Solvable model of the Fokker-Planck equation without detailed balance

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A new model of the Fokker-Planck equation (FPE) without detailed balance is suggested. The potential of the FPE is solved explicitly in the weak-noise limit. A discontinuity of the first derivative of the potential is revealed. The approach of solving the model may be extended to solving general FPE's lacking detailed balance, provided the drift of the FPE can be written in proper form.

I. INTRODUCTION

Fluctuated nonequilibrium systems are often described by Fokker-Planck equations (FPE).[1-5] Recently, much effort has been made to define a nonequilibrium potential based on the asymptotic behavior of the stationary probability distribution of the FPE, which may lack detailed balance.[6-11] Given a FPE,

$$\frac{\partial P(\mathbf{x},t)}{\partial t}=\sum_{i=1}^{n}\frac{-\partial}{\partial x_i}[c_i(\mathbf{x})P(\mathbf{x},t)]+\epsilon\sum_{i=1}^{n}\sum_{j=1}^{n}\left[D_{ij}(\mathbf{x})\frac{\partial^2}{\partial x_i\partial x_j}P(\mathbf{x},t)\right], \tag{1.1}$$

the asymptotic logarithm of the stationary probability distribution

$$\phi_0(\mathbf{x})=\lim_{t\to 0}\{-\epsilon\ln[P(\mathbf{x})]\} \tag{1.2}$$

is regarded as the potential of the FPE system. For clarity and simplicity, in the present paper we consider only the two-dimensional FPE

$$\frac{\partial P(x,y,t)}{\partial t}=-\frac{\partial}{\partial x}[c_1(x,y)P(x,y,t)]-\frac{\partial}{\partial y}[c_2(x,y)P(x,y,t)]+\epsilon\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]P(x,y,t) \tag{1.3}$$

in which the diffusion matrix is assumed to be an identity. The main results in the presentation may be extended to more general cases. Equation (1.3) does not obey detailed balance if $\partial c_2/\partial x\neq\partial c_1/\partial y$.[7,12] Then there is no simple way to specify the potential explicitly.

II. POTENTIAL ABOUT A STABLE POINT

Let us expand the stationary probability distribution of the FPE as

$$P(x,y)=N\exp[-\phi_0(x,y)/\epsilon-\phi_1(x,y)-\epsilon\phi_2(x,y)-\cdots]. \tag{2.1}$$

Inserting (2.1) into (1.3), we may expand the stationary FPE in ϵ. In this paper we only consider FPE's with weak noise $\epsilon\ll 1$, which may model a large number of physical, chemical systems and other practical situations. Thus, in the weak-noise limit, we may retain only the leading order in ϵ in Eq. (1.3) and obtain the following Hamilton-Jacobi equation of the potential:

$$c_1\frac{\partial\phi_0}{\partial x}+c_2\frac{\partial\phi_0}{\partial y}+\left[\frac{\partial\phi_0}{\partial x}\right]^2+\left[\frac{\partial\phi_0}{\partial y}\right]^2=0. \tag{2.2}$$

Suppose the origin is a simple stable point of the dissipative dynamics for vanishing noise

$$\frac{dx}{dt}=c_1(x,y),\quad \frac{dy}{dt}=c_2(x,y), \tag{2.3a}$$

namely, the drift vanishes at the origin

$$c_1(0,0)=c_2(0,0)=0. \tag{2.3b}$$

The real parts of the two eigenvalues of the linear drift matrix

$$\begin{bmatrix}\hat{c}_{1x} & \hat{c}_{1y}\\ \hat{c}_{2x} & \hat{c}_{2y}\end{bmatrix} \tag{2.4}$$

(denoted by λ_1 and λ_2) must be negative

$$\mathrm{Re}(\lambda_1,\lambda_2)<0,$$

where we have

$$\hat{c}_i=c_i(0,0)$$

and

$$\hat{c}_{ix}=\left.\frac{\partial c_i}{\partial x}\right|_{x=y=0},\quad \hat{c}_{iy}=\left.\frac{\partial c_i}{\partial y}\right|_{x=y=0},\ i=1,2.$$

The origin must be a basin of the potential $\phi_0(x,y)$ [remember that $\phi_0(x,y)$ is a Lyapunov function of Eq. (2.3a)[9,12]]. The behavior of the FPE in the vicinity of this point is extremely important since in the stationary state a major portion of probability may center there.

