Field-symmetry-induced phase transitions and analytic continuation in n

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We explicitly show that analytic continuations in the number n of components in n-component cubic models usually break down at finite values of the field at low temperatures due to phase transitions in n. We also consider the replica system and show that replica symmetry must be broken, not only when $n \to 0$, but also when n is a positive integer.

I. INTRODUCTION

Our understanding of critical phenomena in systems with randomness or geometrical constraints is near a state of maturity. Most of these results have been obtained by the use of so-called analytic continuation in the number n (or q) of degrees of freedom of the auxiliary field to $n=0$, the most famous of these being the replica, the polymer, and the percolation limits, among others.[1] Formally, such an unphysical limit is very appealing, as it provides an alternative method for obtaining results with a great deal less work, and it has also provided new insights. However, naive analytic continuation also leads to problems at low temperatures ($T<T_c$),[1-3] most of which have not been fully resolved yet, even though attempts have been made to clarify some of the issues involved.[4-7]

Our aim in the present paper is twofold. We wish (i) to provide an explanation of various pathologies observed below T_c (Refs. 1-3) in terms of a phase transition in n and (ii) to justify our recent conjecture[2,8] regarding a new phase in the polymer problem and also to settle a controversy about it. We accomplish our results by developing a new method of analysis at a mean-field level that should prove useful in other contexts. In the axis model, this phase transition in n manifests itself only when $n<1$ in the form of a phase transition in the T-H plane below T_c, at finite values of the symmetry-breaking field given by $H=H_c\sim|\tau|^{\Delta}$, $\tau=(T-T_c)/T_c$, $\Delta=\frac{3}{2}$, and is a reflection of the fact that the zero-field ground state changes its behavior at $n=1$. In the diagonal model, the phase transition in n occurs at integer n and only affects the phase below H_c. Therefore, the analytic continuation in n cannot be carried out to $n=0$, if we wish to cross $H=H_c$. A similar situation occurs in random Ising system,[1,4,9] as noted by de Almeida and Thouless,[9] and in the polymer problem.[2,8] Continuation in n to $n=0$ above H_c can be carried out without any problem, since the phase transition in n does not affect this phase, as will be demonstrated here.

For the sake of concreteness, we will mostly restrict ourselves to cubic models,[10,11] which have phase transitions in the T-H plane due to a competition between externally imposed ordering and the natural or spontaneous ordering of the system. We show that while the analytic continuation is straightforward in the case of axial cubic symmetry, it is not so in the case of diagonal cubic symmetry. In the latter case, "systems" with noninteger numbers are different from their integral (n integer) counterparts, a result which was quite unexpected. We also consider the replica trick applied to the random Ising model[1,12] and find that replica symmetry should be broken at small n.

The layout of the paper is as follows. We define the cubic model in the next section. The cubic anisotropy term corresponding to Δ breaks the O(n) symmetry. For positive anisotropy ($\Delta>0$), the natural ordering in the model is along one of the main diagonals. We will call this the diagonal model. For negative anisotropy ($\Delta<0$), the natural ordering is along the axes. We will call this the axis model. We consider the effect of an external magnetic field whose symmetry can compete with the natural ordering of the model. The analysis of this section is at a mean-field level. We include the effect of fluctuations in Sec. III. The inclusion of fluctuations does not change the mean-field results of Sec. II. Section IV deals with the issue of the analytic continuation in n. This is the most important section. This section is also very technical and somewhat abstract in nature and requires a certain amount of patience on the part of the reader. In the next section (V), we discuss the nature of analytic continuation in the replica system for the random Ising model. To avoid complications, we restrict our present analysis to the transition temperature, where $r=0$ (see below). The concluding Sec. VI includes a short discussion of the current controversy alluded to above and its resolution, and a summary of our results.

