution of

$$\langle\alpha_n;q\rangle-\alpha(q)=\frac{1}{n}\partial_q\ln Q_q(n) \quad (5.12)$$

[Eqs. (3.7) and (5.10)], then the right-hand side contains an infinitely many exponentially decaying terms even for the simplest case (5.11). Only one of them is the fundamental term, i.e., $\exp(-\gamma_q n)$. This indicates that it is not convenient to deal with $\langle\alpha_n;q\rangle$ for the study of the explicit temporal correlation in $\{u_n\}$. Instead, as has been emphasized previously,[28,29] it is convenient to analyze $Q_q(n)$ itself or its Fourier transform

$$\Xi_q(\omega)=\sum_{n=0}^{\infty}Q_q(n)\cos(\omega n)\ . \quad (5.13)$$

In fact, the poles of $\Xi_q(\omega)$ immediately describe the fundamental characteristic frequencies and the damping rates of characteristic motions in $\{u_n\}$.

APPENDIX A

In this appendix we consider a two-value pure stochastic process

$$u_n=\begin{cases}a\\ b\end{cases} \quad (A1)$$

$(a>b)$, where the generation probability of the value $a\,(b)$ is $p_a\ (p_b)$, $(p_a+p_b=1)$.

The characteristic function is easily obtained as

$$\lambda_q=\frac{1}{q}\ln(p_a e^{qa}+p_b e^{qb})\ ,$$
$$=\frac{a+b}{2}+\frac{1}{q}\ln\left[2\sqrt{p_a p_b}\cosh\left[\frac{a-b}{2}(q-q_*)\right]\right]\ , \quad (A2)$$

where

$$q_*\equiv\frac{1}{a-b}\ln\left[\frac{p_b}{p_a}\right]\ . \quad (A3)$$

Its Legendre transform yields

$$\alpha(q)=\frac{a+b}{2}+\frac{a-b}{2}\tanh\left[\frac{a-b}{2}(q-q_*)\right]\ , \quad (A4)$$

$$\sigma(\alpha)=\frac{a-\alpha}{a-b}\ln\left[\frac{a-\alpha}{a-b}\frac{1}{p_b}\right]+\frac{\alpha-b}{a-b}\ln\left[\frac{\alpha-b}{a-b}\frac{1}{p_a}\right]\ . \quad (A5)$$

Therefore, $\sigma(\alpha;q)$ can be obtained with (A2) and (A5).

The susceptibility is obtained as

$$\chi_q=\chi_{q_*}\mathrm{sech}^2[\sqrt{\chi_{q_*}}(q-q_*)]\ , \quad (A6)$$

$$\chi(\alpha)=(a-\alpha)(\alpha-b)$$
$$=\chi_{q_*}\left[1-\left[\frac{\alpha-\alpha(q_*)}{\sqrt{\chi_{q_*}}}\right]^2\right],\quad b\le\alpha\le a \quad (A7)$$

where

$$\alpha(q_*)=\frac{a+b}{2}\ , \quad (A8)$$

$$\chi_{q_*}=\left[\frac{a-b}{2}\right]^2 \quad (A9)$$

$[\chi_{q+q_*}=\chi_{-q+q_*},\ \chi(\alpha+\alpha(q_*))=\chi(-\alpha+\alpha(q_*))]$. The χ_q and $\chi(\alpha)$ have peaks at $q=q_*$ and $\alpha=\alpha(q_*)$, respectively, and their heights are equal to χ_{q_*}. The width Δ of χ_q is evaluated as

$$\Delta=2\sqrt{\chi_{q_*}}\ . \quad (A10)$$

It should be noted that the susceptibility χ_q in an arbitrary two-value stochastic process is determined by two parameters q_* and χ_{q_*} [Eq. (A6)].

Next we turn to the concrete examples.

1. Coin-tossing model

The first example is the coin-tossing model, where

$$a=1,\quad b=-1,\quad p_a=p_b=\tfrac{1}{2}\ . \quad (A11)$$

This leads to

$$q_* = 0, \quad \alpha(0) = 0, \quad \chi_0 = 1 , \tag{A12}$$

and

$$\chi_q = \text{sech}^2(q) , \tag{A13}$$

$$\chi(\alpha) = 1 - \alpha^2 . \tag{A14}$$

The width of χ_q is $\Delta = 2$.

