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Thomas precession and squeezed states of light

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The Lorentz group, which is the language of special relativity, is a useful theoretical tool in modern optics. Optics experiments can therefore serve as analog computers for special relativity. Possible optics experiments involving squeezed states are discussed in connection with the Thomas precession and the Wigner rotation.

I. INTRODUCTION

While the Lorentz group is the basic language of special relativity,[1] it is an important mathematical tool in many other branches of modern physics. The group Sp(2), which is locally isomorphic to the $(2+1)$-dimensional Lorentz group, is the basic language for linear canonical transformations in phase space.[2–5] It is a useful language in geometrical optics.[6] The importance of this group in modern optics has been emphasized in the current literature.[5,7–9] One of the advantages of using this group in modern optics is that direct experimental tests are possible in optics laboratories.[10,11] Indeed, modern optics can serve as an analog computer for special relativity, just as the *LCR* circuit is an analog of the damped harmonic oscillator with external driving force.[10]

The purpose of the present paper is to elaborate on the suggestions made in an earlier paper[5] on possible experimental tests of some of the relations in the Lorentz group having implications in special relativity. We propose a series of optics experiments concerning the Wigner rotation[1,12] and the Thomas precession.[2,13–15]

To many physicists, the Thomas precession is known as an isolated event of the $\frac{1}{2}$ factor in the spin-orbit coupling in atomic spectroscopy. The Thomas precession is caused by the extra rotation the particle in a circular orbit feels in its own rest frame. As was pointed out in the literature.[12,15] the Thomas effect is a special case of the Wigner rotation.

Among many different representations of the Lorentz group, the Wigner phase space[16] provides us with a simple method for studying the Wigner rotation.[4,5] At the same time, the Wigner phase space is the natural language for coherent and squeezed states.[5] Indeed, these optical states can be represented by circles and ellipses, which are canonical transforms of the circle around the origin in phase space.[5,7,17]

In Sec. II we exploit the local isomorphism between the $(2+1)$-dimensional Lorentz group and the group of homogeneous linear canonical transformations in the Wigner phase space. This allows us to give a 2×2 matrix formulation of coherent and squeezed states. In Sec. III we interpret the formalism of Sec. II in terms of the $(2+1)$-dimensional Lorentz group.

Section IV deals with the Lorentz kinematics of the Thomas rotation in terms of the 2×2 matrix formalism for homogeneous linear canonical transformations. In Sec. V we repeat the procedure of Sec. IV for the Lorentz kinematics of Ref. 10 inspired by the SU(1,1) interferometer of Yurke *et al.*[9] Finally, in Sec. VI we discuss the optical experiments based on the discussions of Secs. IV and V, which may be performed with the experimental techniques available today or in the near future.

II. 2×2 MATRIX FORMULATION OF COHERENT AND SQUEEZED STATES

It has been shown that coherent and squeezed states of light can be formulated in terms of linear canonical transformations of the Wigner phase-space distribution function.[5] This formalism is based on the fact that the group of linear canonical transformations is the group ISp(2), which is a semidirect product of Sp(2) and the group of translations in the two-dimensional phase space.

The purpose of this section is to simplify the mathematics given before and show that Sp(2), which is the group of homogeneous linear transformations in phase space, is enough to deal with the transformation properties of coherent and squeezed states. The basic advantage of the homogeneous Sp(2) is that it has a correspondence with the $(2+1)$-dimensional Lorentz group, while the more complicated group of ISp(2) has no apparent connection with the Lorentz group.

Traditionally, the coherent state or squeezed state is represented by an infinite series involving the solutions of the Schrödinger equation for the harmonic oscillator. The expression for the coherent state is

$$|\alpha\rangle = e^{-\alpha^*\alpha/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \ , \tag{1}$$

where $|n\rangle$ is the nth excited harmonic oscillator state. If we use the expression

$$|n\rangle = [1/(\sqrt{\pi}2^n n!)]^{1/2} H_n(q)\exp(-q^2/2) \ , \tag{2}$$

for the harmonic oscillator wave function, as a function of q, the expression for the coherent state becomes

$$\psi_\alpha(q) = \langle q|\alpha\rangle = \left[\frac{1}{\pi}\right]^{1/4} e^{-[\mathrm{Im}(\alpha)]^2} e^{-(q-\sqrt{2}\alpha)^2/2} \ . \tag{3}$$

The parameter α is a complex number which can best be represented in the two-dimensional complex plane.