II. CUBIC MODEL

We consider the n-component cubic model described by

$$\mathcal{H}_{n,\nu}=\frac{r}{2}\boldsymbol{\phi}^2+\frac{\lambda}{4!}(\boldsymbol{\phi}^2)^2+\frac{\Delta}{4}\phi^4-\mathbf{H}_\nu\cdot\boldsymbol{\phi}\ , \tag{1}$$

where $\boldsymbol{\phi}^2=\sum_{\alpha=1}^{n}\phi_\alpha^2$, $\phi^4=\sum_{\alpha=1}^{n}\phi_\alpha^4$, and the external field is $\mathbf{H}_\nu=(H_\nu/\nu^{1/2})(1,1,\ldots,1,0,0,\ldots,0)$ with $\nu\le n$ nonzero but equal components. As usual, r is proportional to τ and will be considered negative here. Moreover, the cubic anisotropy coefficient Δ is an irrelevant

dangerous variable[11] and cannot be safely set equal to zero below T_c. The presence of this term in (1) breaks the rotational invariance, i.e., the O(n) symmetry of the interaction. For $\Delta>0$ (diagonal model) the spontaneous magnetization $\langle\phi\rangle$ points along the main diagonal. For $\Delta<0$ (axis model) $\langle\phi\rangle$ points along one of the $2n$ axes. As we will see below, this is true as long as $n\geq 1$, but not when $n<1$. For $\Delta=0$, the spontaneous magnetization can point in any direction, in accordance with the O(n) invariance of the interaction.

The above Hamiltonian, provided it is stable, describes cubic models when $\lambda>0$. When $\lambda<0$ and $\Delta>0$, it describes weakly random systems.[13] With the choice $\Delta=n\Delta_0$ and with any $\nu\neq n$, it is easily seen (see Fig. 1) that the above Hamiltonian generates *exactly* the same set of polymer diagrams as the Hamiltonian in the O(n) model in the limit $n\to 0$. For $\Delta_0<0$, this is the mapping proposed by Hilhorst[14] between polymers and the discrete axis model.[6] We have just extended this to the diagonal model. With $\nu=n$, the diagrams like Fig. 1(c) survive as $n\to 0$ and give rise to tree graphs (graphs with *no* loops). This shows that the cubic model (1) is different from the O(n) model as $n\to 0$.[15]

For $\nu\neq n$, we note that all diagrams with branchings [Fig. 1(c)] disappear as $n\to 0$. Since the only effect of the cubic term involving Δ, regardless of its sign, is to give rise to branchings we conclude that the partition function corresponding to the set of diagrams that survive the $n\to 0$ limit *cannot* depend on Δ_0. In other words, the corresponding free energy of the polymer system must be *independent* of Δ_0 in the limit $n\to 0$, provided $\nu\neq n$. We will have occasion to use this observation below.

Let us set $\mathbf{H}_\nu=0$ in (1) and try the following solution:

$$\langle\phi\rangle=\mathbf{M}_\mu=(M_\mu/\mu^{1/2})(1,1,\ldots,1,0,0,\ldots,0) \tag{2}$$

with μ nonzero but equal components. Let F_μ denote the local minimum of energy at $\mathbf{M}_\mu$. For physical n, we find that

$$F_\mu=\frac{-r^2/4}{\lambda/6+\Delta/\mu},\quad M_\mu^2=\frac{-r}{\lambda/6+\Delta/\mu}\ . \tag{3}$$

In the following, we will assume that $\lambda>0$. The analysis can be easily extended to $\lambda<0$. The ordering of various levels depends upon the sign of Δ: $F_n<\cdots<F_2<F_1$ for $\Delta>0$ and $F_1<F_2<\cdots<F_n$ for $\Delta<0$. In zero field, the spontaneous magnetization must be $\mathbf{M}=\mathbf{M}_n$ ($\Delta>0$) or $\mathbf{M}=\mathbf{M}_1$ ($\Delta<0$). In the presence of the external field $\mathbf{H}_\nu$, $\mathbf{M}_\mu$ changes from (2) to $\mathbf{M}_\mu=(\mathbf{m}_1,\mathbf{m}_2,0)$ where $\mathbf{m}_1=m_1(1,1,\ldots,1)$ with n_1 components, $\mathbf{m}_2=m_2(1,1,\ldots,1)$ with n_2 components and the remaining $n-(n_1+n_2)$ components of $\mathbf{M}_\mu$ are zero. For $\mu\geq\nu$, $n_1=\nu$ and $n_2=\mu-\nu$. For $\mu\leq\nu$, $n_1=\mu$ and $n_2=\nu-\mu$. Minimization of (1) now yields [$h=H_\nu/\nu^{1/2}>0$; $Z_\mu=r+\lambda(n_1m_1^2+n_2m_2^2)/6$]