2. Fluctuation dynamics of the local expansion rates

The second concrete example is the fluctuation dynamics of the local expansion rates, i.e., the time series is generated by $u_n = \ln|f'(x_n)|$, where x_n obeys the Bernoulli map (2.2). In this case we have

$$a = \ln p^{-1}, \quad b = \ln(1-p)^{-1}, \quad p_a = p, \quad p_b = 1-p . \tag{A15}$$

The λ_q and $\alpha(q)$ for three values of p are drawn in Fig. 2. Figure 3 shows the order-q fluctuation spectra for several values of q. The characteristic quantities are given by

$$q_* = 1, \quad \alpha(1) = \ln \frac{1}{\sqrt{p(1-p)}} ,$$

$$\chi_1 = \left[\ln \left[\frac{1-p}{p} \right]^{1/2} \right]^2 . \tag{A16}$$

Therefore as $p \to 0$, $\alpha(1)$ and χ_1 diverge as

$$\alpha(1) \simeq \tfrac{1}{2} \ln p^{-1} \to \infty , \tag{A17}$$

$$\chi_1 \simeq \frac{1}{4} (\ln p^{-1})^2 \to \infty . \tag{A18}$$

The width of χ_q is evaluated as

$$\Delta \simeq \frac{4}{\ln p^{-1}} \to 0 \tag{A19}$$

as $p \to 0$. The susceptibility χ_q thus becomes steep at $q = 1$ as $p \to 0$. Simultaneously, $\alpha(0)$ and χ_0 decrease as

$$\alpha(0) = \lambda_0 \simeq p \ln p^{-1} \to 0 , \tag{A20}$$

$$\chi_0 = 2D \simeq p (\ln p^{-1})^2 \to 0 . \tag{A21}$$

Figure 4 shows χ_q and $\chi(\alpha)$ in the scaled form [Eqs. (A6) and (A7)].

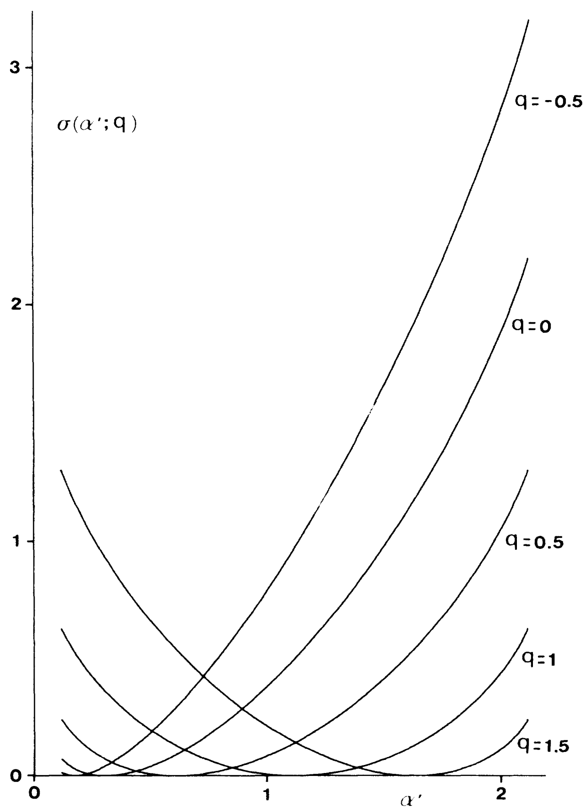

FIG. 3. Order-q fluctuation spectra $\sigma(\alpha';q)$'s for the processed ensemble $S_n(q)$ for the time series of local expansion rates of the Bernoulli map with $p = 0.1$. Each $\sigma(\alpha';q)$ has one minimal value $\sigma = 0$ at $\alpha' = \alpha(q)$.