In Ref. 12 we applied the power-series expansion approach to calculate the potential $\phi_0(x,y)$. It was shown[12] that ϕ_0 can be well expanded about a simple stable point of the corresponding deterministic system. Let us expand c_1, c_2, and ϕ_0 in the power series as

$$\begin{aligned} c_1 &= \hat{c}_{1x}x+\hat{c}_{1y}y+\cdots , \\ c_2 &= \hat{c}_{2x}x+\hat{c}_{2y}y+\cdots , \\ \phi_0 &= \hat{\phi}_{0xx}x^2/2+\hat{\phi}_{0yy}y^2/2+\hat{\phi}_{0xy}xy+\cdots , \end{aligned} \tag{2.5}$$

with

$$\hat{\phi}_{0xx}=\left.\frac{\partial^2\phi_0}{\partial x^2}\right|_{x=y=0} ,$$

and so on. [In Eq. (2.1), the constants in $\phi_0,\phi_1,\ldots$, are all substituted into N.] By inserting (2.5) into Eq. (2.2) and considering only the lowest terms in the power series, one may easily solve $\hat{\phi}_{0xx}$, $\hat{\phi}_{0yy}$, and $\hat{\phi}_{0xy}$ in terms of the drift as (for the calculation, see Ref. 12)

$$\begin{aligned} \hat{\phi}_{0xx} &= (\hat{c}_{1x}+\hat{c}_{2y})[\hat{c}_{2x}(\hat{c}_{1y}-\hat{c}_{2x})-\hat{c}_{1x}(\hat{c}_{1x}+\hat{c}_{2y})] \\ &\quad \times[(\hat{c}_{1x}+\hat{c}_{2y})^2+(\hat{c}_{1y}-\hat{c}_{2x})^2]^{-1} , \\ \hat{\phi}_{0yy} &= -(\hat{c}_{1x}+\hat{c}_{2y}+\hat{\phi}_{0xx}) , \\ \hat{\phi}_{0xy} &= -\hat{c}_{2x}+\hat{\phi}_{0xx}(\hat{c}_{1y}-\hat{c}_{2x})/(\hat{c}_{1x}+\hat{c}_{2y}) , \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \phi'_{0xx}(i) &= -\hat{c}_{2x}\lambda_i(\hat{c}_{2y}-\lambda_i)/d(i) , \\ \phi'_{0yy}(i) &= -\hat{c}_{1y}\lambda_i(\hat{c}_{1x}-\lambda_i)/d(i) , \\ \phi'_{0xy}(i) &= \hat{c}_{1y}\hat{c}_{2x}\lambda_i/d(i), \quad i=1,2 \end{aligned} \tag{2.7}$$

with

$$d(i)=\hat{c}_{1y}(\hat{c}_{1x}-\lambda_i)+\hat{c}_{2x}(c_{2y}-\lambda_i) .$$

It is apparent that with solutions (2.6) we have

$$\hat{\phi}_{0xx}+\hat{\phi}_{0yy}=-(\hat{c}_{1x}+\hat{c}_{2y})=-(\lambda_1+\lambda_2) ,$$

while with the other two sets of solutions [Eqs. (2.7)] we have

$$\hat{\phi}'_{0xx}(i)\hat{\phi}'_{0yy}(i)=\hat{\phi}'^2_{0xy}(i) ,$$

and

$$\hat{\phi}'_{0xx}(i)+\hat{\phi}'_{0yy}(i)=-\lambda_i, \quad i=1,2 .$$

Due to the fact that a stable point must be a probability peak of the stationary distribution, we choose (2.6) as the unique solution for the potential and neglect (2.7) in Ref. 12, since with (2.7) the stationary distribution has no peak on the stable points which is physically unreasonable. Based on (2.6), the terms of higher orders in the power series can be worked out systematically.

III. GLOBAL POTENTIAL OF A NONTRIVIAL EXAMPLE

The approach of power-series expansion is a straightforward method for searching the potential about a stable point of order 1 (cf. Ref. 12). However, in general, it is difficult to apply the expansion approach to study the global behavior of the potential. In particular, it fails if the actual potential is nonanalytic.