$$h+m_1[r+(\lambda\nu/6+\Delta)m_1^2]=0 \quad \text{for } \mu=\nu\ ; \tag{4a}$$

$$\begin{aligned} &h+\Delta m_1m_2(m_1+m_2)=0\ ,\\ &Z_\mu+\Delta(m_1^2+m_1m_2+m_2^2)=0 \quad \text{for } \mu<\nu\ ; \end{aligned} \tag{4b}$$

$$\begin{aligned} &h-\Delta m_1(m_1^2-m_2^2)=0\ ,\\ &Z_\mu+\Delta m_2^2=0 \quad \text{for } \mu>\nu\ ; \end{aligned} \tag{4c}$$

where the first part of (4b) is valid provided $m_1\neq m_2$. These equations are easily analyzed graphically.[16] Equations involving Z_μ describe ellipses and those involving h describe open curves (Fig. 2), and the solutions of (4b) and (4c) are given by their crossings. As h increases, the open curves move outward and eventually lie outside the ellipses. Therefore, the $\mu\neq\nu$ levels cease to exist for large h. Here, only the $\mu=\nu$ level exists, for which $\mathbf{M}_\nu$ is parallel to $\mathbf{H}_\nu$, with m_1 given by (4a).

At $h=0$, the stable solution at low temperatures corresponds to either $m_1=m_2$ ($\mu=n$) for $\Delta>0$ or $m_2=0$ ($\mu=1$) for $\Delta<0$. These solutions correspond to the points D_0 and A_0 in Fig. 2, and are obtained from (4b) and (4c). For $h=0$, the solution of (4a) is *not* physical, i.e., stable. As h is raised, the solutions of (4b) and (4c) eventually disappear and we must consider the only remaining solution that is given by (4a). Therefore, at large enough h, to be determined below, the stable solution corresponds to the one given by (4a).

Consider the zero-field stable solutions D_0 and A_0 (Fig. 2). As h increases, they move towards D_c and A_c, respectively. At D_c, $m_2=0$ and $H_\nu=H_c=\Delta\nu^{1/2}m_1^3\sim|\tau|^{3/2}$, and (4a) and (4c) become identical. As a matter of fact, it is easily seen that $F_\mu=F_\nu$ for all $\mu\neq\nu$ at D_c, where $m_2=0$. Below H_c, $\mu=n$ is the lowest level.

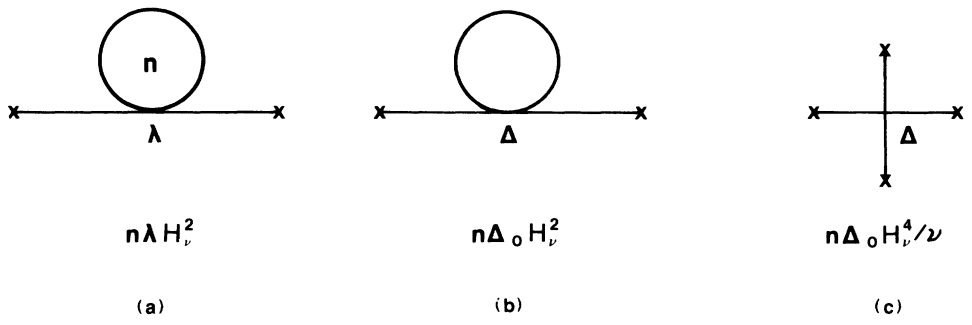

FIG. 1. Various diagrams with their corresponding weights. Each cross ($\times$) contributes $H_\nu/\nu^{1/2}$ and each diagram must be summed over ν components of $\mathbf{H}_\nu$.