If we use the Wigner phase space distribution function defined as

$$W(q,p) = \frac{1}{\pi}\int \psi_\alpha^*(q+y)\psi_\alpha(q-y)e^{2ipy}dy \ , \tag{4}$$

the coherent state can be written as a function of two scalar variables q and p,

$$W(q,p) = \frac{1}{\pi}\exp[-(q-a)^2-(p-b)^2] \ , \tag{5}$$

where $a=\sqrt{2}\,\mathrm{Re}(\alpha)$ and $b=\sqrt{2}\,\mathrm{Im}(\alpha)$. This function is concentrated within a circular region described by the equation

$$(q-a)^2+(p-b)^2=1 \ . \tag{6}$$

If $\alpha=0$, then both a and b vanish, and the preceding circle is centered around the origin. This is the vacuum state. If we multiply α by $e^{i\theta/2}$, the preceding circle becomes rotated around the origin, and the resulting equation is

$$(q-a')^2+(p-b')^2=1 \ , \tag{7}$$

where

$$a' = a\cos(\theta/2) - b\sin(\theta/2) \ ,$$

$$b' = a\sin(\theta/2) + b\cos(\theta/2) \ .$$

The squeezed state means that the preceding circle is linearly deformed in such a manner that the area is preserved. The squeeze along the q direction means that q becomes $(e^{\eta/2})q$, and p is replaced by $(e^{-\eta/2})p$. The result is that the circle in Eq. (6) becomes an ellipse specified by

$$e^{-\eta}(q-e^{\eta/2}a)^2+e^{\eta}(p-e^{-\eta/2}b)^2=1 \ . \tag{8}$$

Recently Han *et al.*[5] indicated that the group of linear canonical transformations in the Wigner phase is possibly the natural language for coherent and squeezed states. It was pointed out further that, since the group Sp(2) of homogeneous linear canonical transformations is locally isomorphic to the (2+1)-dimensional Lorentz group, modern optical laboratories can serve as an analog computer for Lorentz transformations.[5,10] Let us summarize the formalism given in Ref. 5.

The group of homogeneous linear canonical transformations is Sp(2) and can be represented by the 2×2 real matrices. Indeed, the generators of the group of homogeneous linear canonical transformations in the two-dimensional phase space are

$$K_1 = \begin{bmatrix} i/2 & 0 \\ 0 & -i/2 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & i/2 \\ i/2 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix} . \tag{9}$$

These operators satisfy the commutation relations

$$[K_1,K_2]=-iL, \quad [K_1,L]=-iK_2, \quad [K_2,L]=iK_1 \ . \tag{10}$$

K_1 and K_2 generate squeezes along the q direction and the direction which makes a 45° angle with the q axis, respectively. L is the generator of rotations around the origin. The preceding commutation relations are invariant under the sign change of K_i to $-K_i$. For this reason, the word squeeze can mean the deformation given in Eq. (8) or its inverse. In this paper, we choose the convention given in Eq. (9).

The rotation matrix $R(\phi)$ can be written as

$$R(\phi) = e^{-i\phi L} = \begin{bmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{bmatrix} . \tag{11}$$

Likewise, K_1 generates the squeeze along the x direction. Its matrix form is

$$S(0,\eta) = e^{-i\eta K_1} = \begin{bmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{bmatrix} . \tag{12}$$

The squeeze along the $\phi/2$ direction is

$$S(\phi,\lambda) = R(\phi)S(\lambda,0)R(-\phi) = \begin{bmatrix} \cosh(\lambda/2)+(\cos\phi)\sinh(\lambda/2) & (\sin\phi)\sinh(\lambda/2) \\ (\sin\phi)\sinh(\lambda/2) & \cosh(\lambda/2)-(\cos\phi)\sinh(\lambda/2) \end{bmatrix} . \tag{13}$$