Let us study an interesting solvable model

$$\begin{aligned} c_1(x,y) &= x(a^2-x^2-y^2)+y(b^2-x^2+y^2) , \\ c_2(x,y) &= y(a^2-x^2-y^2)-x(b^2-x^2+y^2) . \end{aligned} \tag{3.1}$$

It is obvious that the function

$$\psi(x,y)=a^2(x^2+y^2)/2-(x^2+y^2)^2/4 \tag{3.2}$$

satisfies Eq. (2.2). However, it can serve as the potential $\phi_0(x,y)$ only in case of $a^2<b^2$ (as will be confirmed later). In the case of

$$a^2>b^2 , \tag{3.3}$$

Eq. (3.2) no longer represents the potential. Then, for vanishing noise, the dissipative system possesses two simple stable fixed points

$$x_{S_{1,2}}=\pm\sqrt{(a^2+b^2)/2}, \quad y_{S_{1,2}}=\pm\sqrt{(a^2-b^2)/2} \tag{3.4}$$

and two saddles

$$x_{D_{1,2}}=\pm\sqrt{(a^2+b^2)/2}, \quad y_{D_{1,2}}=\mp\sqrt{(a^2-b^2)/2} . \tag{3.5}$$

The solution (3.2) gives no probability peaks about the stable points (3.4), and is physically unacceptable just as the solutions (2.7). [A direct calculation may confirm that (3.2) is identical to (2.7) about (3.4).]

To find the correct form of the potential in the case of (3.3), we rewrite Eq. (1.3) in polar coordinates

$$\begin{aligned} \frac{\partial P(r,\theta,t)}{\partial t} &= -\frac{\partial}{\partial r}[r(a^2-r^2)P(r,\theta,t)]+\epsilon\left[\frac{\partial^2}{\partial r^2}\right]P(r,\theta,t) \\ &\quad +\frac{\partial}{\partial\theta}\{[b^2-r^2\cos(2\theta)]P(r,\theta,t)\}+\epsilon\left[\frac{\partial^2}{\partial\theta^2}\right]P(r,\theta,t)/r^2+(\epsilon/r)\left[\frac{\partial}{\partial r}\right]P(r,\theta,t) . \end{aligned} \tag{3.6}$$

In the weak-noise limit, the last term, which can never be important, may be neglected. For the stationary solution, Eq. (3.6) can be reduced to

$$0=-\left[\frac{\partial}{\partial r}\right][r(a^2-r^2)P(r,\theta)]+\epsilon\left[\frac{\partial^2}{\partial r^2}\right]P(r,\theta)+\left[\frac{\partial}{\partial\theta}\right]\{[b^2-r^2\cos(2\theta)]P(r,\theta)\}+(\epsilon/r^2)\left[\frac{\partial^2}{\partial\theta^2}\right]P(r,\theta)\ . \quad (3.7)$$

Considering only the leading terms in the weak-noise limit, the stationary probability distribution is assumed to be factorized as

$$P(r,\theta)=N\exp[-f_1(r)/\epsilon-f_2(\theta)/\epsilon]\ . \quad (3.8)$$

Hence, Eq. (3.7) can be transformed to two independent equations:

$$\left[r(a^2-r^2)+\frac{df_1(r)}{dr}\right]\frac{df_1(r)}{dr}-\epsilon\left[\frac{d}{dr}\right]\left[r(a^2-r^2)+\frac{df_1(r)}{dr}\right]=0\ , \quad (3.9a)$$

$$\left[[b^2-a^2\cos(2\theta)]+\frac{1}{a^2}\frac{df_2(\theta)}{d\theta}\right]\frac{df_2(\theta)}{d\theta}-\epsilon\left[\frac{d}{d\theta}\right]\left[[b^2-a^2\cos(2\theta)]+\frac{df_2(\theta)}{d\theta}\right]=0\ . \quad (3.9b)$$

Solution (3.2) is nothing but the solution of the equations

$$r(a^2-r^2)+\frac{df_1(r)}{dr}=0\ ,\qquad \frac{df_2(\theta)}{d\theta}=0\ , \quad (3.10)$$

which, indeed, satisfies Eq. (3.9) if one only considers the leading terms in ϵ. On the basis of the previous arguments, this solution should be neglected. Nevertheless, Eqs. (3.9) can be reduced to another pair of equations, namely,

$$r(a^2-r^2)+\frac{df_1(r)}{dr}=0\ , \quad (3.11a)$$

$$b^2-a^2\cos(2\theta)+\frac{df_2(\theta)}{d\theta}=0\ , \quad (3.11b)$$

leading to an alternative solution

$$f_1=\int r(a^2-r^2)dr,\quad f_2=\int[b^2-a^2\cos(2\theta)]d\theta\ . \quad (3.12)$$