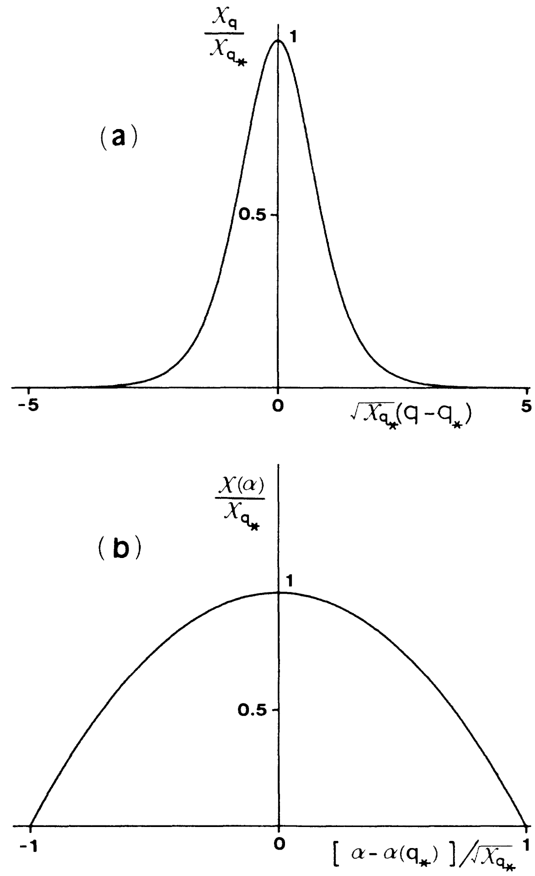

FIG. 4. Susceptibilities χ_q and $\chi(\alpha)$ for two-value pure random processes. They are universal functions [Eqs. (A6) and (A7)].

APPENDIX B

In this appendix we further give two examples. In contrast to those in Appendix A, they have temporal correlations.

1. Symbolic dynamics

The first example is symbolic dynamics,[8]

$$u_n=\begin{cases}0, & 0<x_n<\beta^{-1}\\ 1, & \beta^{-1}\le x_n<1\end{cases} \tag{B1}$$

generated by the β transformation

$$x_{n+1}=\begin{cases}\beta x_n, & 0<x_n<\beta^{-1}\\ \beta x_n-1, & \beta^{-1}\le x_n<1\end{cases} \tag{B2}$$

with $\beta=(\sqrt{5}+1)/2$ ($\beta^2-\beta-1=0$) which allows the unstable period-2 cycle. The largest eigenvalue of the evolution matrix

$$H_q=\begin{bmatrix}\beta^{-1} & e^q\\ \beta^{-2} & 0\end{bmatrix} \tag{B3}$$

gives the characteristic function

$$\lambda_q=\frac{1}{q}\ln\left[\frac{(1+4e^q)^{1/2}+1}{\sqrt{5}+1}\right] . \tag{B4}$$

Its Legendre transformation is easily calculated as

$$\alpha(q)=\frac{1}{2}\left[1-\frac{1}{(1+4e^q)^{1/2}}\right] , \tag{B5}$$

$$\sigma(\alpha)=\alpha\ln\left[\frac{\alpha}{1-\alpha}\right]+(1-2\alpha)\ln\left[\frac{1-2\alpha}{1-\alpha}\right]+\ln\left[\frac{\sqrt{5}+1}{2}\right] . \tag{B6}$$

$\sigma(\alpha;q)$ is determined with (B4) and (B6). We obtain

$$\chi_q=\frac{e^q}{(1+4e^q)^{3/2}} , \tag{B7}$$

$$\chi(\alpha)=\alpha(1-2\alpha)(1-\alpha),\quad 0\le\alpha\le\tfrac{1}{2} . \tag{B8}$$

The susceptibility χ_q has a single peak at $q=-\ln 2$.

2. Stochastic model

The next illustration is the stochastic model for the symmetry change as in the band-splitting phenomena,[30] where u_n can take two values $+1$ or -1. Let κ be the transition probability from the $+$ ($-$) state to the $-$ ($+$) state in a unit step. The $\{u_n\}$ has temporal correlation except for the special case $\kappa=\frac{1}{2}$ where this model reduces to the coin-tossing model. If we apply this model to the symmetry dynamics in the vicinity of the band-splitting phenomena, κ is taken to be small.[31] This leads to the intermittency characteristics in the sense that once the $+$ ($-$) state appears, the state continues in a long duration because the transition probability is small. Hereafter we put $\kappa<\frac{1}{2}$. The stochastic dynamics is determined by the evolution matrix

$$H=\begin{bmatrix}1-\kappa & \kappa\\ \kappa & 1-\kappa\end{bmatrix} . \tag{B9}$$

With the largest eigenvalue of the extended evolution matrix[31]

$$H_q=\begin{bmatrix}(1-\kappa)e^q & \kappa e^{-q}\\ \kappa e^q & (1-\kappa)e^{-q}\end{bmatrix} \tag{B10}$$

($H=H_0$), the characteristic function is obtained as[31]

$$\lambda_q=\frac{1}{q}\ln(\cosh(q)\{1-\kappa+[\kappa^2+(1-2\kappa)\tanh^2(q)]^{1/2}\}) . \tag{B11}$$