Under the transformation of $S(\phi,\lambda)$, the circle of Eq. (6) becomes

$$e^{-\lambda}\left[(q-a')\cos\frac{\phi}{2}+(p-b')\sin\frac{\phi}{2}\right]^2 + e^{-\lambda}\left[(q-a')\sin\frac{\phi}{2}-(p-b')\cos\frac{\phi}{2}\right]^2 = 1 \ , \tag{14}$$

with

$$a'=\left[\cosh\frac{\lambda}{2}+(\cos\phi)\sinh\frac{\lambda}{2}\right]a+\left[(\sin\phi)\sinh\frac{\lambda}{2}\right]b\ ,$$

$$b'=\left[(\sin\phi)\sinh\frac{\lambda}{2}\right]a+\left[\cosh\frac{\lambda}{2}-(\cos\phi)\sinh\frac{\lambda}{2}\right]b\ .$$

Then the repeated squeezes result in

$$S(\phi,\lambda)S(0,\eta)=S(\theta,\xi)R(\Omega)\ , \tag{15}$$

where

$$\cosh\xi=(\cosh\eta)\cosh\lambda+(\sinh\eta)(\sinh\lambda)\cos\phi\ ,$$

$$\tan\frac{\Omega}{2}=\frac{(\sin\phi)[\tanh(\lambda/2)][\tanh(\eta/2)]}{1+[\tanh(\lambda/2)][\tanh(\eta/2)](\cos\phi)}\ ,$$

$$\tan\theta=\frac{(\sin\phi)[\sinh\lambda+(\tanh\eta)(\cosh\lambda-1)\cos\phi]}{(\sinh\lambda)\cos\phi+(\tanh\eta)[1+(\cosh\lambda-1)(\cosh\phi)^2]}\ .$$

Equation (15) implies that two successive squeezes do not make one squeeze but result in a squeeze preceded by a phase change in α. In this paper, we apply this relation to some specific circumstances under which optical experiments may be carried out.

III. (2+1)-DIMENSIONAL LORENTZ GROUP

The commutation relations given in Eq. (10) are identical to those for the (2+1)-dimensional Lorentz group in the space of (x,y,t). We shall hereafter call this group O(2,1). There are many groups which are locally isomorphic to this Lorentz group. Table I contains the generators of some of widely used groups in physics satisfying the same set of commutation relations as those for the group O(2,1).

This group is smaller and simpler than the (3+1)-dimensional Lorentz group which appears to be the full space-time symmetry group. However, as we shall see later in this section, O(2,1) contains all the essential features of Wigner's little groups. In addition, as explained in Sec. II, this group serves as the basic language for modern optics, in addition to the modern approach to classical mechanics. Indeed, the group O(2,1) is a physically rich group.

With this point in mind, let us interpret the mathematics of Sec. II in terms of the Lorentz transformations. The Lorentz boost along the x direction is represented by

$$S(0,\eta)=\begin{bmatrix}\cosh\eta & 0 & \sinh\eta\\ 0 & 1 & 0\\ \sinh\eta & 0 & \cosh\eta\end{bmatrix}. \tag{16}$$

This matrix is mathematically different from $S(0,\eta)$ of Eq. (12), and performs a different physical operation. However, we use the same notation $S(0,\eta)$ for both Eq. (12) and Eq. (16).

The rotation around the origin in the x-y plane is given by

$$R(\phi)=\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{17}$$

Here again, $R(\phi)$ is used for both Eq. (11) and Eq. (17).

The boost along the direction which makes an angle ϕ with the x axis is

$$S(\phi,\eta)=\begin{bmatrix}1+(\cosh\eta-1)\cos^2\phi & (\cosh\eta-1)(\sin\phi)\cos\phi & (\sinh\eta)\cos\phi\\ (\cosh\eta-1)(\sin\phi)\cos\phi & 1+(\cosh\eta-1)\sin^2\phi & (\sinh\eta)\sin\phi\\ (\sinh\eta)\cos\phi & (\sinh\eta)\sin\phi & \cosh\eta\end{bmatrix}. \tag{18}$$

TABLE I. Table of generators of the groups which are locally isomorphic to the (2+1)-dimensional Lorentz group. The first row consists of the generators of homogeneous linear canonical transformations in phase space. The second line consists of those in the Schrödinger representation discussed in Ref. 5. The third row gives the generators of Sp(2), which is the basic language for the present paper. The fourth row gives the generators SU(1,1) which was used in Ref. 9.