It is appreciated that (3.12) has probability peaks, indeed, in the deterministic stable points. Unfortunately, it does not preserve the periodicity condition in θ, namely,

$$P(r,2\pi+\theta)\neq P(r,\theta)\ .$$

Thus, it still remains physically unacceptable. To overcome such a difficulty, we apply the Gaussian distribution approximation about $r=|a|$, modify (3.8) to

$$P(r,\theta)=NP_1(r,\theta)P_2(\theta)=NP_2(\theta)\exp\{-(r-|a|)^2/[2\epsilon\sigma(\theta)]\}\ , \quad (3.13)$$

and directly start from Eq. (3.7). We arrive at another two independent equations

$$2a^2\sigma(\theta)+\{[b^2-a^2\cos(2\theta)]/2\}\frac{d\sigma(\theta)}{d\theta}+1=0\ , \quad (3.14a)$$

$$\frac{-\partial}{\partial\theta}\{[b^2-a^2\cos(2\theta)]P_2(\theta)\}+\frac{\epsilon}{a^2}\frac{\partial^2}{\partial\theta^2}P_2(\theta)=0\ . \quad (3.14b)$$

The unique solution of (3.14a), which satisfies the periodicity condition $\sigma(2\pi+\theta)=\sigma(\theta)$, reads

$$\sigma(\theta)=1/(2a^2)\ . \quad (3.15a)$$

$P_1(r,\theta)$ is the very Gaussian approximation of (3.2). Equation (3.14b) can be solved as

$$P_2(\theta)=P_2(\theta)_i,\quad \pi+\theta_{D_i}>\theta>\theta_{D_i},\quad i=1,2$$

$$P_2(\theta)_i=\exp[-a^2v(\theta)_i/\epsilon]\left[1+\int_{\theta_{D_i}}^{\theta}J(a^2/\epsilon)\exp[a^2v(\theta)_i/\epsilon]d\theta\right]\ , \quad (3.15b)$$

with

$$v(\theta)_i=\int_{\theta_{D_i}}^{\theta}[b^2-a^2\cos(2\theta)]d\theta\ ,$$

where θ_{D_i} are the angles of the saddle points D_1 and D_2. The current J is fixed by requiring $P_2(2\pi+\theta)=P_2(\theta)$, namely,

$$\exp(-\pi b^2a^2/\epsilon)\times\left[1+J(a^2/\epsilon)\int_{\theta_{D_i}}^{\pi+\theta_{D_i}}d\theta\exp[a^2v(\theta)_i/\epsilon]\right]=1\ , \quad (3.16)$$

leading to

$$J=[\exp(\pi b^2a^2/\epsilon)-1]/\int_{\theta_{D_i}}^{\pi+\theta_{D_i}}d\theta\exp[(a^2/\epsilon)v(\theta)_i]\ .$$

To end this section the following points should be made.

(i) The solution (3.13) is correct only in the vicinity of $r=|a|$. However, the global properties of the stationary probability distribution can be represented by it quite well since in the weak-noise limit the total probability far from $r=|a|$ is negligibly small. By global property we mean the relative heights of the probability peaks.

(ii) In the limit $\epsilon \to 0$, (3.13) can be further reduced by the saddle-point approximation and the Gaussian approximation as

$$P(r,\theta)=N\exp[-h_1(r)/\epsilon-h_2(\theta)/\epsilon] , \tag{3.17}$$

with

$$h_1(r)=a^2(r-|a|)^2$$

and

$$h_2(\theta)=\begin{cases} a^2v(\theta)_i & \text{in region I} \\ 0 & \text{in region II} , \end{cases} \tag{3.18}$$

where region I is defined as

$$\theta_i'>\theta>\theta_{D_i}, \quad i=1,2 \tag{3.19}$$

with θ_i' being given by

$$v(\theta_i')_i=0 . \tag{3.20}$$

Region II is the remainder.