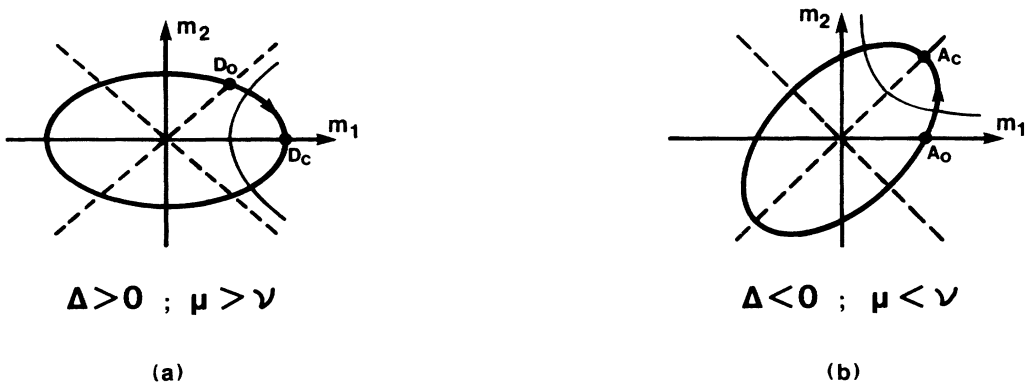

FIG. 2. Graphical solution for (a) $\Delta>0$, $\mu>\nu$ and (b) $\Delta<0$, $\mu<\nu$. For $\Delta<0$, the axes of the ellipse, in general, do not coincide with $m_1=\pm m_2$.

Above H_c, $\mu=\nu$ is the only surviving level and, therefore, the stable level. There is a phase transition at H_c between these two levels. For the axis model ($\Delta<0$), the situation is somewhat complicated [Fig. 2(b)]. A detailed analysis[16] shows that $\mu=1$ is stable below some $H_c\sim|\tau|^{3/2}$ whereas $\mu=\nu$ is stable above it, with a phase transition at H_c between the two levels. This is not surprising in view of Fig. 2(b). It should also be evident that there are no phase transitions if $\nu=1$ for $\Delta<0$ and $\nu=n$ for $\Delta>0$.

A similar graphical analysis shows that levels $\mu>\nu$ ($\Delta<0$) give negative m_2 and levels $\mu<\nu$ ($\Delta>0$) give negative susceptibility $\partial m_1/\partial h$, as h increases. Therefore, these levels are unphysical for physical values of n and need not be considered.

III. FLUCTUATIONS

Let us now include the effect of fluctuations.[17] Expressing the αth component of $\mathbf{M}_\nu$ by $M_\alpha+\zeta_\alpha$, where ζ_α is the fluctuating part, we find that the $(n-\nu)$ transverse fluctuations (T) ($\alpha=\nu+1,\ldots,n$) have the mass squared r_T given by

$$r_T=H/M_\nu-\Delta M_\nu^2/\nu\ . \tag{5a}$$

The rest of the components are conveniently described by the following linear combinations: the longitudinal fluctuation $L=\sum_{\alpha=1}^{\nu}\zeta_\alpha$ and the $(\nu-1)$ transverse fluctuations (t), described by $\xi_k=\sum_{\alpha=1}^{k}\zeta_\alpha-k\zeta_{k+1}$, $k=1,2,\ldots,\nu-1$, with their masses given by[17]

$$r_L=r+3M_\nu^2(\lambda/6+\Delta/\nu) \tag{5b}$$

for the longitudinal fluctuation, and

$$r_t=H/M_\nu+2\Delta M_\nu^2/\nu \tag{5c}$$

for the $(\nu-1)$ transverse fluctuations. We observe that for $\Delta\neq0$, the two transverse modes T and t have different masses. We also note from (5) that with $\nu=n$, the transverse modes remain massive as $n\to0$, provided $\Delta=n\Delta_0$. Therefore, $\nu=n$, $n\to0$ is different from $\nu\neq n$, $n\to0$, as explained above [Fig. 1(c)]. For $\nu=n$, the T modes do not exist, whereas the t modes cease to exist for $\nu=1$. For $\Delta>0$, the T modes become critical when $r_T=0$. For $\Delta<0$, the t modes become unstable at some small fields. Therefore, there must be some sort of phase transitions at some field $H_\nu=H_c$. These phase transitions are in accordance with the above analysis of (4). Moreover, (5) also shows that no phase transition at nonzero field will occur if the external field is compatible with the natural ordering: $\nu=n$ for $\Delta>0$; $\nu=1$ for $\Delta<0$. For $\nu=1$, the axis model has no phase transition, but the diagonal model does, because r_T becomes negative at small H_ν. Since the two models are equivalent at $n=0$ above H_c, we are forced to conclude that a phase transition must also occur in the H-T plane for the axis model as $n\to0$ when $\nu=1$, as shown below.