Lorentz group	J_3	K_1	K_2
Wigner representation	$\frac{i}{2}\left[p\frac{\partial}{\partial q}-q\frac{\partial}{\partial p}\right]$	$\frac{i}{2}\left[q\frac{\partial}{\partial q}-p\frac{\partial}{\partial p}\right]$	$\frac{i}{2}\left[q\frac{\partial}{\partial p}+p\frac{\partial}{\partial q}\right]$
Schrödinger representation	$\frac{1}{4}\left[x^2-\left[\frac{\partial}{\partial x}\right]^2\right]$	$-\frac{i}{4}\left[2x\frac{\partial}{\partial x}+1\right]$	$\frac{1}{4}\left[x^2+\left[\frac{\partial}{\partial x}\right]^2\right]$
Sp(2) SL(2,R)	$\frac{1}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$
SU(1,1)	$\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

The physics of Eq. (13) and the physics of Eq. (18) are different, but we can use $S(\phi,\eta)$ for both.

Since the generators of O(2,1) and Sp(2) satisfy the same set of commutation relations, the preceding three matrices satisfy the algebraic relation given in Eq. (13). If the Lorentz transformation $S(\phi,\lambda)S(0,\eta)=S(\theta,\xi)R(\Omega)$ is applied to a particle at rest, $R(\Omega)$ does not affect its momentum but changes the direction of its spin. This rotation of a particle at rest is called the Wigner rotation.

The group O(2,1) is a subgroup of O(3,1), and its symmetry is somewhat restricted compared with the full Lorentz symmetry of O(3,1). However, the group O(2,1) contains most of the essential features of Wigner's little groups of the Poincaré group.[1] The little group is the maximal subgroup of the Lorentz group which leaves the momentum and energy of a given particle invariant.

In order to grasp the physical picture of this group, let us note first that the rotation has to be included in the Lorentz group. Furthermore, it is possible to form a series of boosts and rotations to form a Lorentz transformation which brings the momentum-energy vector back to its original form. Is this transformation necessarily an identity transformation? The answer to this question is definitely *no*.

Let us consider, for example, a massive particle at rest. Its momentum is clearly invariant under rotations. The little group for a massive particle at rest is clearly the rotation group. What is then the little group for a moving massive particle? Since we can obtain this moving particle by boosting the particle at rest, the little group in this case is naturally a Lorentz-boosted rotation group.[18]

The question then is whether the study of transformations in the x-y plane can describe the scenario in the three-dimensional space. The answer is *yes*. Since the study of the little groups start with a fixed momentum, we can define the x axis as the direction of the momentum. As we did in this section, the y axis is perpendicular to the momentum. On the other hand, there are other directions perpendicular to the momentum, such as the z direction which is perpendicular to both x and y. This means that we have to take into the account rotation around the x axis. This is a rather simple matter because the boost along and the rotation around the x direction commute with each other.

IV. THOMAS CONFIGURATION

The effect of this Wigner rotation is seen in atomic spectra as the Thomas precession. Undoubtedly, it is also an important factor in nuclear and hadronic spectra where the relativistic effects are more prominent. The exact Lorentz kinematics has been discussed in the literature in terms of the conventional 4×4 matrix formalism.[12,15] On the other hand, in view of the correspondence between the mathematics of squeezed states and that of Lorentz transformations, we can study the Thomas rotation from optics experiments. For this purpose, we use the mathematics of squeezed states in order to study the Thomas rotation.

Let us consider a system of three Lorentz boosts, as is described in Fig. 1. We start with a massive particle at rest whose momentum-energy vector is