(iii) Now Eq. (3.2) is replaced by the potential (3.17) plotted in Fig. 1. The first derivative of the potential is discontinuous at θ_i' and θ_{D_i}. Raising b^2, the flat part in Fig. 1 enlarges. As $b^2>a^2$, the fixed points disappear via saddle-node bifurcation, and a limit cycle arises. Consequently, Eq. (3.17) becomes identical to Eq. (3.2). Thus, Eq. (3.2) is the potential indeed in the case of $a^2<b^2$ as we declared before.

(iv) Both Eqs. (3.2) and (3.12) satisfy (2.2) in the leading order in ϵ. However, as $a^2>b^2$ both solutions are physically unreasonable. With (3.2), the distribution preserves the periodicity in θ while it has no probability peaks in the deterministic attractors. With (3.12), the situation is just the opposite. It is remarkable that with (3.17) we find (3.2) in region I while (3.12) holds in region II, and then (3.17) satisfies (2.2) as well as all the physical requirements. As solution (3.17) jumps from region I to region II, discontinuity of the first derivative of the potential arises.

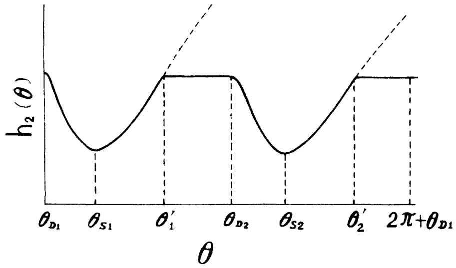

FIG. 1. Potential of Eq. (3.6) on the cycle $r=|a|$. θ_{D_1}, θ_{D_2} and θ_{S_1}, θ_{S_2} are the angles of the saddle and stable points of (3.1), respectively. θ_i' are given by Eq. (3.20).

IV. POTENTIAL OF GENERAL FPE

Up to now, one knows very few nontrivial solvable examples of FPE without detailed balance. Equation (3.1) provides an interesting new model which does not manifest detailed balance while it is solvable, indeed, in the weak-noise limit. In this section we will find that the model is very important because the main ideas extracted may be extended to some rather general systems.

According to Eq. (2.2), we may divide the drift as

$$\begin{aligned} c_1(x,y)&=G_1(x,y)+d_1(x,y) , \\ c_2(x,y)&=G_2(x,y)+d_2(x,y) , \end{aligned} \tag{4.1}$$

where $\mathbf{G}=(G_1,G_2)$ is the gradient of a function $\psi(x,y)$

$$\frac{\partial G_1(x,y)}{\partial y}=\frac{\partial G_2(x,y)}{\partial x}=\frac{\partial^2\psi(x,y)}{\partial x\,\partial y} \tag{4.2}$$

and $\mathbf{d}=(d_1,d_2)$ is the circulation orthonormal to the gradient

$$G_1(x,y)d_1(x,y)+G_2(x,y)d_2(x,y)=0 . \tag{4.3}$$

Based on (4.2) and (4.3), we can rewrite Eq. (4.1) in the form

$$\begin{aligned} c_1(x,y)&=Q(x,y)a_1(x,y)+k(x,y)a_2(x,y) , \\ c_2(x,y)&=Q(x,y)a_2(x,y)-k(x,y)a_1(x,y) , \end{aligned} \tag{4.4}$$

with

$$\frac{\partial(Qa_1)}{\partial y}=\frac{\partial(Qa_2)}{\partial x}=\frac{\partial^2\psi(x,y)}{\partial x\,\partial y} , \tag{4.5}$$

leading to

$$Q_ya_1=Q_xa_2$$

on the curve $Q=0$. The zero-dimensional set points, if they exist,

$$a_1(x,y)=a_2(x,y)=0 \tag{4.6}$$

represent the minima, maxima, or saddle points of the potential, according to whether the points are the stable, unstable, or saddle points of the corresponding dissipative system for vanishing noise, respectively. On the one-dimensional set

$$k(x,y)=0 \tag{4.7}$$

(indicated by Γ_2, if it exists) the circulation alters its direction. If the set

$$Q(x,y)=0 \tag{4.8}$$

(denoted by Γ_1, whenever it occurs) is not empty and is a one-dimensional closed line, and, moreover, it does not intersect with the set Γ_2 [the intersection of Γ_1 with the zero-dimensional set (4.6) is exceptional and can be excluded from our consideration]. Γ_1 must be the limit cycle of the deterministic system

$$\frac{dx}{dt}=c_1(x,y), \quad \frac{dy}{dt}=c_2(x,y) .$$

According to (2.2), Γ_1 must be a one-dimensional extreme of the potential. In this paper we only consider stable limit cycles which are the minima of the potential.

