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Excess spontaneous emission in non-Hermitian optical systems. I. Laser amplifiers

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Petermann first predicted in 1979 the existence of an excess-spontaneous-emission factor in gain-guided semiconductor lasers. We show that an excess spontaneous emission of this type, and also a correlation between the spontaneous emission into different cavity modes, will in fact be present in all open-sided laser resonators or optical lens guides. These properties arise from the non-self-adjoint or non-power-orthogonal nature of the optical resonator modes. The spontaneous-emission rate is only slightly enhanced in stable-resonator or index-guided structures, but can become very much larger than normal in gain-guided or geometrically unstable structures. Optical resonators or lens guides that have an excess noise emission necessarily also exhibit an "excess initial-mode excitation factor" for externally injected signals. As a result, the excess spontaneous emission can be balanced out and the usual quantum-noise limit recovered in laser amplifiers and in injection-seeded laser oscillators, but not in free-running laser oscillators.

I. INTRODUCTION

The conventional second-quantized analysis of spontaneous emission in a laser cavity or in any other system involving interaction between atoms and electromagnetic radiation leads to a general principle that the rate of spontaneous emission from a collection of atoms into any individual resonant-cavity or transmission-line mode will always be exactly equal to the downward stimulated transition rate that would be produced in the same atoms by a signal energy of one extra photon in the same electromagnetic mode. This principle can be used to derive fundamental conclusions relating to thermal equilibrium, blackbody radiation density, Johnson-Nyquist or thermal noise in lossy linear systems, laser-amplifier noise figure, and quantum-noise fluctuations in laser oscillators.

Petermann first predicted in 1979 a so-called "excess-spontaneous-emission factor" or excess-noise factor with a value greater than one extra photon per mode in gain-guided semiconductor lasers.[1] This prediction was initially controversial, since spontaneous emission at a rate corresponding to more than one extra photon per mode would seem to violate the accepted principle of quantum-noise theory mentioned above. Moreover, application of the same argument to passive or loss-guided systems appears to predict an excess thermal noise in such systems, in apparent violation of elementary thermodynamics.

These difficulties were resolved, at least for loss-guided systems, in an excellent analysis by Haus and Kawakami,[2] who pointed out that such gain or loss-guided systems also exhibit correlations between the noise emission into different propagating modes. This is in contrast to the familiar situation with power-orthogonal normal modes, where the spontaneous emission into each mode is uncorrelated with the emission into any other mode. Haus and Kawakami showed that the correlations between emission into different modes in the loss-guided case are just sufficient to recover the usual blackbody radiation or Johnson-Nyquist thermal-noise results in these loss-guided systems. Several other derivations of Petermann's excess-noise factor have also been presented by other authors;[3-13] most of these are reviewed in Haus and Kawakami. An earlier discussion of spontaneous emission in open resonators and its application to laser amplifiers and oscillators, in much the same spirit as the present paper, has also been given by Henry.[14]

In this paper we demonstrate that an excess spontaneous emission rate per mode and also noise correlations between the emission into different modes are, in fact, to be expected in *all* open-sided laser resonators, optical lens guides, and similar optical structures, exactly as predicted by Petermann and by Haus and Kawakami for the gain-guided case. These excess-noise properties have nothing necessarily to do with gain or loss guiding, or with wave-front curvature. They arise entirely from the non-Hermitian and hence non-power-orthogonal or biorthogonal nature of the transverse eigenmodes in such structures.[15,16] These excess-noise emission and mode-correlation effects remain small (normalized values close to unity or zero, respectively) in conventional stable optical resonators or index-guided systems. They become of significant magnitude, however, in gain-guided or loss-guided systems, or in unstable-resonator or unstable-lens-guide systems having significant geometric magnification per pass. In such systems, the excess emission factor can become as large as hundreds to thousands of time above the usual value, even at moderate Fresnel numbers for the unstable systems.

The crucial step in the analysis in this paper comes in using the real Fox and Li transverse eigenmodes of the optical system[15] as the basis set for expanding the fields in the structure, rather than assuming an idealized set of power-orthogonal normal modes as is usually (and incorrectly) done in most laser analyses. The biorthogonal rather than power-orthogonal nature of the real cavity modes then turns out to be responsible for the excess-noise and correlated-emission properties. The approach in this paper is very similar to the Haus-Kawakami approach, but the results now apply to a much broader class of general, open-sided, non-loss-guided optical structures and the analogous periodic lens guides.

We also demonstrate that any such systems having significant excess-noise emission will necessarily also exhibit a corresponding "excess initial mode excitation" in their response to externally injected signals. By sending in a properly shaped "adjoint-coupled" external signal, for example, it is possible to excite such systems with more initial power or energy in any given transverse eigenmode of the system than is present in the injected external signal itself. As a result, in a laser amplifier or an injection-seeded laser oscillator with optimum signal injection, one can always recover the minimum quantum-limited noise performance predicted by conventional theories, i.e., one equivalent input noise photon per resolution time for the lowest-loss mode, despite the excess spontaneous emission with the laser amplifier.

The situation is less satisfactory for free-running unstable-resonator laser oscillators, however. In the second part of this paper we will show that a laser oscillator can always be expected to exhibit quantum noise effects or Schawlow-Townes noise fluctuations which are larger than the usually stated values by just the excess-spontaneous-emission ratio or excess-noise factor calculated in this paper. Fortunately, this excess-noise factor is close to unity for lasers using ordinary stable laser cavities, or using pure index guiding. For gain-guided lasers, however, or for either hard-edged or variable-reflectivity unstable resonators, the excess-noise enhancement can be as large as 100 to 1000 times for moderate values of geometrical magnification and Fresnel number.

II. NOISE ANALYSIS FOR PERIODIC LENS-GUIDE SYSTEMS

In this paper we will use an extended semiclassical analysis to derive the excess-noise properties for a laser

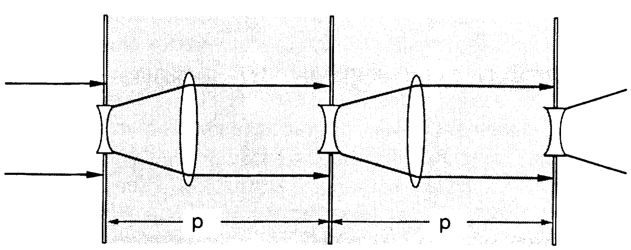

FIG. 1. Example of a periodic lens guide filled with absorbing or amplifying atomic medium, such as analyzed in this paper. The lens guide shown is a geometrically unstable confocal system to emphasize the characteristics of such systems. The absorbing apertures separating each period of the structure represent the diffractive loss out the sides or past the edges of the structure in each period. As such, they are taken to be perfectly absorbing but noise free or effectively at zero temperature.

amplifier produced by spontaneous emission from inverted laser atoms within the optical structure. It will be most convenient for this purpose to consider amplification down a cascaded periodic lens-guide system, as in Fig. 1, and to express all signal and noise quantities in the frequency domain. In a subsequent paper we will modify the analysis to apply to the time buildup of laser oscillation in a resonant laser cavity.

A. Analytical formulation

We wish to analyze, therefore, the propagation of waves traveling in the forward direction down a periodic optical waveguide or lens guide completely filled with an amplifying (or possibly absorbing) atomic medium, as illustrated in Fig. 1. This could, of course, also represent amplification through successive passes around an optical resonator containing the same atomic medium. We have made the example shown in Fig. 1 a geometrically unstable lens guide with a sizable magnification per period, to emphasize the characteristics of such systems. The analysis is intended to apply, however, to either stable or unstable systems, with the assumption that all such systems will have some finite-diameter aperture or mirror, and hence at least some amount of diffractive energy loss past the mirror edges or out the sides of the periodic structure.

We begin as usual with the scalar wave equation

$$\nabla^2\mathcal{E}(\mathbf{r},t)-\mu\sigma\frac{\partial\mathcal{E}(\mathbf{r},t)}{\partial t}-\mu\epsilon\frac{\partial^2\mathcal{E}(\mathbf{r},t)}{\partial t^2}=\mu\frac{\partial^2 p_N(\mathbf{r},t)}{\partial t^2}\ , \tag{1}$$

where $\mathcal{E}(\mathbf{r},t)$ gives the real E field of the propagating wave in the medium as a function of space coordinates $\mathbf{r}$ and time t. The conductivity σ (which is negative for an amplifying medium) represents the net linear stimulated emission or absorption by the atomic medium with which the E field interacts, and μ and ϵ are the magnetic and dielectric permeabilities of the medium. All these quantities are independent of position within the structure, i.e., there is no gain or index guiding (though this could easily be added, at the cost of some complexity in the analysis, and with no change in the overall results).