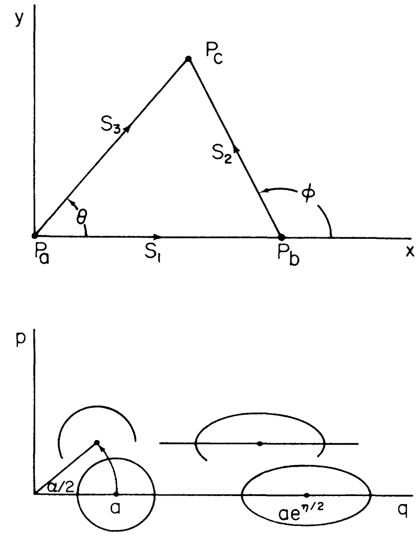

FIG. 1. Lorentz kinematics for the Thomas precession. S_1 bring a particle at rest to that with momentum-energy vector P_b. S_2 brings P_b to P_c. S_3 brings the particle at rest directly to P_c. If we start from P_b, the transformation $S_1(S_3)^{-1}S_2$ will leave this momentum-energy vector invariant, and will be the Lorentz-boost rotation: $S_1R(\alpha)(S_1)^{-1}$. In the Wigner phase space, which is the basic language of coherent and squeezed states, the ellipse of Eq. (37) is transformed into Eq. (40) with α replaced by $-\alpha$.

$$P_a=(0,0,m) \tag{19}$$

in the space-time vector convention: $x^{\mu}=(x,y,t)$, where we omit the z component which is not affected by the transformations discussed in this paper. S_1 boosts the preceding momentum-energy vector along the x axis with the boost parameter η,

$$P_b=S_1P_a=m(\sinh\eta,0,\cosh\eta)\ , \tag{20}$$

where S_1 is the 3×3 matrix $S(0,\eta)$ of Eq. (16). For algebraic convenience, we can also use the 2×2 matrix of Eq. (12). Since we are interested in both the squeezed states and Lorentz transformations, we shall use the 2×2 matrix representation throughout this section.

The second boost transforms P_b into P_c, whose momentum has the same magnitude as that of P_b but makes an angle θ with the direction of P_b,

$$P_c=S_2P_b=m((\sinh\eta)\cos\theta,(\sinh\eta)\sin\theta,\cosh\eta)\ , \tag{21}$$

where $S_2=S(\phi,\lambda)$, with

$$\begin{aligned}\phi&=(\pi+\theta)/2\ ,\\ \lambda&=2\tanh^{-1}\{[\sin(\theta/2)]\tanh\eta\}\ .\end{aligned} \tag{22}$$

The 2×2 matrix corresponding to the Lorentz transformation S_2 is

$$\begin{bmatrix} \cosh(\lambda/2)+[\sin(\theta/2)]\sinh(\lambda/2) & -[\cos(\theta/2)]\sinh(\lambda/2) \\ -[\cos(\theta/2)]\sinh(\lambda/2) & \cosh(\lambda/2)-[\sin(\theta/2)\sinh(\lambda/2)] \end{bmatrix} . \tag{23}$$

The momentum of P_c is the x-y plane. We can also obtain P_c by rotating P_b around the z axis by θ, as is illustrated in Fig. 1. The third boost S_3 brings P_a directly to P_c,

$$P_c = S_3 P_a , \tag{24}$$

where

$$S_3 = S(\theta,\eta) = \begin{bmatrix} \cosh(\eta/2)+(\cos\theta)\sinh(\eta/2) & (\sin\theta)\sinh(\eta/2) \\ (\sin\theta)\sinh(\eta/2) & \cosh(\eta/2)-(\cos\theta)\sinh(\eta/2) \end{bmatrix} . \tag{25}$$

Even if the successive transformations $S_2 S_1$ bring the particle at rest to the same momentum state as S_3 does, they are not identical transformations. Indeed,

$$S(\phi,\lambda)S(0,\eta) = S(\theta,\eta)R(\alpha) , \tag{26}$$

where

$$R(\alpha) = \begin{bmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix} ,$$

with

$$\tan(\alpha/2) = \frac{[\tan(\theta/2)](\cosh\eta - 1)}{\cosh\eta + [\tan(\theta/2)]^2} . \tag{27}$$

The Thomas effect comes from the successive transformations $S(\theta,\eta)[S(0,\eta)]^{-1}$ applied to P_b. Since Eq. (26) can be written as

$$S(\theta,\eta) = S(\phi,\lambda)S(0,\eta)R(-\alpha) ,$$

$S(\theta,\eta)[S(0,\eta)]^{-1}$ takes the form

$$S(\theta,\eta)S(0,-\eta) = S(\phi,\lambda)T(\theta,\eta) , \tag{28}$$

where $T(\theta,\eta)$ is the Thomas factor, where[15]

$$T(\theta,\eta) = S(0,\eta)R(-\alpha)S(0,-\eta) . \tag{29}$$

This is a Lorentz-boosted rotation, and leaves the momentum P_b invariant. However, it is not an identity matrix. This causes a rotation of the spin in the Lorentz frame in which the particle is at rest. This is exactly the Thomas rotation, which is a special case of the Wigner rotation.