That there must be some sort of phase transition as $h\to0$ is also evident from the following simple argument. The free energy in the ground state at $h=0$ is given by F_n if $\Delta>0$ and F_1 if $\Delta<0$ for physical n's. Therefore, in the absence of any phase transition, F_0 ($\Delta>0$) and F_1 ($\Delta<0$) will be the corresponding free energies as $n\to0$. For all $\nu\neq n$, $n\to0$, the cubic model describes the *same* polymer system whose free energy *cannot* depend upon Δ_0 as was discussed before [see the discussion just before (2)]. However, F_0 certainly depends on Δ_0 and *cannot* describe the polymer system. Moreover, $F_0\neq F_1$, which contradicts our claim that as $n\to0$ ($\nu\neq n$) both models describe the same polymer system and, therefore, must have the same free energies. The only conclusion that can be drawn is that the polymer analogy *cannot* work all the way down to $h=0$.

IV. ANALYTIC CONTINUATION

We must now face the issue of the analytic continuation in n, with *fixed* ν. We find that for all real $n\geq\nu$, the ordering of the levels is given by (3), but with the following important difference. Levels with $\mu>n$, which do not exist for physical systems, appear under analytic continuation, as their multiplicities are given by the binomial coefficients $\binom{n}{\mu}$ which vanish whenever n is an integer less than μ, but not when n is not an integer. Therefore, these levels must be included in our analytic continuation, as was done by Bray and Moore[12] and which allowed them

to take the $\mu \to \infty$ limit in their investigation of the Ising spin glass. For $\Delta < 0$, these levels do not create any problem as they lie above the $\mu = n$ level. In the following, we will, therefore, ignore these levels in the axis model. However, for $\Delta > 0$, these levels lie below n and the ground state corresponds to $\mu \to \infty$. For integer n, the phase transition is between the $\mu = \nu$ and the $\mu = n$ levels. For noninteger n, it is between the $\mu = \nu$ and the $\mu \to \infty$ levels. These transitions occur at $H_c \sim |\tau|^{3/2}$ and are due to the competition between two *different* symmetries. The analytic continuation in n is not possible because of the nonanalyticity when n is integer, except above H_c, where the system is always described by the $\mu = \nu$ level. This nonanalyticity is a reflection of different ground states at integer and noninteger n's. This problem in analytic continuation in n would persist even if we choose $\nu = n$.

Let us now consider the axis model. It is only the $\mu = n$ level that requires analytic continuation in n. For $n > \nu$, $\mathbf{M}_n = (\mathbf{m}_1, \mathbf{m}_2)$ where $m_1 = m_1'/n^{1/2}$, $m_2 = m_2'/n^{1/2}$ are given by (4c). We have introduced m_1' and m_2' for convenience as we will see below. At $H_\nu = 0$, $m_1' = m_2' = M_n$ given in (3) and remain *finite* and nonzero, even as $n \to 0$. This is one of the reasons for introducing the primed quantities. As n is continued across $n = \nu$ to $n < \nu$, let us take the solutions of (4c) to define $\mathbf{M}_n$ for nonzero h. We find that m_1' and m_2' are given by

$$h - \Delta_0 m_1' (m_1'^2 - m_2'^2)/n^{1/2} = 0 ,$$

$$r + (\lambda/6)[m_2'^2 + \nu(m_1'^2 - m_2'^2)/n] + \Delta_0 m_2'^2 = 0 ,$$

where we have introduced the primed quantities in (4c). From the above equations, it is easily seen that m_1' satisfies the following equation:

$$r m_1' + \left[\frac{\lambda}{6} + \Delta_0\right]\left[m_1'^3 - \frac{n^{1/2} h}{\Delta_0}\right] + \frac{h\lambda\nu}{6\Delta_0 n^{1/2}} = 0 .$$