The polarization $p_N(\mathbf{r},t)$ on the right-hand side of the equation is a random-noise polarization (dipole moment per unit volume) representing the spontaneous emission from the laser medium. The value of this noise polarization is derived from a heuristic argument in Appendix A.

For the amplifier case it is most convenient to assume a traveling-wave expansion in the form

$$\begin{aligned}\mathcal{E}(\mathbf{r},t)&=\mathrm{Re}\tilde{E}(\mathbf{r},\omega)\exp[j(\omega t-\beta z)],\\ p_N(\mathbf{r},t)&=\mathrm{Re}\tilde{P}_N(\mathbf{r},\omega)\exp[j(\omega t-\beta z)]\ ,\end{aligned} \tag{2}$$

where $\beta=\omega(\mu\epsilon)^{1/2}$. We omit writing an explicit Fourier integration over frequency ω since the system is linear, the noise signals at different frequencies are uncorrelated, and we will be examining the behavior only in a narrow bandwidth about any given carrier frequency ω. We can then make a slowly-varying-envelope approximation for the phasor amplitude $\tilde{E}(\mathbf{r},\omega)$, drop the explicit dependence on the frequency ω, and simplify the wave equation to the extended paraxial form

$$\nabla_T^2\tilde{E}-2j\beta\left[\frac{\partial\tilde{E}}{\partial z}-\alpha\tilde{E}\right]=-\omega^2\mu\tilde{P}_N(\mathbf{r}) , \quad (3)$$

where ∇_T^2 indicates the Laplacian operator with respect to the transverse coordinates x,y and $\alpha=-\eta_0\sigma/2$ is the gain coefficient in the atomic medium, with $\eta_0=(\mu/\epsilon)^{1/2}$ being the characteristic impedance for the medium.

As shown in Appendix A, the phasor amplitude $\tilde{P}_N(\mathbf{r})$ for the noise polarization will have a δ-function correlation in space with an amplitude given by

$$\langle\tilde{P}_N(\mathbf{r})\tilde{P}_N^*(\mathbf{r}')\rangle=\frac{16\hbar\alpha B}{\omega\eta_0}\frac{N_2}{N_2-N_1}\delta(\mathbf{r}-\mathbf{r}')$$
$$=\frac{16\rho_I\hbar\alpha B}{\omega\eta_0}\delta(\mathbf{r}-\mathbf{r}') , \quad (4)$$

where N_1 and N_2 are lower- and upper-level population densities, and B is the (Hertzian) bandwidth over which the noise power is measured. We also introduce the symbol ρ_I as a shorthand for the additional noise-emission factor $N_2/(N_2-N_1)$ produced by incomplete inversion in the atomic medium. Equation (4) is essentially the same equation as has been used by other authors.[1,2,14] Note that the mean-square amplitude of this spontaneous emission is directly proportional to the atomic gain coefficient α, since we assume the inversion N_2/N_1 is fixed and independent of position. We have written the gain coefficient and population ratios here as if the atomic medium were an inverted or amplifying medium. If the signs of the gain coefficient, population difference, and Boltzmann temperature are all inverted, however, the noise polarization magnitude will still remain positive, and all of the subsequent results will apply equally well to a lossy or absorbing atomic system.

B. Expansion in propagation eigenmodes

We then expand the total field in the optical system in terms of the "cold" propagating modes or transverse eigenmodes $\tilde{u}_N(\mathbf{s},z)$ of the propagating structure[15] in the form

$$\tilde{E}(\mathbf{r})=\sum_n\tilde{c}_n(z)\tilde{u}_n(\mathbf{r})=\sum_n\tilde{c}_n(z)\tilde{u}_n(\mathbf{s},z) , \quad (5)$$

where from here on we will use the notation $\mathbf{r}\equiv(\mathbf{s},z)$ in order to distinguish the transverse coordinates $\mathbf{s}=(x,y)$ or $\mathbf{s}=(r,\theta)$ in the lens guide from the axial coordinate z. The functions $\tilde{u}_n(\mathbf{r})\equiv\tilde{u}_n(\mathbf{s},z)$ are intended to be the usual transverse eigenmodes, or "Fox and Li modes," of the optical resonator or lens guide without the amplifying medium. (For the case of two transverse dimensions the index n should really be a double set, i.e., $\tilde{u}_{nm}(\mathbf{s},z)$, but we write only a single index for simplicity.)

These transverse eigenmodes $\tilde{u}_n$ are solutions of the homogeneous wave equation, i.e., of Eq. (3) with the gain and noise polarization terms omitted, but subject to all the boundary conditions of the optical structure itself. In practice, these transverse eigenmodes are usually obtained not by solving the differential Eq. (3), but as eigensolutions of an integral equation[15]

$$\tilde{u}_n(\mathbf{s},z+p)=\int_A\tilde{K}(\mathbf{s},\mathbf{s}_0,z)\tilde{u}_n(\mathbf{s}_0,z)d\mathbf{s}_0=\tilde{\gamma}_n\tilde{u}_n(\mathbf{s},z) , \quad (6)$$

where $\tilde{K}(\mathbf{s},\mathbf{s}_0)$ is a propagation kernel, generally similar to Huygen's integral, but including any finite mirrors, apertures, and intracavity optics in the optical structure. This integral then describes propagation through one period of the empty optical system, starting at any arbitrary reference plane z and propagating to the corresponding reference plane at $z+p$ one period p (or one cavity round trip) later. The differential $d\mathbf{s}_0\equiv dx_0dy_0$ is integrated over the full cross section A of the resonator or lens guide at the selected reference plane z.

The eigenvalue $\tilde{\gamma}_n$ gives the complex amplitude reduction and phase shift for the nth order eigenmode after propagation through one period in an optical lens guide or one round trip in an optical resonator. The fractional power loss for the nth mode in one period due to diffraction losses is $1-|\tilde{\gamma}_n|^2$, while the phase angle of $\tilde{\gamma}_n$ gives the added phase variation over and above the basic $\exp(-j\beta p)$ propagation factor for the same mode.

The integral Eq. (6) directly provides the transverse eigenmodes $\tilde{u}_n(\mathbf{s},z)$ only at the one arbitrarily chosen reference plane z. These modes of course exist at all other planes z and their (generally slow) variation with z can be determined either by shifting the kernel to a different reference plane z or, more easily, by forward propagation of the modes from the initial reference plane z. The eigenvalues $\tilde{\gamma}_n$ and all the results to be obtained below are entirely independent of the choice of z.

C. Biorthogonality properties

The crucial factor in this analysis is that the propagation integral (6) is in general not a Hermitian operator for any of the usual open-sided structures used as optical resonators or lens guides. One might think that this integral operator should be Hermitian, since it is used to find a solution to the fully Hermitian wave equation. The boundary conditions on the wave equation for open-sided resonators are not Hermitian, however, and this shows up in the integral equation as a non-Hermitian form for the integral operator.

Because the operator is not Hermitian, neither the completeness nor even the existence of a set of eigensolutions $\tilde{u}_n$ to the integral Eq. (6) can be rigorously guaranteed. We must take the existence of such eigenmodes and their usefulness as a basis set, therefore, as matters of empirical (or numerical) observation rather than matters of rigorous mathematical proof. Because of the non-Hermitian character of the integral operator, the transverse modes in these open-sided optical systems are also in general not "normal modes," i.e., they are not power orthogonal or self-adjoint to each other, with the result that

$$\int_A\tilde{u}_n(\mathbf{s},z)\tilde{u}_m^*(\mathbf{s},z)d\mathbf{s}\neq\delta_{nm} . \quad (7)$$

Within one period between hard-edged apertures of finite mirrors, however, we can at least scale the amplitude of each individual mode so that they are individually power normalized, in the sense that

$$\tilde{A}_{nn} \equiv \int_A \tilde{u}_n(\mathbf{s},z)\tilde{u}_n^*(\mathbf{s},z)d\mathbf{s}=1 \ . \tag{8}$$

Of course, as any such mode passes through the finite aperture between successive periods of the structure (or bounces off the output mirror in the cavity case), it loses energy to diffraction losses, so that the power in the mode is reduced by $|\tilde{\gamma}_n|^2$ in the next period.