V. CONFIGURATION OF YURKE, McCALL, AND KLAUDER

As another illustrative example of the Wigner rotation, Han and Kim[10] studied the Lorentz kinematics inspired by the work of Yurke *et al.*[9] They considered a series of boosts and rotations whose net effect is to leave the momentum and energy of a given massive particle invariant, and this kinematics is described in Fig. 2. We shall hereafter call this Lorentz kinematics the Yurke configuration.

In order to be consistent with the purpose of this paper, we discuss here the Yurke configuration using the mathematics of squeezed states. Let us start with a massive particle whose momentum-energy vector is

$$P = m(\sinh\eta, 0, \cosh\eta) . \tag{30}$$

We can get this momentum by boosting the particle at rest by applying the boost operator of Eq. (16), whose 2×2 counterpart is given in Eq. (12). We then rotate the system by $\theta/2$. In the present 2×2 representation, the rotation matrix is

$$R(\theta/2) = \begin{bmatrix} \cos(\theta/4) & -\sin(\theta/4) \\ \sin(\theta/4) & \cos(\theta/4) \end{bmatrix} . \tag{31}$$

Next, we boost the system along the negative y direction

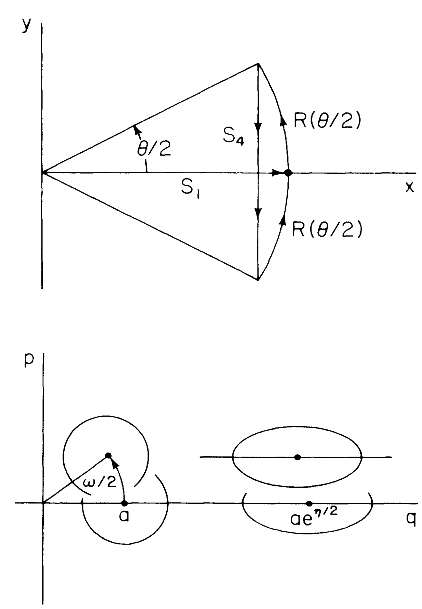

FIG. 2. Lorentz kinematics based on the mathematics of the SU(1,1) interferometer of Yurke *et al.* The starting point is a massive particle moving along the x direction with its momentum-energy vector given by Eq. (20). This momentum is rotated around the z axis, boosted along the y axis, and then rotated around the z axis, as shown in this figure. The net effect is a transformation which does not change the initial momentum. This is not an identity transformation, but a Lorentz-boosted rotation. In the Wigner phase space, which is the language of coherent and squeezed states, the ellipse of Eq. (37) is transformed into that of Eq. (41).

as is shown in Fig. 2, using the matrix equivalent to

$$S(-\pi/2,\lambda)=\begin{bmatrix}\cosh(\lambda/2) & -\sinh(\lambda/2)\\ -\sinh(\lambda/2) & \cosh(\lambda/2)\end{bmatrix} . \tag{32}$$

The boost parameter λ can be calculated from θ and η, and the result is

$$\lambda=\tanh^{-1}[(\tanh\eta)\sin(\theta/2)] . \tag{33}$$

Finally, we rotate the system by $\theta/2$ to return to the initial momentum of Eq. (30). The net effect of these transformations is $R(\theta/2)S(-\pi/2,\lambda)R(\theta/2)$. As was pointed out by Yurke *et al.*,[9] this expression is the same as a Lorentz-boosted rotation,

$$R(\theta/2)S(-\pi/2,\lambda)R(\theta/2)=S(0,\eta)R(\omega)S(0,-\eta) , \tag{34}$$

where

$$\tan(\omega/2)=[\tan(\theta/2)]/\cosh\eta . \tag{35}$$

In their recent paper,[10] Han and Kim studied the kinematics in detail using the 4×4 Lorentz transformation matrices. They concluded that this rotation is also a Wigner rotation in the sense that the rotation takes place in the Lorentz frame where the particle is at rest. By studying this kinematics using the 2×2 matrix formalism, we are able to suggest a corresponding experiment with squeezed states.