The Legendre transform yields

$$\alpha(q)=\frac{(1-\kappa)\tanh(q)}{[\kappa^2+(1-2\kappa)\tanh^2(q)]^{1/2}} , \tag{B12}$$

$$\sigma(\alpha)=\frac{\alpha}{2}\ln\left[\frac{h(\alpha,\kappa)+\kappa\alpha}{h(\alpha,\kappa)-\kappa\alpha}\right]+\ln\left[\frac{(1-\alpha^2)^{1/2}}{h(\alpha,\kappa)+\kappa}\right] , \tag{B13}$$

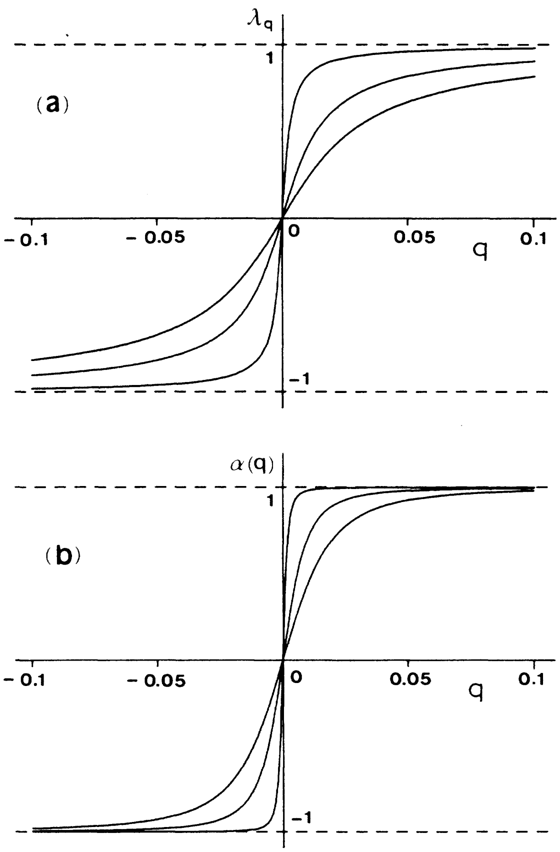

FIG. 5. Characteristic function λ_q and the quantity $\alpha(q)$ for the stochastic model in Appendix B 2. The values of κ are 0.02, 0.01, and 0.002. As κ becomes small, the slopes of λ_q and $\alpha(q)$ at $q=0$ become monotonously steep.

where

$$h(\alpha,\kappa)=[\kappa^2+(1-2\kappa)(1-\alpha^2)]^{1/2} . \tag{B14}$$

The dynamics symmetry gives $\lambda_{-q}=-\lambda_q$, $\alpha(-q)=-\alpha(q)$, and $\sigma(-\alpha)=\sigma(\alpha)$. Figure 5 illustrates how the shapes of λ_q and $\alpha(q)$ change as κ is decreased.

$\sigma(\alpha;q)$ is given in terms of (B11) and (B13) $[\sigma(-\alpha;-q)=\sigma(\alpha;q)]$. In Fig. 6, $\sigma(\alpha;q)$'s are drawn for several values of q. The susceptibilities are obtained as

$$\chi_q=\chi_0\mathrm{sech}^2(q)\left[1+\frac{1-2\kappa}{\kappa^2}\tanh^2(q)\right]^{-3/2} , \tag{B15}$$

$$\chi(\alpha)=\chi_0(1-\alpha^2)\left[1-\frac{1-2\kappa}{(1-\kappa)^2}\alpha^2\right]^{1/2} , \tag{B16}$$

with

$$\chi_0=\frac{1-\kappa}{\kappa} \tag{B17}$$

$[\chi_{-q}=\chi_q,\ \chi(-\alpha)=\chi(\alpha)]$. It should be noted that as $\kappa\to 0$, χ_q and $\chi(\alpha)$ tend to diverge as κ^{-1} near the peaks $q=0$ and $\alpha=0$, respectively. Figure 7 is the scaling plots of χ_q and $\chi(\alpha)$ for three values of κ (see below).