Whenever Γ_1 and Γ_2 intersect each other, the closed line Γ_1 is no longer a limit cycle, and $\psi(x,y)$, given by Eq. (4.2), is not the potential of the FPE

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}[(Qa_1+ka_2)P]-\frac{\partial}{\partial y}[(Qa_2-ka_1)P]+\epsilon\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]P \quad (4.9)$$

though it is, indeed, an exact solution of Eq. (2.2). The situation is exactly the same as that described in Sec. III for $a^2>b^2$.

To study this matter in detail, let us define new curve coordinates as

$$s=s(x,y)\ ,\qquad n=n(x,y)\ , \quad (4.10)$$

with s and n coordinates being orthonormal to each other at each intersection

$$s_x(x,y)n_x(x,y)+s_y(x,y)n_y(x,y)=0\ . \quad (4.11)$$

The set

$$n(x,y)=0$$

must be identical to the set

$$Q(x,y)=0\ .$$

Moreover, we require that the transformation from (x,y) to (n,s) is unitary on the cycle Γ_1. (This condition is not necessary. We impose the unitarity condition on the transformation on Γ_1 only for the sake of simplicity of the result.)

$$\bar{n}_x^2+\bar{n}_y^2=1\ , \quad (4.12a)$$

$$\bar{s}_x=-\bar{n}_y,\quad \bar{s}_y=\bar{n}_x\ . \quad (4.12b)$$

By $\bar{n}$ and $\bar{n}_x,\ldots,$ we mean

$$\bar{n}=n|_{\text{on }\Gamma_1},\quad \bar{n}_x=\left.\frac{\partial n}{\partial x}\right|_{\text{on }\Gamma_1},$$

and so on.

In the new coordinates $\partial/\partial x,\partial/\partial y$ and $\partial^2/\partial x^2,\partial^2/\partial y^2$ can be transformed to

$$\frac{\partial}{\partial x}=n_x\frac{\partial}{\partial n}+s_x\frac{\partial}{\partial s}\ ,\qquad \frac{\partial}{\partial y}=n_y\frac{\partial}{\partial n}+s_y\frac{\partial}{\partial s}\ , \quad (4.13)$$

and

$$\overline{\epsilon\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]}=\overline{\epsilon\left[\frac{\partial^2}{\partial n^2}+\frac{\partial^2}{\partial s^2}\right]}\ . \quad (4.14)$$

In (4.14) we neglect all the terms of first-order derivative like $n_{xx}(\partial/\partial n)$, $s_{xx}(\partial/\partial s)$, $n_{yy}(\partial/\partial n)$, $s_{yy}(\partial/\partial s)$, and so on in the weak-noise limit. Finally, in new coordinates the stationary FPE (4.9) acquires the form

$$0=-n_x\frac{\partial}{\partial n}[c_1p(n,s)]-n_y\frac{\partial}{\partial n}[c_2p(n,s)]-s_x\frac{\partial}{\partial s}[c_1p(n,s)]-s_y\frac{\partial}{\partial s}[c_2p(n,s)]+\epsilon\left[\frac{\partial^2}{\partial n^2}+\frac{\partial^2}{\partial s^2}\right]p(n,s)\ . \quad (4.15)$$

By applying the Gaussian distribution approximation in n we assume that the stationary probability distribution takes the form

$$P(n,s)=NP_1(n,s)P_2(s)=NP_2(s)\exp\{-n^2/[2\epsilon\sigma(s)]\}\ . \quad (4.16)$$

By inserting (4.16) into (4.15) and keeping only the leading terms in the weak-noise limit, we may reduce (4.15) to two independent equations:

$$1+\overline{(n_xc_1+n_yc_2)_n}\sigma(s)+(\bar{s}_x\bar{c}_1+\bar{s}_y\bar{c}_2)\frac{d\sigma(s)}{ds}=0\ , \quad (4.17)$$

$$\frac{\partial}{\partial s}[(\bar{s}_x\bar{c}_1+\bar{s}_y\bar{c}_2)P_2(s)]+\epsilon\frac{\partial^2}{\partial s^2}P_2(s)=0\ . \quad (4.18)$$