In spite of its name, what we are applying Eq. (34) to a state different from that the state discussed in the original paper of Yurke *et al.*[9] Our starting point to which Eq. (34) is applicable is a squeezed state, while Yurke *et al.* applies Eq. (34) to a coherent state. While Yurke *et al.* use the complex 2×2 representation SU(1,1), our treatment is based on the real 2×2 representation of Sp(2). The basic advantage of the real representation is that it allows us to draw two-dimensional figures.

VI. POSSIBLE EXPERIMENTS

We are now ready to propose a series of specific experiments based on the Lorentz kinematics discussed in the preceding sections. The rotation by θ around the origin of the x-y plane is equivalent to the rotation in the q-p plane by $\theta/2$. The Lorentz boost along the x direction is equivalent to the squeeze along the q axis. We are establishing this connection because it is possible to study the Lorentz group by performing optics experiments.

On the other hand, the optics experiment has its limitations. It is by now a routine process to produce coherent states. It is also easy to change the phase of the α parameter. However, it is a relatively new technology to produce squeezed states from a coherent state. It may be possible to squeeze a squeezed state in the near future. Since there is a loss of intensity during the squeezing process, it is not practical to have more than two successive squeezes.[19]

Let us start with the Thomas transformation on a spinless particle at rest. The two repeated boosts S_2S_1 are not S_3, but is S_3 preceded by a rotation. However, the spinless particle at rest is invariant under rotations. Therefore, in this particular case, S_2S_1 should give the same effect as that of S_3. In optics, the particle without spin appears as a vacuum which is represented by a circle centered around the origin in the Wigner phase space. This circle is invariant under rotations around the origin. Thus if we apply Eq. (26) to the vacuum, $R(\alpha)$ does not give any effect. This means that we squeeze the vacuum twice, the result is simply a squeezed vacuum.

This is not true if the particle has a spin which is affected by rotations. We have to take into account the Thomas rotation which precedes $S(\theta,\eta)$ in Eq. (26). The particle at rest with its intrinsic spin corresponds to a coherent state, which in the Wigner phase space can be described by the equation

$$(q-a)^2+p^2=1 . \tag{36}$$

If the particle moves along the x direction with the velocity parameter η, then it corresponds to the squeezed state represented by the ellipse

$$e^{-\eta}(q-ae^{\eta/2})^2+e^{\eta}p^2=1 . \tag{37}$$

In order to avoid more than two successive squeezes, we write Eq. (26) as

$$S(0,\eta)R(-\alpha)S(0,-\eta)=S(\phi,-\lambda)S(\theta,\eta)S(0,-\eta) . \tag{38}$$

The operator $S(0,-\eta)$ unsqueezes the ellipse of Eq. (37) to the circle of Eq. (36). Then the net result is

$$S(0,\eta)R(-\alpha)=S(\phi,-\lambda)S(\theta,\eta) , \tag{39}$$

applicable to the coherent state represented by the circle centered around $(q,p)=(a,0)$. This circle is not invariant under the rotation $R(-\alpha)$ around the origin, and the operation of the left-hand side of the preceding equation should give a new squeezed state represented by the ellipse

$$e^{-\eta}[q-a(\cos\alpha)e^{\eta/2}]^2+e^{\eta}[p+a(\sin\alpha)e^{-\eta/2}]^2=1 . \tag{40}$$

The Lorentz group tells us that the operation of the right-hand side should also give the same result. We can test this by performing two successive squeeze operations. It is remarkable that the ellipse of Eq. (40) has the same eccentricity as the starting ellipse of Eq. (37), and its major axis is parallel to that of the starting ellipse. Indeed, the net result is only a translation of the ellipse of Eq. (37) to that of Eq. (40). This is another advantage of using Eq. (39) instead of Eq. (26).