Rather than being orthogonal to each other, the eigenmodes $\tilde{u}_n(\mathbf{s},z)$ are instead biorthogonal to a set of transposed or adjoint eigenfunctions $\tilde{\phi}_n(\mathbf{s},z)$ in the fashion

$$\int_A \tilde{u}_n(\mathbf{s},z)\tilde{\phi}_m(\mathbf{s},z)d\mathbf{s}=\delta_{nm} \ , \tag{9}$$

where the adjoint eigenmodes $\tilde{\phi}_n(\mathbf{s},z)$ are solutions of the transposed eigenequation

$$\tilde{\phi}_n(\mathbf{s},z)=\int_A \tilde{K}^T(\mathbf{s},\mathbf{s}_0,z)\tilde{\phi}_n(\mathbf{s}_0,z)d\mathbf{s}_0=\tilde{\gamma}_n\tilde{\phi}_n(\mathbf{s},z) \ . \tag{10}$$

Here the integral kernel $\tilde{K}^T(\mathbf{s},\mathbf{s}_0,z)$ is the transpose (in $\mathbf{s}$ and $\mathbf{s}_0$) of the kernel in Eq. (6). The adjoint eigenfunctions $\tilde{\phi}_n(\mathbf{s},z)$ are thus physically different—distinctly different in some cases—from the original eigenfunctions $\tilde{u}_n(\mathbf{s},z)$, although the eigenvalues $\tilde{\gamma}_n$ are identical for corresponding modes in the two sets.

In physical terms, it can be shown that if $\tilde{K}$ represents the propagation kernel for forward propagation through one period or segment of an optical waveguide, from one arbitrary reference plane to the same plane one period later, then the transposed kernel $\tilde{K}^T$ corresponds to propagation in the opposite or reverse direction along the same lens guide, between the same two reference planes. Figure 2 illustrates this difference for a strongly unstable lens guide. The solid lines in this figure indicate the approximate outer boundaries for the forward set of propagation eigenmodes $\tilde{u}_n(\mathbf{s},z)$, and the dashed lines the approximate outer boundaries for the reverse set of propagation eigenmodes $\tilde{\phi}_n(\mathbf{s},z)$.

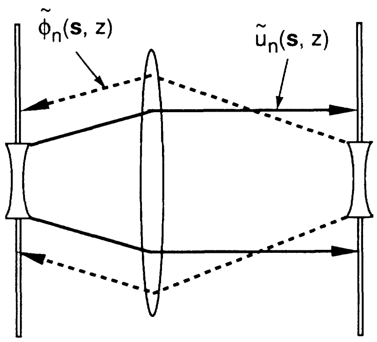

FIG. 2. In an unstable optical lens guide with a large magnification, such as the confocal example shown here, the forward eigenmodes $\tilde{u}_n(\mathbf{s},z)$ will be confined more or less within the volume bounded by the solid lines, while the adjoint eigenmodes $\tilde{\phi}_n(\mathbf{s},z)$ in this example will lie generally within the volume bounded by the dashed lines. Both sets of eigenmodes will, however, have complicated beam profiles with complex Fresnel ripples and some spillover outside the regions indicated. Note that the ordinary and adjoint eigenmodes will both have the same geometric magnification and the same complex eigenvalues $\tilde{\gamma}_n$ per period.

We should note that the adjoint modes $\tilde{\phi}_n$ are equally non-normal, i.e.,

$$\int_A \tilde{\phi}_n(\mathbf{s},z)\tilde{\phi}_m^*(\mathbf{s},z)d\mathbf{s}\neq\delta_{nm} \ . \tag{11}$$

More importantly, these modes cannot be individually power normalized to unity in the same fashion as the $\tilde{u}_n$ modes, for suppose we write

$$\tilde{\phi}_n(\mathbf{s},z)\equiv\tilde{u}_n^*(\mathbf{s},z)+\Delta\tilde{u}_n^*(\mathbf{s},z) \tag{12}$$

and also the complex conjugate of this, so that $\Delta\tilde{u}_n(\mathbf{s},z)$ is the difference between a given eigenmode $\tilde{u}_n(\mathbf{s},z)$ and the corresponding $\tilde{\phi}_n^*(\mathbf{s},z)$ having the same eigenmode $\tilde{\gamma}_n$. The biorthogonality of $\tilde{u}_n$ and $\tilde{\phi}_n$ and the power normalization of $\tilde{u}_n$ and $\tilde{u}_n^*$ then lead to

$$\int_A \tilde{u}_n\Delta\tilde{u}_n^*d\mathbf{s}=\int_A \tilde{u}_n^*\Delta\tilde{u}_n d\mathbf{s}=0 \ , \tag{13}$$

and so the power normalization of the adjoint eigenmodes becomes

$$\int_A \tilde{\phi}_n\tilde{\phi}_n^*d\mathbf{s}=1+\int_A \Delta\tilde{u}_n\Delta\tilde{u}_n^*d\mathbf{s}\geq 1 \ . \tag{14}$$

This integral is necessarily greater than unity unless the difference $\Delta\tilde{u}_n(\mathbf{s},z)\equiv\tilde{\phi}_n^*(\mathbf{s},z)-\tilde{u}_n(\mathbf{s},z)$ between the regular and adjoint eigenmodes is identically zero.

D. Formal solution

If we substitute the mode expansion of Eq. (5) into the reduced wave Eq. (3), multiply both sides by any one adjoint mode function $\tilde{\phi}_n(\mathbf{s},z)$ and integrate over the transverse cross section A, the wave equation separates into individual equations for the complex amplitude of each transverse mode in the form

$$\frac{d\tilde{c}_n(z)}{dz}=\alpha\tilde{c}_n(z)-j\tilde{p}_n(z) \ , \tag{15}$$

where the Langevin noise term $\tilde{p}_n(z)$ that drives each nth mode coefficient $\tilde{c}_n(z)$ is given by

$$\tilde{p}_n(z)=\frac{\omega\eta_0}{2}\int_A \tilde{P}_N(\mathbf{s},z)\tilde{\phi}_n(\mathbf{s},z)d\mathbf{s} \ . \tag{16}$$

The formal solution to (15), given an input coefficient $\tilde{c}_n(0)$ at $z=0$, is

$$\tilde{c}_n(z)=\exp(\alpha z)\tilde{c}_n(0)-j\int_0^z\exp[\alpha(z-z')]\tilde{p}_n(z')dz' \ . \tag{17}$$

The crucial point here, as will become more evident later, is that the gain factor α for the nth mode coefficient $\tilde{c}_n$ involves an overlap integral of $\tilde{u}_n$ and $\tilde{\phi}_n$, which evaluates to unity, while the Langevin noise term involves $\tilde{\phi}_n$ and $\tilde{\phi}_n^*$, whose overlap integral (14) is always greater than or equal to unity. This alters the fundamental relationship between gain and noise emission per unit length for strongly non-Hermitian structures where the transposed eigenfunctions $\tilde{\phi}_n(\mathbf{s},z)$ are distinctly different from the original eigenfunctions $\tilde{u}_n^*(\mathbf{s},z)$.

E. Noise correlations and excess noise factors

The total power $I(z)$ flowing down the lens guide in the $+z$ direction at any plane z can then be written as

$$I(z)=\frac{1}{2\eta_0}\int_A \tilde{E}(\mathbf{s},z)\tilde{E}^*(\mathbf{s},z)d\mathbf{s} = \frac{1}{2\eta_0}\sum_n \tilde{c}_n(z)\tilde{c}_n^*(z)+\frac{1}{2\eta_0}\sum_{\substack{n,m\\ n\neq m}} \tilde{A}_{nm}\tilde{c}_n(z)\tilde{c}_m^*(z) \ . \tag{18}$$

The first sum in the second line is the usual sum over the "powers per individual mode," assuming the $\tilde{u}_n(\mathbf{s},z)$ functions to be normalized to unity. The second sum (taken over all values of n and m except $n=m$) expresses the "cross powers" between modes. The constants $\tilde{A}_{nm}$ represent the overlap integrals between different transverse eigenmodes as given by

$$\tilde{A}_{nm}\equiv\int_A \tilde{u}_n(\mathbf{s},z)\tilde{u}_m^*(\mathbf{s},z)d\mathbf{s} \ . \tag{19}$$

These cross-mode integrals can be shown to be independent of axial position z within one period of the lens guide, and the diagonal elements have values $\tilde{A}_{nn}\equiv 1$ for the normalization we have selected.