As $n \to 0$, m_1' diverges as $n^{-1/6}$ for any h and the corresponding free energy F_n becomes unbounded from below (except at $H_\nu = 0$). If we accept this analytic continuation, the entire phase below T_c ($r < 0$, $h > 0$) is determined by the *infinitely* deep $\mu = n$ level and there is *no* hope of any polymer analogy below T_c for any $H_\nu \neq 0$. Note that the free energy for the polymer system must be bounded from below. That something must be wrong with this continuation is also evident when we note that (4c) does not reduce to (4a) when $n = \nu$. Therefore, we need to obtain another continuation. If we choose for $\mathbf{M}_n$, $n < \nu$, the form $\mathbf{M}_n = (\mathbf{m}_1, \mathbf{m}_2, 0)$ valid for $\mu < \nu$, $\mu = n$, no problem arises.[15] Here $\mathbf{m}_1$ has $\mu = n$ equal components $m_1'/n^{1/2}$, $\mathbf{m}_2$ has $\nu - n$ equal components m_2 and the remaining $n - \nu$ components are zero:

$$\mathbf{m}_1 = (m_1'/n^{1/2})(1, 1, \ldots, 1) ,$$

$$\mathbf{m}_2 = m_2(1, 1, \ldots, 1) .$$

The values of m_1' and m_2 are given by (4b). For large enough fields ($H_\nu > H_c$) this level $\mu = n$ does not exist, and (4a) yields the stable solution $\langle \phi \rangle = \mathbf{M}_\nu$. This change of form of the $\mu = n$ level causes nonanalyticity at $n = \nu$ and amounts to a "phase transition" in n at $n = \nu$. It does not affect the phase above H_c, which is determined by the level $\mu = \nu$ and not by the level $\mu = n$. Since the level $\mu = 1$ is the stable level below H_c, this phase transition in n also does not affect the phase below H_c, as long as $n \geq 1$. Therefore, let us consider $n < 1$, and set $\nu = 1$ for simplicity. For $n < 1$, the ground state at $H_\nu = 0$ must have $\mathbf{M} = \mathbf{M}_n$ since $F_n < F_1$. There should be a phase transition from $\mathbf{M} = \mathbf{M}_1$ at large fields to $\mathbf{M} = \mathbf{M}_n$ at very small fields at some $H_c \sim |\tau|^{\Delta}$, $\Delta = \frac{3}{2}$, due to the competition between two symmetries, similar to what happens for physical n's above. Of course, we could have anticipated such a phase transition by noting that the ground state at $H_\nu = 0$ changes from $\mu = 1$ to $\mu = n$ as n becomes less than unity. The above phase transition at H_c is a reflection of this fact.[15]

V. REPLICA SYSTEM

Let us now apply our method to the replica system with the corresponding effective Hamiltonian[1,12]

$$\mathcal{H}_n = \frac{r}{4}\mathrm{Tr}Q^2 - \frac{w}{6}\mathrm{Tr}Q^3 - \frac{u}{8}\mathrm{Tr}Q^4 + \frac{u}{4}\sum_{\alpha,\beta} Q_{\alpha\beta}^2 Q_{\alpha\delta}^2 - \frac{u}{12}\sum_{\alpha,\beta} Q_{\alpha\beta}^4 , \quad (6)$$

where we have used the notations of Bray and Moore.[12] Let us try the following solutions in (6): $Q = Q_{(\mu)}$ with $Q_{\alpha\beta} = q$, $\alpha \neq \beta$, $\alpha, \beta \leq \mu$, and $Q_{\alpha\beta} = 0$ otherwise. The choice of $\mu = n$ gives the usual mean-field solution used in the literature.[1] A simple algebra yields

$$F_\mu = \tfrac{1}{4}\mu(\mu-1)q^2\left[r - \tfrac{2}{3}\bar{w}q + \frac{\bar{u}}{2}q^2\right] \quad (7)$$

valid for $\mu \geq 2$, and where $\bar{w} = w(\mu - 2)$ and $\bar{u} = u[\frac{7}{12} - (\mu - \frac{5}{2})^2]$. We will consider only those values of μ or n for which $\bar{u}$ is positive so that F_μ is stable: $2.5 - (\frac{7}{12})^{1/2} < \mu$ or $n < 2.5 + (\frac{7}{12})^{1/2}$. To simplify our present argument, we consider only $r = 0$. The minimum of F_μ and the corresponding stable q_μ are given by

$$F_\mu = -\frac{\mu(\mu-1)(\mu-2)^2 w^2}{[\frac{7}{12} - (\mu - \frac{5}{2})^2]^3 u^3}, \quad q_\mu = \frac{\bar{w}}{\bar{u}} \geq 0 . \quad (8)$$

Now, it is easy to see that for $n > 3$, F_n is stable, whereas F_3 is stable below $n = 3$, with a phase transition in n at $n = 3$. Even though this result is obtained for $\mathcal{H}_n$ in (6), we believe this to be a genuine feature of the theory. We also suggest that this phase transition in n reflects itself in the de Almeida–Thouless (A-T) line[9] in the replica trick, a situation similar to the polymer problem discussed above.