Both equations (4.17) and (4.18) can be solved explicitly. Equation (4.17) gives rise to

$$\sigma(s)=\exp\left[-\int_{S_{D_1}}^{s}f_2(s')ds'\right]\times\left[B+\int_{S_{D_1}}^{s}ds'f_1(s')\exp\left[\int_{S_{D_1}}^{s'}f_2(s'')ds''\right]\right]\ , \quad (4.19)$$

where

$$f_1(s)=-1/(\bar{s}_x\bar{c}_1+\bar{s}_y\bar{c}_2)$$

and

$$f_2(s)=\overline{(n_xc_1+n_yc_2)_n}/(\bar{s}_x\bar{c}_1+\bar{s}_y\bar{c}_2)\ .$$

The constant B can be fixed by requiring $\sigma(T+s)=\sigma(s)$ where T is the period of s. The solution of (4.18) can be worked out as

$$P_2(s)=\exp[-v(s)/\epsilon]\times\left[1+\int_{S_{D_1}}^{s}ds(J/\epsilon)\exp[v(s)/\epsilon]\right]\ , \quad (4.20)$$

where the constant J is fixed by the periodicity condition $P_2(s)=P_2(s+T)$. The function $v(s)$ is given by

$$v(s)=\int_{S_{D_1}}^{s}(\bar{s}_x\bar{c}_1+\bar{s}_y\bar{c}_2)ds\ , \quad (4.21)$$

where S_{D_i}, $i=1,2,\ldots,$ are the saddle points of (2.3a), and D_1 is chosen such that in all stable points on circle Γ_1, the potential takes minimal values.

Expressing the drift in the form of (4.4), we may directly write down (4.19) as

$$\sigma(s)=\bar{Q}_n(\bar{n}_x\bar{a}_1+\bar{n}_y\bar{a}_2)=\bar{Q}_n\sqrt{\bar{a}_1^2+\bar{a}_2^2}\ , \quad (4.22)$$

which is nothing but $\bar{\psi}_{nn}$ with $\psi(x,y)$ being given by (4.2). Thus, the exponent $n^2/[2\epsilon\sigma(s)]$ is the very $\psi(x,y)$ [see (4.2)] in the Gaussian distribution approximation. In deriving (4.22) we make use of (4.11), (4.12), and the following identities:

$$(Qa_1)_y=(Qa_2)_x \ ,$$

$$\bar{a}_1\bar{n}_y=\bar{a}_2\bar{n}_x \ .$$

Inserting (4.22) into (4.17) one may be convinced of its validity.

The structure of Eq. (4.16) is the same as Eq. (3.13) with polar coordinates (r,θ) replaced by curve coordinates (n,s). The figure of the potential (4.20) must be somewhat like Fig. 1 as long as Γ_1 and Γ_2 intersect each other. About stable points of deterministic dynamics one finds a basin of the potential while in the saddle points the first derivative of the potential must undergo discontinuity. By utilizing the saddle-point approximation approach, we reduce (4.20) to

$$P_2(s)=\exp[-h_2(s)/\epsilon] \ ,$$

$$h_2(s)=\begin{cases} v(s)_i, & s_i'>s>s_{D_i} \\ 0 & \text{otherwise} \ , \end{cases} \tag{4.23}$$

where

$$v(s)_i=v(s)-v(s_{D_i})$$

and s_i' is defined by

$$v(s_i')_i=0 \ .$$

The solution (4.23) is plotted schematically in Fig. 2. Equation (4.23) together with (4.22) give the potential about the cycle Γ_1. About Γ_1 the global behavior of the potential, i.e., the relative depths of the potential basins can be revealed by (4.23). As the intersection of Γ_1 and Γ_2 disappears, the second term in (4.18) is negligibly small in comparison with the first term and can be ruled out and the solution of (4.18) is, simply,

$$P_2(s)=\text{const}=1 \ .$$

Thus the potential of the FPE (4.9) is provided by (4.2).