From Eq. (15), the modes are driven by the Langevin noise polarizations $\tilde{p}_n(z)$ as well as, possibly, by coherent input signals. The coefficients $\tilde{c}_n(z)$ are thus in the general case random variables, so that average powers must be obtained by calculating expectation values of the form $\langle\tilde{c}_n\tilde{c}_n^*\rangle$ or $\langle\tilde{c}_n\tilde{c}_m^*\rangle$. We thus write for the total power at plane z

$$I(z)=\frac{1}{2\eta_0}\sum_n \langle\tilde{c}_n(z)\tilde{c}_n^*(z)\rangle + \frac{1}{2\eta_0}\sum_{\substack{n,m\\ n\neq m}}\langle\tilde{c}_n(z)\tilde{c}_m^*(z)\rangle\tilde{A}_{nm} \ . \tag{20}$$

But from Eq. (17) the expectation values with the Langevin noise terms included will be given in general by

$$\langle\tilde{c}_n(z)\tilde{c}_m^*(z)\rangle=e^{2\alpha z}\langle\tilde{c}_n(0)\tilde{c}_m^*(0)\rangle + e^{2\alpha z}\int_0^z dz'\int_0^z dz'' e^{-\alpha(z'+z'')} \times\langle\tilde{p}_n(z')\tilde{p}_m^*(z'')\rangle \ , \tag{21}$$

including the case with $n=m$. In writing this we make the (very reasonable) physical assumption that in a one-way amplifier the spontaneous emission $\tilde{p}_n(z)$ at any plane $z>0$ will be entirely uncorrelated with the input signal, or noise, contained in the input wave $\tilde{c}_n(0)$ at $z=0$, i.e., that $\langle\tilde{c}_n(0)\tilde{p}_m^*(z)\rangle\equiv 0$ for $z\geq 0$.

The noise term inside the integral in Eq. (21) can then be written, using Eq. (16), in the form

$$\langle\tilde{p}_n(z')\tilde{p}_m^*(z'')\rangle=\left[\frac{\omega\eta_0}{2}\right]^2 \times\int_A d\mathbf{s}'\int_A d\mathbf{s}''\langle\tilde{P}_N(\mathbf{r}')\tilde{P}_N^*(\mathbf{r}'')\rangle \times\tilde{\phi}_n(\mathbf{s}',z')\tilde{\phi}_m^*(\mathbf{s}'',z'') \ . \tag{22}$$

But, since the original noise polarization $\tilde{P}_N(\mathbf{r})$ is δ-function correlated in all spatial coordinates, as given in Eq. (4), we can do the transverse integrations immediately and reduce this to

$$\langle\tilde{p}_n(z')\tilde{p}_m^*(z'')\rangle=4\rho_I\hbar\omega\eta_0\alpha B\tilde{B}_{nm}\delta(z'-z'') \ , \tag{23}$$

where we define the $\tilde{B}_{nm}$ to be the (non-z-varying) overlap integrals

$$\tilde{B}_{nm}\equiv\int_A\tilde{\phi}_n(\mathbf{s},z)\tilde{\phi}_m^*(\mathbf{s},z)d\mathbf{s} \ . \tag{24}$$

So long as the adjoint eigenfunctions $\tilde{\phi}_n(\mathbf{s},z)$, like the original eigenfunctions $\tilde{u}_n(\mathbf{s},z)$, are not power orthogonal, the off-diagonal elements $\tilde{B}_{nm}$ for $n\neq m$ will have finite values, and there will be correlation between the Langevin noise terms $\tilde{p}_n(z)$ and $\tilde{p}_m(z)$ driving different transverse eigenmodes in the lens guide.

For the on-diagonal elements, in addition we will adopt the notation

$$K_n\equiv\tilde{B}_{nn}=\int_A\tilde{\phi}_n(\mathbf{s},z)\tilde{\phi}_n^*(\mathbf{s},z)d\mathbf{s} \tag{25}$$

and refer to K_n henceforth as the Petermann excess-noise factor for the nth mode. We have already shown that $K_n\geq 1$ in all cases. The value of K_n approaches unity only for near-ideal systems with nearly power-orthogonal modes and near-zero diffraction losses.

F. Noise generation within one period

Putting the Langevin noise sources into Eq. (21) and completing the integration then leads to

$$\langle\tilde{c}_n(z)\tilde{c}_m^*(z)\rangle=e^{2\alpha z}\langle\tilde{c}_n(0)\tilde{c}_m^*(0)\rangle +(e^{2\alpha z}-1)2\eta_0\tilde{B}_{nm}\rho_I\hbar\omega B \ . \tag{26}$$

Putting this into Eq. (20) gives for the power growth over distance p within one period of the lens guide

$$I(p)=G\left[\sum_n\frac{1}{2\eta_0}\langle\tilde{c}_n(0)\tilde{c}_n^*(0)\rangle + \sum_{\substack{n,m\\ n\neq m}}\frac{1}{2\eta_0}\langle\tilde{c}_n(0)\tilde{c}_m^*(0)\rangle\tilde{A}_{nm}\right] +(G-1)\rho_I\hbar\omega B\left[\sum_n K_n+\sum_{\substack{n,m\\ n\neq m}}\tilde{A}_{nm}\tilde{B}_{nm}\right] \ , \tag{27}$$

where $G\equiv\exp(2\alpha p)$ is the net power gain due to the atoms within one period from plane z to plane $z+p$. The first two terms obviously give simply the input power $I(z=0)$ multiplied by the power gain $G\equiv\exp(2\alpha p)$. The second pair of terms gives the spontaneous emission or noise power generated in and amplified by the laser medium within one period.

Most of the essential results of this analysis are contained in this expression. In particular, the usual equivalent input noise power term $\rho_I \hbar\omega B$ [interpreted as one photon per resolution time, per mode, times the incomplete inversion factor $\rho_I \equiv N_2/(N_2-N_1)$] appears in the second term of Eq. (27) as usual, but here it is multiplied by the excess-noise factors K_n for each mode, as well as the dimensionless noise correlation factors $\tilde{B}_{nm}$. We will introduce one further analytical result before discussing this further.

G. Effects of apertures

The results given thus far apply within one period of the periodic lens guide, in the "free-space" region between two finite apertures or mirrors. At any such apertures the fields are truncated or absorbed and thus suffer diffraction losses. It is simplest, and costs little in generality, to think of the lens guide as having just one such aperture or coupling point per period, with the coordinate system chosen so that $z=0+$ comes just after one such aperture, and $z=p-$ comes just before the next succeeding aperture.

When the wave passes through any such aperture, each of its eigenmodes is reduced in amplitude by the corresponding eigenvalue amplitude $|\tilde{\gamma}_n|$. We can think of these diffraction losses at the apertures either as reducing the normalized amplitudes $\tilde{u}_n$ of the eigenmodes by the complex eigenvalue $\tilde{\gamma}_n$, and hence the overlap integrals $\tilde{A}_{nm}$ by the amount $\tilde{\gamma}_n\tilde{\gamma}_m^*$ at each successive aperture; or alternatively as reducing the expansion coefficients $\tilde{c}_n$ by the same amount at each successive aperture. We choose the latter approach, so that the normalizations of $\tilde{u}_n$ given earlier will be preserved everywhere. (Note that if we elected to change the normalization of the $\tilde{u}_n$'s by the ratio $\tilde{\gamma}_n$ on going through each aperture moving in the $+z$ direction, we would correspondingly have to change the ratios of the $\tilde{\phi}_n$'s by the inverse ratio $1/\tilde{\gamma}_n$.)