Let us discuss the asymptotics for $\kappa\to 0$. This refers to the approach to the band-splitting point if we apply the present model to the band-splitting phenomena.[26,31] We take the limit $\kappa\to 0$ by keeping q/κ and α finite in order to single out the characteristics in those regions. This immediately gives the scaling laws[31,32]

$$\lambda_q=L(q/\kappa),\quad L(x)=\frac{(1+x^2)^{1/2}-1}{x} , \tag{B18}$$

$$\alpha(q)=\frac{q}{\kappa}\left[1+\left[\frac{q}{\kappa}\right]^2\right]^{-1/2} , \tag{B19}$$

$$\sigma(\alpha)=\kappa[1-(1-\alpha^2)^{1/2}] . \tag{B20}$$

The κ evaluates the convergence radius of λ_q around $q=0$. Therefore, one obtains, for $\kappa\to 0$,

$$\sigma(\alpha;q)=\kappa B\left[\alpha,\frac{q}{\kappa}\right] , \tag{B21}$$

$$\chi_q=\kappa^{-1}\left[1+\left[\frac{q}{\kappa}\right]^2\right]^{-3/2} , \tag{B22}$$

$$\chi(\alpha)=\kappa^{-1}(1-\alpha^2)^{3/2} , \tag{B23}$$

where

$$B(y,x)=(1+x^2)^{1/2}-(1-y^2)^{1/2}-xy . \tag{B24}$$

The scaling function $B(y,x)$ is concave with respect to y and has a minimal value $B=0$ at $y=x(1+x^2)^{-1/2}$.

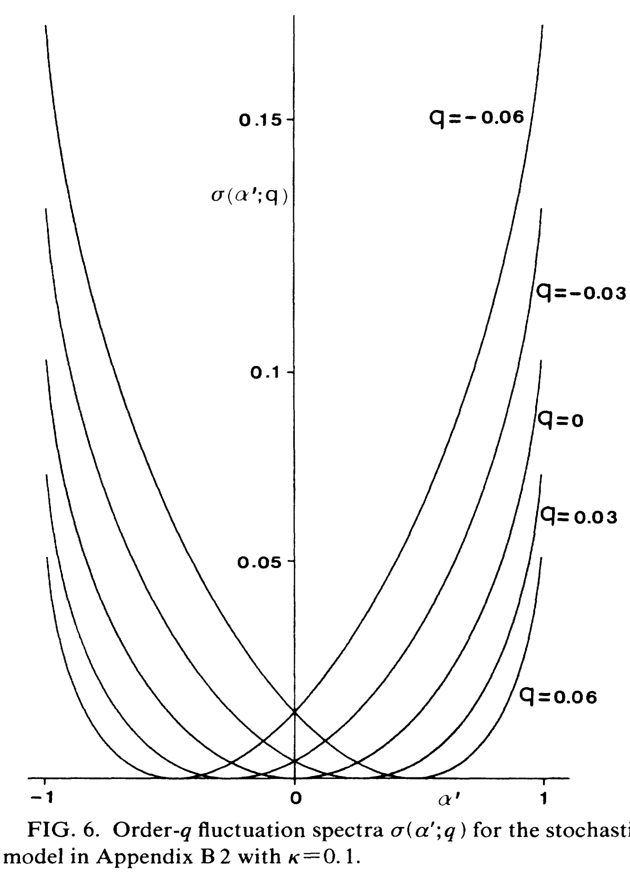

FIG. 6. Order-q fluctuation spectra $\sigma(\alpha';q)$ for the stochastic model in Appendix B 2 with $\kappa=0.1$.

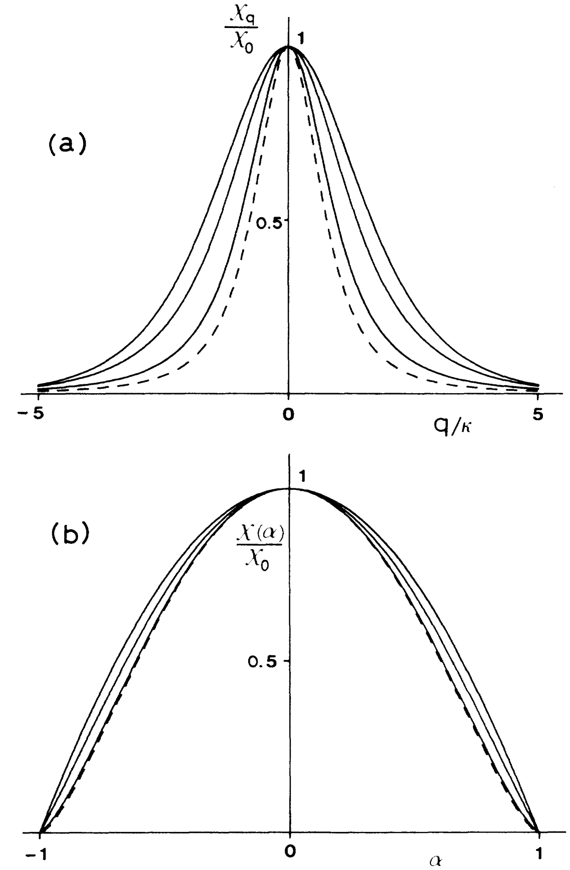

FIG. 7. Susceptibilities χ_q and $\chi(\alpha)$ for the stochastic model in Appendix B 2 in the scaling forms. The solid lines in (a) and (b) are the exact results (B15) and (B16) for $\kappa=0.48$, 0.4, and 0.2. As $\kappa\to 0$, the monotonously approach the scaling functions (B22) and (B23), denoted by the dashed lines in the figures.