VI. CONCLUSIONS

The above analysis suggests very strongly that analytic continuations in n must fail in general at some nonzero H_c below T_c in the "ordered" phase due to a phase transition in n. At present, we do not know how to extend

our method to the continuous $O(n)$ model. (However, see below.) The locus of H_c near T_c is described by $H_c \sim |\tau|^{\Delta}$, and is similar in shape to the de Almeida–Thouless (A-T) line[9] in the Ising spin-glass or the curve AC in the $O(n)$ model.[2] For the axis model, the analytic continuation in n causes a problem only below H_c for $n<1$ because the $\mu=n<1$ level controls the physics below H_c but not above it. As a matter of fact, it can be shown[15] in a mean-field calculation that the phase below H_c in the discrete version of the axis model[6,15] has no polymer analogue, suggesting that the phase below H_c may have nothing to do with the original system (polymers, random systems, etc.). It must be obtained by some independent methods[8,18] that deal directly with the systems under consideration without any formal limits.

Our results here should also settle the controversy raised recently[19,20] about the new phase in the polymer problem.[2,8] It is not hard to understand the origin of the controversy. These authors in Ref. 19 put too much faith in the ϵ-expansion calculation of the equation of state of the O(n) model, a calculation believed to be certainly correct for all $n \geq 1$. However, there is *no* guarantee that the analytic continuation to $n<1$ of this ϵ-expansion equation of state will be correct. As a matter of fact, it certainly cannot be correct as it *violates* the constraint[15] $x \geq -1$ on the scaling variable $x=\tau/M^{1/\beta}$ (M is the magnetization) below some curve given by $H=H_c \sim |\tau|^{\Delta}$ in the scaling regime for all $n<1$. The fact that the ϵ-expansion calculation, which is believed to be under control for $n>1$, is *no longer* under control for $n<1$ when $H<H_c$ is presumably an indication of a phase transition in n at $n=1$. We must invoke new rules of calculation for $n<1$ to obtain sensible results. This suggests a phase transition in n at $n=1$. This situation is similar to the situation observed in the cubic model. This phase transition then manifests itself in the H-T plane as a phase transition across H_c as discussed above. There is another serious problem with the analytic continuation of the ϵ-expansion calculation in Ref. 19. The continuation violates a thermodynamic relation: this relation shows that $(\partial M/\partial\tau)_H$ must *vanish* whenever the longitudinal susceptibility vanishes.[20] The equation of state in Ref. 19 violates this. The same is true of the counterexamples given by Wheeler *et al.*[19] to show that $x<-1$. In our previous work,[2] we have argued for a new phase below H_c due to various pathologies. It has become clear only now that these pathologies are due to a phase transition in n at $n=1$. It is this phase transition in n which manifests itself in a new phase below H_c in the H-T plane for $n<1$. The authors in Ref. 19 overlook the phase that appears below H_c and claim that the continuation in n does not break down. We have explicitly demonstrated here and in Ref. 15 that this is not the case.

We can investigate what must happen under analytic continuation in general, by considering all possible local "minima" at low temperatures. A warning must be offered at this point. It is possible that we have not discovered all the local minima, which may be futile in some cases. But this does not invalidate our explicit demonstration here. In general, one must ensure that all minima have been considered, before it can be argued that the continuation does not break down.

In summary, we have explicitly demonstrated the occurrence of a phase transition in n under analytic continuation in n. This phase transition exhibits itself in the form of a phase transition across $H=H_c \sim |\tau|^{\Delta}$, $\Delta=\frac{3}{2}$ in the mean-field calculation, in the low-temperature phase ($\tau<0$) only; it does not affect the high-temperature phase. In the axis model, this phase transition in the H-T plane occurs only for $n<1$. In the diagonal model, it occurs at integer n. A similar phase transition is also expected in the replica system. Therefore, the theory cannot by analytically continued to $n=0$ in the region below H_c. The theory can be continued to $n=0$ above H_c. The polymer limit $n=0$ identifying magnetic systems with polymers can be taken without any problem only above H_c, but not below it, as shown here, due to the phase transition at H_c.

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