When the aperture transmission is taken into account, the net gain and net noise generation in one period, going from an input plane at $z=0$ just after one aperture to an output plane at $z=p+$ just after the succeeding aperture, must be computed by extending Eq. (17) with the factor

$$\tilde{c}_n(p+)=\tilde{\gamma}_n\tilde{c}_n(p-) \; . \tag{28}$$

Note that by definition of the eigenmodes there is no mode conversion at the aperture; the amplitude of each expansion coefficient is merely reduced by its eigenvalue $\tilde{\gamma}_n$. The signal and noise power at the output from one complete period is then written as

$$I(p+)=G\left[\sum_n \frac{1}{2\eta_0}|\tilde{\gamma}_n|^2\langle\tilde{c}_n(0)\tilde{c}_n^*(0)\rangle+\sum_{\substack{n,m\\ n\neq m}}\frac{1}{2\eta_0}\tilde{\gamma}_n\tilde{\gamma}_m^*\langle\tilde{c}_n(0)\tilde{c}_m^*(0)\rangle\tilde{A}_{nm}\right]$$
$$+(G-1)\rho_I\hbar\omega B\left[\sum_n|\tilde{\gamma}_n|^2K_n+\sum_{\substack{n,m\\ n\neq m}}\tilde{\gamma}_n\tilde{\gamma}_m^*\tilde{A}_{nm}\tilde{B}_{nm}\right] , \tag{29}$$

in analogy to Eq. (27).

H. Initial wave excitation factor

Suppose we now think of launching an initial wave or mixture of transverse eigenmodes with a transverse field distribution $\mathcal{E}_{\text{in}}(\mathbf{s},t)=\text{Re}\tilde{E}_{\text{in}}(\mathbf{s})e^{j\omega t}$ at an input plane $z=0$ at the input end of the periodic lens guide, so that

$$\tilde{E}_{\text{in}}(\mathbf{s})=\sum_n \tilde{c}_n(0)\tilde{u}_n(\mathbf{s},0) \; . \tag{30}$$

Lacking a completeness proof for the eigenmodes $\tilde{u}_n$ we cannot claim that any arbitrary distribution $\tilde{E}_{\text{in}}(\mathbf{s})$ can be expanded in this fashion; but we assume that any useful input wave will be given by such an expansion.

Suppose the input field distribution $\tilde{E}_{\text{in}}(\mathbf{s})$ carries unity power, and we wish to achieve the maximum possible initial amplitude $\tilde{c}_q(0)$ for one particular transverse eigenmode with $n=q$. To accomplish this in this particular case, we should not "spatially mode match" the injected signal $\tilde{E}_{\text{in}}(\mathbf{s})$ into the desired mode $\tilde{u}_q(\mathbf{s},0)$ as is often done in more conventional situations, i.e., we do not want the usual condition that $\tilde{E}_{\text{in}}(\mathbf{s})=\text{const}\times\tilde{u}_q(\mathbf{s},0)$. Rather, we should match the input field into the complex conjugate of the *adjoint* mode corresponding to the desired eigenmode, i.e., the condition for maximum excitation of mode $\tilde{u}_q$ is

$$\tilde{E}_{\text{in}}(\mathbf{s})=(2\eta_0/\tilde{B}_{pp})^{1/2}\tilde{\phi}_q(\mathbf{s},0) \; , \tag{31}$$

where $\tilde{B}_{qq}\equiv K_q$ is the overlap integral for the $\tilde{\phi}_q$ function defined in Eqs. (24) and (25).

The initial mode-expansion coefficients $\tilde{c}_n(0)$ will then be, from Eqs. (24) and (25) and the biorthogonality relation,

$$\tilde{c}_n(0)=\int_A \tilde{E}_{\text{in}}(\mathbf{s})\tilde{\phi}_n(\mathbf{s},0)d\mathbf{s} = \begin{cases}(2\eta_0/\tilde{B}_{qq})^{1/2}\tilde{B}_{nq}, & n\neq q\\ (2\eta_0\tilde{B}_{qq})^{1/2}\equiv(2\eta_0K_q)^{1/2}, & n=q \; .\end{cases} \tag{32}$$

The total power $I(0+)$ in the system just after the input plane, as calculated using Eq. (18), will necessarily still by unity. The total power in the selected mode $n=q$ by itself will, however, be given by

$$I(0)|_{q\text{ th mode}}=\frac{1}{2\eta_0}|\tilde{c}_q(0)|^2=K_q \; . \tag{33}$$

Since $K_q>1$ in general, there will be *more initial power per mode* put into the selected eigenmode than there is in the injected signal to start with. This enhanced excitation of the coefficient $\tilde{c}_q$ will necessarily be accompanied by finite excitation of other coefficients $\tilde{c}_n$, $n \neq q$, and hence there will be still further excess power excitation into all the other eigenmodes.

Conservation of energy is maintained in this situation by the cross-power terms that appear in the second summation in Eq. (18). At least some of these terms will necessarily be negative, so that the total power remains at unity. The excess initial wave excitation of the selected mode $n = q$ is nonetheless real and meaningful, and exactly equal (in power) to the excess noise factor K_q for the same mode.

III. DISCUSSION AND CONCLUSIONS

A. Laser amplifier noise figure

Let us first show that, despite the excess spontaneous emission factor, one can still recover the usual quantum-limited noise figure characteristic of any laser amplifier or, in fact, any other kind of linear phase-preserving signal amplifier. In any given optical resonator or lens-guide structure one particular transverse eigenmode, conventionally designated the $n = 0$ mode, will have smaller diffraction losses and hence a larger eigenvalue $\tilde{\gamma}_0$ than all other eigenmodes. Suppose we arrange to inject maximum intensity into this lowest-loss mode at $z = 0$ using an input wave with total (external) power I_{in}, as described just above, and allow the resulting mode mixture to propagate through multiple sections. We assume the system has net power gain, i.e., $G|\tilde{\gamma}_n|^2 > 1$ at least for the lowest-order mode and possibly also for higher-order modes.

Then, because the lowest-order mode has higher net gain per period than any other eigenmode, after a sufficiently large number of periods N all the higher-order eigenmodes will have small amplitudes relative to the $n = 0$ eigenmode, and the field in the lens guide will become predominantly the $n = 0$ eigenmode. The output power in this case, after N periods, will be given by the cascaded expression

$$\begin{aligned} I(N_p) &= (G|\tilde{\gamma}_0|^2)^N K_0 I_{\text{in}} \\ &\quad + \sum_{k=0}^{N-1} (G|\tilde{\gamma}_0|^2)^k (G-1)|\gamma_0|^2 K_0 \rho_I \hbar\omega B \\ &= G_N K_0 I_{\text{in}} + (G_N - 1)\frac{(G-1)|\tilde{\gamma}_0|^2}{G|\tilde{\gamma}_0|^2 - 1} K_0 \rho_I \hbar\omega B \ , \end{aligned} \tag{34}$$

where $G_N \equiv (G|\tilde{\gamma}_0|^2)^N$ is the overall power gain for the $n = 0$ eigenmode through N complete periods in cascade. The second line of this formula is evidently a generalization, valid for N periodic sections in cascade, of the usual noise figure expression for a single-mode laser amplifier of total gain G_T, namely,

$$I_{\text{out}} = G_T I_{\text{in}} + (G_T - 1)\rho_I \hbar\omega B \ . \tag{35}$$

This result displays, however, some significant generalizations.

First, the ratio $(G-1)|\tilde{\gamma}_0|^2/(G|\tilde{\gamma}_0|^2 - 1)$ in the noise term on the second line simply gives the slightly more complicated (but still standard) noise form that results if the total gain $G_T = G_N \equiv (G|\tilde{\gamma}_0|^2)^N$ of an amplifier results from N noisy amplifier sections of gain G cascaded with noiseless attenuators of transmission $|\tilde{\gamma}_0|^2$ between sections. Second, and more important, the effective input noise power $K_0 \rho_I \hbar\omega B$ appearing with the $G_N - 1$ gain term is the usual ρ_I photons per mode multiplied by the (potentially large) excess-noise factor K_0 for the $n = 0$ mode. But, finally, the input signal power I_{in} from an external signal source is (if properly adjoint coupled) also multiplied by the same factor K_0, as a result of the "excess initial mode excitation factor." Exactly the usual laser-amplifier noise figure is thus recovered, despite the excess spontaneous emission per mode.

As a practical matter, it is not clear that anyone would want to build a cascaded laser amplifier using an unstable lens-guide structure having large magnification and hence large diffraction losses per period. The above analysis applies equally well, however, to the signal buildup with time in injection-seeded laser oscillators or regenerative amplifiers, and these devices do in practice employ unstable resonator structures. This analysis demonstrates that for an injection-seeded, pulsed oscillator one should first try to make the single-pass gain G reasonably large compared to the mode eigenvalue $|\tilde{\gamma}_0|^2$ in order to optimize noise performance.