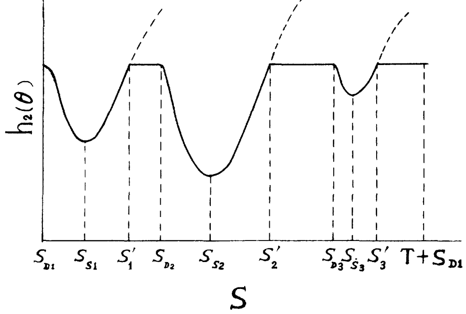

FIG. 2. Schematical figure of the potential (4.23). S_{D_i} and s_{S_i} are the s coordinates of the saddle and stable points of the deterministic equations $\dot{x}=c_1$, $\dot{y}=c_2$, on the cycle Γ_1. s_i' is given by $v(s_i')_i=0$ with $v(s)_i$ being given by (4.21) and (4.23).

It is just what should happen when a limit cycle arises.

Representing the drift in the new form (4.4), we have made an interesting discovery. There are two kinds of saddle points: The first comes from

$$a_1(x,y)=a_2(x,y)=0 \tag{4.24}$$

and the second from the intersections of

$$Q(x,y)=k(x,y)=0 \ . \tag{4.25}$$

These two kinds of saddle points cannot be distinguished from each other in the point of view of deterministic path. However, they are essentially different in the point of view of the potential. For the former, the potential has real local saddle points as concluded by Graham *et al.* For the latter the discontinuity of the first derivative of the potential arises. We expect that the dynamic behavior about the two kinds of saddle points might be considerably different since the potentials of the two kinds of points have substantially distinctive forms. We have briefly investigated this matter by simulating two forced dynamic systems

$$\begin{aligned} \frac{dx}{dt}&=(ax-x^3)+D(ay-y^3) \ , \\ \frac{dy}{dt}&=(ay-y^3)-D(ax-x^3)+\nu\cos(2\pi\omega t) \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} \frac{dx}{dt}&=x(a^2-x^2-y^2)+Dy(b^2-x^2+y^2) \ , \\ \frac{dy}{dt}&=y(a^2-x^2-y^2)-Dx(b^2-x^2+y^2)+\nu\cos(2\pi\omega t) \ . \end{aligned} \tag{4.27}$$

In the case of $a^2>b^2$ and $a>0$, the former unforced equations have saddle points of the first kind while the latter have saddle points of the second kind. The numerical results of Eqs. (4.26) and (4.27) are essentially different. By testing the forced equations (4.26), and varying the control parameters a, ν, D, and ω in a wide range, we find complicated bifurcations as well as chaotic motions. However, we have never found quasiperiodic motion and the Farey bifurcation sequence. On the contrary, with (4.27), we can readily find quasiperiodic motion as well as the Farey sequences. The road to chaos via quasiperiodicity can be found easily. These features are typical for periodically forced limit cycle systems. However, in the absence of the external force ($\nu=0$), the free system (4.27) has no limit cycle at all. The only similarity between the unforced (4.27) and limit cycle systems is that they have a somewhat similar potential. It manifests that the difference between the potentials of the two kinds of saddle points may be meaningful. Nevertheless, this point should be further investigated.

V. CONCLUSION

Provided the drift can be expressed in the form of (4.4), we are able to completely solve the stationary solution of the FPE without detailed balance in the weak-noise limit. The solution is (4.2) if the set of the intersection of $Q=0$ and $k=0$ is empty (the set $Q=0$ must be closed cycles or, in certain critical cases, points whenever the system is bounded) or (4.22) and (4.23) if it is not.

In general, it is difficult to specify the drift in the form of (4.4). Nevertheless, the condition can be considerably loosened. For instance, using (4.19) and (4.20), one can specify the potential in the vicinity of Γ_1 when Γ_1 is given while (4.4) is not. In particular, if a limit cycle exists and the cycle is provided by studying the deterministic equations, then a set of curve coordinates can be defined accordingly, and (4.19) leads to the steady solution of the FPE about the cycle. It is emphasized that a cycle $Q=0$ may exist in the absence of any limit cycle. In this case, the potential about Γ_1 may be solved [cf. (4.19) and (4.20)] without knowing (4.4). The cycle $Q=0$ must be composed of an even number of segments connecting saddles and the neighbor stable point attractors successively. This kind of cycles can be easily detected numerically or analytically, taking various approximations. However, to identify closed cycles to Γ_1, one should justify the relevant saddle points to be of the second class in advance. Thus it is an interesting open problem to distinguish the two kinds of saddle points of the deterministic equations without solving them. The rough numerical results on (4.26) and (4.27) may stimulate the research in this direction.

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