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[15]If $\kappa=\infty$, the time series $\{u_n\}$ is Gaussian. On the contrary, for $\kappa=0$, $\{u_n\}$ is called the pure intermittency process (Ref. 13 and Inoue *et al.* in Ref. 25), and there is no linear region in λ_q near $q=0$. These are ideal processes. Usually κ is finite. This means that a time series usually has these two different aspects.

[16]In contrast to the fact that $\rho_n(\alpha')$ near the peak position is determined at most with the double time correlation function, the tail regions cannot be determined with time correlation functions of any finite order.

[17]If F is chosen as $F(\alpha_n)=\delta(\alpha_n-\alpha')$, Eq. (3.5) reduces to $\rho_n(\alpha';q)=\langle\delta(\alpha_n-\alpha');q\rangle$.

[18]Let $M_q(n)$ be expanded in terms of cumulants as $\ln M_q(n)=\langle \exp(qn\alpha_n)-1\rangle_c$. Since the cumulants are written by the set of time correlation functions of $\{u_n\}$, the combination of (3.8) with (3.7) shows that the susceptibility $\chi_{q,n}$ is determined by those time correlation functions. Especially, as $q\rightarrow 0$, Eqs. (3.7) and (3.8) yield $\chi_{0,n}=n^{-1}\sum_{j=0}^{n-1}\sum_{l=0}^{n-1}C(|j-l|)$. Therefore, we have the well-known result $\chi_{0,\infty}=\sum_{n=-\infty}^{\infty}C(n)$ [Eq. (2.2)].

[19]In Ref. 8 the quantity corresponding to the heat capacity is introduced by $C=d\alpha(q)/dq^{-1}$ (<0) by noting that q and $\alpha(q)$ play the roles of the internal energy and the inverse temperature, respectively, in the statistical-thermodynamics formalism. Since we consider the response to the infinitesimal change of q, it is convenient to use the derivative $\chi_q=d\alpha(q)/dq$ (>0). Clearly one gets $C=-q^2\chi_q$.

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[21]If we consider the susceptibility $\chi_{q,n}^F$, i.e., $\chi_{q,n}^F\equiv\partial_q\langle F(\alpha_n);q\rangle$, it is determined by the correlation function as $\chi_{q,n}^F=n\langle(\alpha_n-\langle a_n;q\rangle)[F(\alpha_n)-\langle F(\alpha_n);q\rangle];q\rangle$.

[22]If $\{u_n\}$ is Gaussian, then $\{\alpha_n\}$ is also Gaussian. We immediately get $\lambda_q=\lambda_0+Dq$ for all of q. This leads to $\alpha(q)=\alpha(0)+2Dq$, $\sigma(\alpha')=\sigma(\alpha';0)=(\alpha'-\alpha(0))^2/4D$, and $\sigma(\alpha';q)=(\alpha'-\alpha(q))^2/4D$, i.e., $\chi_q=\chi(\alpha)=2D$. In the Gaussian process the susceptibility χ_q does not depend on the parameter q.

[23]More generally speaking, by taking into account the direct temporal correlation, this is written as

$$\rho_n(\alpha';q)\simeq(2\pi n^{-1}\partial_q\langle\alpha_n;q\rangle)^{-1/2}\times\exp\left[-\frac{(\alpha'-\langle\alpha_n;q\rangle)^2}{2\partial_q\langle\alpha_n;q\rangle}n\right]$$

with the Gaussian approximation.

[24]In a previous paper (Ref. 9), we have formulated the global characterization of spatial fluctuations in a continuum medium. The present approach can be straightforwardly extended to a discrete system (a lattice system).

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[32]Near the band-splitting point in the double-well potential system under the external periodic excitation (Ref. 26), the same scaling laws for λ_q and $Q_q(n)$ hold. See A. Yamaguchi, H. Fujisaka, and M. Inoue (unpublished).