But even if this is done, there will be a (large) excess-noise factor K_0 in the effective input noise or initial noise in such structures. In effect, there will be K_0 initial noise photons per mode, rather than just one effective photon, in an unstable injection-seeded laser oscillator. This excess-noise factor per mode can, however, be exactly balanced by the excess initial mode excitation factor K_0, if adjoint coupling is employed. With adjoint coupling one really can inject more initial energy into one selected cavity eigenmode at $t = 0$ than is available in the external signal one uses as the injection signal.

B. Thermal noise in lossy systems

We consider next the opposite limit of lossy or absorbing systems, and discuss how the excess spontaneous emission should not lead to any violation of usual thermal-equilibrium considerations.

All the results in this paper will apply equally well to a periodic lens guide filled with lossy or absorbing atoms, rather than amplifying atoms, if one simply reverses the signs of σ, α, and the population inversion $N_2 - N_1$ in all the formulas. It may be more convenient in the absorbing case to rephrase the noise power per transverse mode $\rho_I \hbar\omega$ in all the expressions as a thermal noise power given by

$$\rho_I \hbar\omega = \frac{N_2}{N_1 - N_2}\hbar\omega = \frac{\hbar\omega}{e^{\hbar\omega/kT_a} - 1} \equiv kT_{\text{eq}} \ , \tag{36}$$

where T_{eq} is an equivalent temperature which becomes identical to the Boltzmann temperature $T_a \equiv k^{-1}\ln(N_1/N_2)$ in the limit $\hbar\omega \ll kT_a$. We can then use $kT_{\rm eq}$ as a shorthand for the more general quantum result for thermal excitation of a single mode.

In an absorbing periodic lens guide in thermal equilibrium, not only the lowest-loss eigenmode but in general some substantial number of higher-order, higher-loss eigenmodes will be thermally excited to significant amplitudes by spontaneous emission from the absorbing atoms. To calculate the total steady-state noise power in the lossy case, therefore, all modes and mode cross correlations must be retained. The mode coefficient $\tilde{c}_n$ at the output of, say, the $(k+1)$-the period of the lens guide, just after the end aperture, can be written, using a combination of Eqs. (15) and (23), as

$$\tilde{c}_n(p+;k+1)=g\tilde{\gamma}_n\tilde{c}_n(p+;k)-jg\tilde{\gamma}_n\tilde{\Lambda}_n^{(k)} , \tag{37}$$

where $g\equiv e^{-\alpha p}$ is the amplitude attenuation due to the atoms in one period. The quantity $\tilde{\Lambda}_n^{(k)}$ given by

$$\Lambda_n^{(k)}\equiv\int_{kp}^{(k+1)p}e^{\alpha z}\tilde{p}_n(z)dz \tag{38}$$

is a Langevin equivalent noise source at the input to the kth period of the lens guide produced by the spontaneous emission within that period. Since the spontaneous emission within different periods of the lens guide will be uncorrelated we can find, using earlier formulas, that

$$\langle\tilde{\Lambda}_n^{(k)}\tilde{\Lambda}_m^{*(l)}\rangle=(e^{2\alpha p}-1)2\eta_0\tilde{B}_{nm}kT_{\rm eq}B\delta_{kl} . \tag{39}$$

The mode coefficient $\tilde{c}_n$ after N periods will thus be given by

$$\tilde{c}_n(p+;N)=(g\tilde{\gamma}_n)^N\tilde{c}_n(p+;0)-j\sum_{k=1}^{N}(g\tilde{\gamma}_n)^k\tilde{\Lambda}_n^{(N-k)} \tag{40}$$

and from Eqs. (26) and (39) we can obtain, after N periods,

$$\begin{aligned}\langle\tilde{c}_n(p+;N)\tilde{c}_m^*(p+;N)\rangle\\ &=(G\tilde{\gamma}_n\tilde{\gamma}_m^*)^N\langle\tilde{c}_n(p+;0)\tilde{c}_m^*(p+;0)\rangle\\ &\quad+[1-(G\tilde{\gamma}_n\tilde{\gamma}_m^*)^N]\frac{(1-G)\tilde{\gamma}_n\tilde{\gamma}_m^*}{1-G\tilde{\gamma}_n\tilde{\gamma}_m^*}2\eta_0\tilde{B}_{nm}kT_{\rm eq}B .\end{aligned} \tag{41}$$

The factor $G\equiv g^2=e^{-2\alpha p}$ due to the atomic absorption in one period is of course less than unity. After a large enough number of periods so that $(G\tilde{\gamma}_n\tilde{\gamma}_m^*)^N\approx 0$, therefore, Eq. (41) will approach the stationary thermal-equilibrium limit

$$\langle\tilde{c}_n(p+)\tilde{c}_m^*(p+)\rangle=\frac{(1-G)\tilde{\gamma}_n\tilde{\gamma}_m^*}{1-G\tilde{\gamma}_n\tilde{\gamma}_m^*}2\eta_0\tilde{B}_{nm}kT_{\rm eq}B . \tag{42}$$

One can then substitute this into (20) to obtain the thermal or blackbody noise power propagating along the lossy line in the steady state.

The periodic structures we are considering consist of periodic segments of atomic medium having power transmission G and noise temperature $T_{\rm eq}$, separated by apertures with power transmission $|\tilde{\gamma}_n|^2$ but zero noise temperature. That is, in our model the apertures between periods account for the diffraction losses out the sides of the finite diameter lens guide, and we assume the energy lost in this fashion is simply radiated out into a "cold infinity" with no thermal noise coming back. Hence the apertures themselves emit no thermal noise. There may also in general be lenses, curved mirrors, and other lossless optical elements imbedded in the lossy atomic medium, but these are assumed to be transparent or lossless and hence also contribute no additional noise.

The bracketed ratio $(1-G)\tilde{\gamma}_n\tilde{\gamma}_m^*/(1-G\tilde{\gamma}_n\tilde{\gamma}_m^*)$ appearing in Eq. (42) can then be identified as a generalization of the usual reduction in effective noisiness or apparent noise temperature for a system which contains two loss mechanisms, only one of which has a finite temperature $T_{\rm eq}$ associated with it (cf. Appendix B). Let us assume for simplicity that the atomic loss per section is large enough so that $2\alpha p\gg 1$ or $G\ll 1$. The stationary thermal noise power emerging through any output aperture after many segments of such a lens guide can then be written as

$$I(p+)=kT_{\rm eq}B\left[\sum_n|\tilde{\gamma}_n|^2K_n+\sum_{\substack{n,m\\ n\neq m}}\tilde{\gamma}_n\tilde{\gamma}_m^*\tilde{A}_{nm}\tilde{B}_{nm}\right] . \tag{43}$$

This evidently expresses a sum over transverse modes in which individual modes may carry *more* than the normal thermal-equilibrium power $kT_{\rm eq}B$ per mode, since $|\tilde{\gamma}_n|^2K_n$ may be greater than 1 for strongly unstable systems. There are, however, also cross-power terms of magnitude $\tilde{\gamma}_n\tilde{\gamma}_m^*\tilde{A}_{nm}\tilde{B}_{nm}$ for $n\neq m$, many of which one presumes must have negative values.

It is of interest to consider the relationship between these results and the usual ideas of thermal equilibrium and blackbody radiation in absorbing systems. Consider, for example, a semi-infinite length of a periodic unstable lens guide extending back to $z=-\infty$ and terminating at a final output aperture at $z=0$ in a zero-temperature half plane, as shown in Fig. 3(a). One can view the final output aperture of this structure as an finite aperture through which one can look back into a more or less black semi-infinite holhraum filled with absorbing atoms at temperature T_a.

Equation (43) then tells how much total thermal noise power should be emerging from this aperture, summed over all the transverse modes of the lens-guide structure (assuming for simplicity that the atomic loss per period is large compared to the diffraction loss). For a conventional power-orthogonal waveguide, such as a loss-filled rectangular waveguide with dimensions large compared to a wavelength, it is simple to confirm that Eq. (43) will give exactly the usual thermal or blackbody radiation to be expected from an aperture A looking into a hohlraum at temperature T_a, or from a blackbody surface of the same area with unit emissivity. That is, for an ordinary power-orthogonal waveguide the K_n's and $\tilde{\gamma}_n$'s will have magnitude unity; the $\tilde{A}_{nm}$'s and $\tilde{B}_{nm}$'s will all be zero for

$n \neq m$; and there will be just $2\pi A/\lambda^2$ modes (counting polarizations) in the first summation. Hence the total noise power emerging from the aperture in bandwidth B will be

$$I = \frac{2\pi A}{\lambda^2} kT_{eq} B = \frac{2\pi A}{\lambda^2} \frac{hfB}{\exp(hf/kT_a) - 1} , \tag{44}$$

which is the usual blackbody result.