The Lorentz kinematics of the Yurke configuration has been discussed in Ref. 10. Unlike the case discussed in the original paper of Yurke *et al.*, the starting point to which Eq. (34) is applicable is a squeezed state which is unsqueezed by $S(0,-\eta)$. Therefore the right-hand side of Eq. (34) is the rotation $R(\omega)$ applicable to a coherent state followed by the squeeze $S(0,\eta)$. The left-hand side starts with the rotation $R(\theta/2)$ of the squeezed state of Eq. (37), followed by a squeeze and another rotation. If we start with the squeezed state given in Eq. (37), the end result should be

$$e^{-\eta}[q-a(\cos\omega)e^{\eta/2}]^2+e^{\eta}[p-a(\sin\omega)e^{-\eta/2}]^2=1 \; . \tag{41}$$

This can also be checked in optics laboratories.

So far, we have been discussing the question of whether the experiment performed according to one side of the equation gives the same result as what the other side gives. If this is confirmed experimentally, we can next check the measured values to see if they are consistent with those predicted by the Lorentz group.

In the Thomas configuration, we are measuring the angle α for given θ and η, and check the measured value with the value predicted by the Lorentz group. For the Yurke configuration, we measure ω and compare it with the value calculated from Eq. (35). If these comparisons work out, we can venture into a more ambitious program of combining both the Thomas and Yurke configurations. It is indeed interesting to note that the angles α and ω satisfy the sum rule

$$\theta=\alpha+\omega \tag{42}$$

by performing the calculation of the addition formula

$$\tan[(\alpha+\omega)/2]=\frac{\tan(\alpha/2)+\tan(\omega/2)}{1-[\tan(\alpha/2)]\tan(\omega/2)} \; . \tag{43}$$

Equation (41) can be written as

$$\frac{\alpha}{\theta}+\frac{\omega}{\theta}=1 \; . \tag{44}$$

Figure 3 describes the Thomas relation of Eq. (27), the Yurke relation of Eq. (35), and the preceding sum rule. The best aspect of this figure is that the curves given there can be checked in optics laboratories.

Throughout this paper, we have used the words "vacuum" and "squeezed vacuum" somewhat uncritically. The vacuum in the present case means a state with no photons. The squeezed vacuum is clearly not a vacuum state, because it is a linear combination of a vacuum state as well as many multiphoton states.

Indeed, the vacuum has many different implications in physics. We are concerned here with a possible confusion with the vacuum in quantum electrodynamics and the vacuum considered in the study of the Casimir effect. In general, the vacuum state is defined to be the lowest energy state. However, it is not a zero-energy state due to the uncertainty principle, as in the case of the harmonic oscillator.

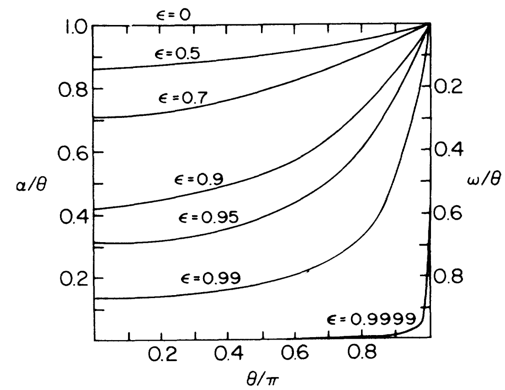

FIG. 3. Wigner rotation angles α and ω vs the lab-frame angle θ for fixed values of the velocity parameter $\varepsilon=\tanh\eta$. This graph represents also the sum rule $(\alpha+\omega)/\theta=1$.

In QED with its renormalization program based on the Hilbert space consisting of square-integrable functions, there is a well-established procedure for dealing with the vacuum.[20] The QED vacuum contains vacuum fluctuations. If photons are confined to a finite space, the vacuum state depend on boundary conditions. The physical consequence of this boundary condition is discussed in textbooks on quantum field theory,[21] and was observed experimentally.[22] This is known as the Casimir effect.

While we constantly mention the word "vacuum" when studying squeezed states, it may be worthwhile to study its possible connection with the above-mentioned aspects of the vacuum. In particular, it may be possible to study the Casimir effect using coherent and squeezed states.

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[19]In counting the number of squeezes, we do not include repeated squeezes in the same direction. For instance, $S(0,\eta)$ can be written as $S(0,\eta/2)S(0,\eta/2)$ or as $S(0,\eta/3)S(0,\eta/3)S(0,\eta/3)$. This is still one squeeze.

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