One may then ask whether this will also be true for the nonorthogonal example shown in Fig. 3(a), if the summations of Eq. (43) can be evaluated. Haus and Kawakami[2] were in fact able to verify this for the analogous gain-guided system shown in Fig. 3(b), which is also a nonorthogonal or biorthogonal waveguiding system, by appealing to the completeness properties of the infinite set of eigenmodes which one can verify will exist in such a gain-guided structure.

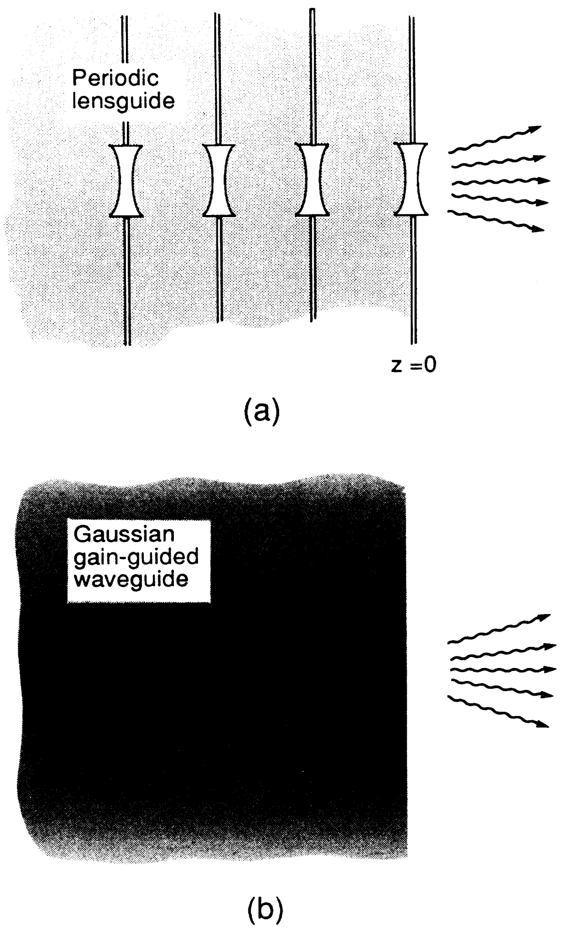

FIG. 3. (a) A semi-infinite length of hard-edged unstable periodic lens guide terminating at an aperture in an infinite plane. The periodic lens guide is filled with absorbing and hence spontaneously emitting atoms at a finite temperature T, but the lenses and dividing walls between sections are at zero temperature. (b) The analogous situation for a semi-infinite length of gain-guided or loss-guided waveguide. The radially varying lossy medium extends out to infinity and also has a uniform finite temperature T.

Unfortunately, for the case of more complex hard-edged unstable lens guides such as the one shown in Fig. 3(a), no rigorous proof of the completeness or even the existence of the lens-guide eigenmodes is available; and therefore one is unable at present to confirm that ordinary blackbody emission will be recovered from Eq. (43) for such a structure. From numerical studies one finds empirically that such an unstable lens-guide structure, rather than supporting $\approx 2\pi A/\lambda^2$ transverse modes, seems to posses only a small finite number (say 5–10) of transverse eigenmodes $\tilde{u}_n$ with moderate power losses. Moreover, the excess noise factors K_n for these moderate-loss modes are found to be large compared to unity, and to increase very rapidly with increasing mode number n. Beyond this it is not clear whether still higher-order modes exist or not. If they do, they have extremely small eigenvalues, with $|\tilde{\gamma}_n| \leq 0.001$, which makes them very difficult to calculate numerically; and they presumably also have very large excess-noise factors. The numerical difficulty in calculating these higher-order terms makes even the purely numerical exploration of Eq. (43) very difficult.

One can be sure from the results of Haus and Kawakami that the cross-mode correlation terms in Eqs. (41) and (43) will be important and that they will remove the apparent violation of thermodynamics produced by the excess noise factors K_n at least for simple gain-guided lens guides. Whether one will recover exactly the usual blackbody results using Eq. (43) for hard-edged unstable lens guides, however, is not clear. Further investigation of the unresolved questions concerning mode properties and noise properties in this situation would be very interesting.

C. Relationship to amplified spontaneous emission

Finally, there is an intriguing question as to what relationship if any there may be between the Petermann excess-noise factor for unstable systems as derived here and a recent discussion of amplified spontaneous emission (ASE) in Cassegrainian amplifiers given by Eimerl.[17] Using purely ray-optic arguments Eimerl notes that in these systems there can by rays of amplified spontaneous emission which make approximately twice as many bounces within the amplifier as do those rays associated with the usual signal beam traveling through the system. Hence there are ASE components which experience the square of the power gain which the usual signal experiences. This excess gain for some of the ASE components can perhaps be interpreted as saying that the net spontaneous emission from the amplifier will be significantly enhanced over its value in conventional geometrically stable or single-pass laser amplifiers, in a fashion which seems very comparable to the Petermann enhancement described in this paper.

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APPENDIX A: NOISE POLARIZATION TERM

We give here a brief heuristic derivation of the δ-function-correlated noise polarization term introduced in Eq. (4). The approach followed here is very analogous to that used earlier by Haus[18] and others.

The physical situation envisioned is a collection of two-level atoms with upper- and lower-level population densities N_2 and N_1. Since the spontaneous emission from each individual atom should be randomly phased and uncorrelated with nearby atoms, we assume this spontaneous emission can be represented by a noise polarization (i.e., a noiselike electric dipole moment per unit volume) whose phasor amplitude, call it $\tilde{P}_N(\mathbf{r})$, will be δ correlated in space in the form

$$\langle \tilde{P}_N(\mathbf{r})\tilde{P}_N^*(\mathbf{r}')\rangle = C_1\delta(\mathbf{r}-\mathbf{r}') \ , \tag{A1}$$

where the correlation volume for the three-dimensional δ function $\delta(\mathbf{r}-\mathbf{r}')$ is of atomic dimensions, and hence much smaller than an optical wavelength.

To determine the constant C_1 we note that the phasor amplitude $\tilde{\mu}_N$ of the oscillating electric dipole moment due to the total spontaneous emission in any small volume element V will is given by

$$\tilde{\mu}_N \equiv \int_V \tilde{P}_N(\mathbf{r})d\mathbf{r} \tag{A2}$$

where we take the volume element V to be large compared to the correlation volume for $\tilde{P}_N(\mathbf{r})$ but still small compared to an optical wavelength. The average power radiated by a noise dipole $\tilde{\mu}_N e^{j\omega t}$ from the small volume V is then given, from standard em theory, by

$$I_{\text{av}} = \frac{\eta_0\omega^4\langle\tilde{\mu}_N\tilde{\mu}_N^*\rangle}{12\pi c^2} = \frac{\eta_0\omega^4 C_1 V}{12\pi c^2} \ , \tag{A3}$$

where c is the velocity of light in the medium and we have used (A1) in deriving the second equality. But this emission should correspond to the spontaneous emission from the number of upper-level atoms N_2V in this small volume, assuming each atom radiates at a spontaneous emission rate (or Einstein A coefficient) γ_{rad}. Hence we can also write

$$I_{\text{av}} = \frac{\gamma_{\text{rad}}N_2 V\hbar\omega}{3} \ , \tag{A4}$$

where the factor of 3 in the denominator is inserted because the atomic dipoles will in general by randomly polarized and we want the emission only into one sense of polarization, not all polarization directions.

The constant C_1 corresponding to the total spontaneous emission into the full linewidth of a reasonably narrow atomic transition is thus given by

$$C_1(\text{full linewidth}) = \frac{4\pi c^2\hbar\gamma_{\text{rad}}N_2}{\eta_0\omega^3} \ . \tag{A5}$$

We are more inserted, however, in the noise emission only into a narrow frequency band at the center of the transition. If the transition is Lorentzian with a full width at half maximum (FWHM) linewidth $\Delta\omega_a$, and we consider only a small bandwidth $d\omega \equiv 2\pi B$ at line center, the noise within this bandwidth will be given by

$$C_1(\text{narrow band}) = \frac{d\omega}{\pi\,\Delta\omega_a/2}C_1(\text{full linewidth}) \ . \tag{A6}$$

In addition, the midband power gain coefficient 2α for a randomly polarized Lorentzian transition with radiative decay γ_{rad} and FWHM linewidth $\Delta\omega_a$ can be written as

$$2\alpha = \frac{\lambda^2}{2\pi}\frac{\gamma_{\text{rad}}}{\Delta\omega_a}(N_2-N_1) \ . \tag{A7}$$

Combining all of these gives the desired result

$$C_1(\text{bandwidth } B) = \frac{N_2}{N_2-N_1}\frac{16\hbar\alpha B}{\omega\eta_0} \ . \tag{A8}$$

Though derived for a Lorentzian transition, this result is in fact very general, for any kind of atomic response producing a net absorption coefficient α at frequency ω.

To verify this we could equally well note that the blackbody-radiation energy density U_{BR} in a narrow bandwidth B in any medium at temperature T can be written as

$$U_{\text{BR}} = \frac{\epsilon}{2}|\tilde{E}_{\text{BR}}|^2 = \frac{16\pi^2}{\lambda^3}\frac{\hbar B}{\exp(\hbar\omega/kT)-1} \ , \tag{A9}$$

where $\tilde{E}_{\text{BR}}$ is the phasor amplitude of the blackbody E field in bandwidth B. The energy absorbed from these fields in a volume V of an atomic medium of effective conductivity σ, and hence absorption coefficient $\alpha = \eta_0\sigma/2$, will be

$$I_{\text{abs}} = \frac{\sigma|\tilde{E}_{\text{BR}}|^2 V}{2} \ . \tag{A10}$$

Equating one-third of this narrow-band power absorption to the power emission of (A3) again gives directly the same result as (A8).

APPENDIX B: TWO-SECTION TRANSMISSION LINE NOISE

Consider a periodic transmission system composed of alternating segments of single-mode transmission line of length l_1 and l_2 having power transmissions $L_1 = \exp(-2\alpha_1 l_1)$ and $L_2 = \exp(-2\alpha_2 l_2)$, and noise temperatures T_1 (finite) and $T_2 = 0$, respectively. Using standard arguments, one can show that the thermal noise power $I(N)$ emerging after N segments of this line, observed at the output end of an l_2 segment, will be

$$I(N)=(L_1L_2)^N I(0)+[1-(L_1L_2)^N]\frac{(1-L_1)L_2}{1-L_1L_2}kT_iB$$
$$\approx\frac{(1-L_1)L_2}{1-L_1L_2}kT_1B \quad \text{if } L^N \ll 1 \ . \tag{B1}$$

This formula is the single-transverse-mode analog of Eqs. (41) and (42). It leads to the three limiting cases

$$I_N \approx \begin{cases} kT_1B, & \alpha_2 l_2 \to 0, \quad L_2 \to 1 \\ L_2 kT_1 B, & \alpha_1 l_1 \gg 1, \quad L_1 \to 0 \\ \dfrac{\alpha_1 l_1}{\alpha_1 l_1+\alpha_1 l_2}kT_1B, & \alpha_1 l_1, \alpha_2 l_2 \ll 1, \quad L_1, L_2 \to 1 \end{cases} \tag{B2}$$

and all three of these limits are obviously physically correct.

[1]K. Petermann, IEEE J. Quantum Electron. **QE-15**, 566 (1979).
[2]H. A. Haus and S. Kawakami, IEEE J. Quantum Electron. **QE-21**, 63 (1985).
[3]Y. Suematsu and K. Furuya, Trans. IECE Jpn. **E60**, 467 (1977).
[4]W. Streifer, D. R. Scifres, and R. D. Burhham, Electron. Lett. **17**, 933 (1981).
[5]E. Patzak, Electron. Lett. **18**, 278 (1982).
[6]D. Marcuse, Electron. Lett. **18**, 920 (1982).
[7]A. Yariv and S. Margalit, IEEE J. Quantum Electron. **QE-18**, 1831 (1982).
[8]T. Mamine, J. Appl. Phys. **54**, 2103 (1983).
[9]J. Arnaud, Electron. Lett. **19**, 688 (1983).
[10]J. Arnaud, Electron. Lett. **19**, 798 (1983).
[11]G. P. Agrawal, J. Opt. Soc. Am. Commun. **1**, 406 (1984).
[12]M. Newstein, IEEE J. Quantum. Electron. **QE-20**, 1270 (1984).
[13]M. B. El Mashade and J. Arnaud, IEEE J. Quantum Electron. **QE-22**, 505 (1986).
[14]C. H. Henry, J. Lightwave Technol. **LT-4**, 288 (1986).
[15]See, for example, A. E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986), Chaps. 14–23, especially Sec. 21.7
[16]A. E. Siegman, Opt. Commun. **31**, 369 (1979).
[17]David Eimerl, Appl. Opt. **26**, 1594 (1987).
[18]H. A. Haus, J. Appl. Phys. **32**, 493 (1961).

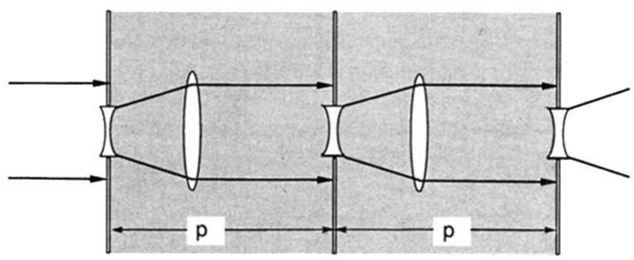

FIG. 1. Example of a periodic lens guide filled with absorbing or amplifying atomic medium, such as analyzed in this paper. The lens guide shown is a geometrically unstable confocal system to emphasize the characteristics of such systems. The absorbing apertures separating each period of the structure represent the diffractive loss out the sides or past the edges of the structure in each period. As such, they are taken to be perfectly absorbing but noise free or effectively at zero temperature.

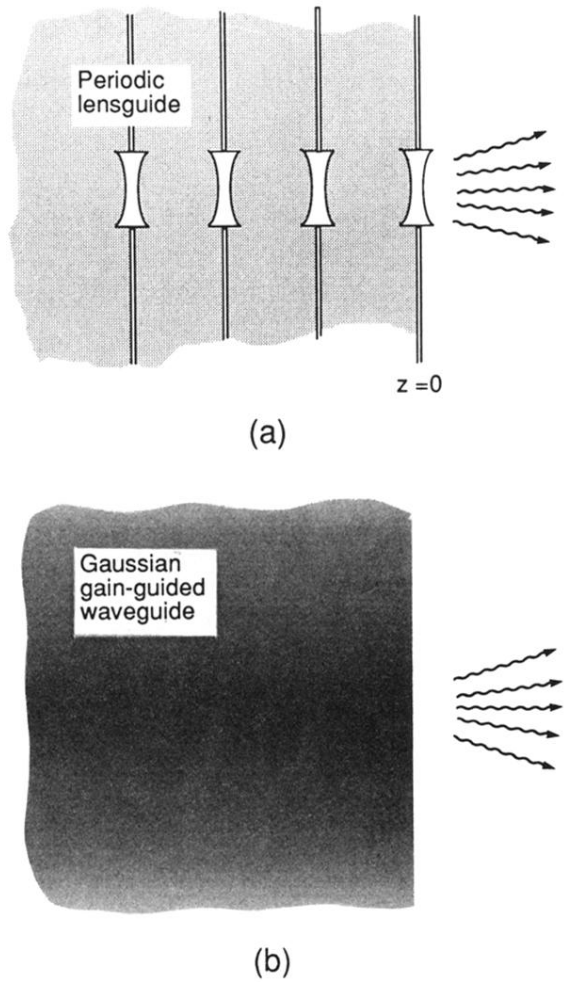

FIG. 3. (a) A semi-infinite length of hard-edged unstable periodic lens guide terminating at an aperture in an infinite plane. The periodic lens guide is filled with absorbing and hence spontaneously emitting atoms at a finite temperature T, but the lenses and dividing walls between sections are at zero temperature. (b) The analogous situation for a semi-infinite length of gain-guided or loss-guided waveguide. The radially varying lossy medium extends out to infinity and also has a uniform finite temperature T.