Natural linewidths of a laser with a saturable absorber and a dye laser

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(Received 27 May 1987)

The natural linewidths of a laser with a saturable absorber and of a dye laser have been calculated using a Fokker-Planck approach.

I. INTRODUCTION

The photon statistics of a laser with a saturable absorber and of a dye laser have been studied extensively.[1–6] Many interesting features of these systems have been predicted. For instance, it is shown that these systems exhibit a first-order phase transition as opposed to the second-order phase transition exhibited by an ordinary single-mode laser.[3,4]

In this paper we evaluate the natural linewidth of a laser with a saturable absorber and a dye laser. We start with the equations of motion for the reduced density matrix of the field that are obtained by suitably generalizing the Scully-Lamb theory.[5,6] These equations are transformed into equivalent diffusion equations for the coherent-state representation. The phase-diffusion constant gives the natural linewidth. Our results indicate that the absorber in the laser with a saturable absorber contributes to an increase in the linewidth, whereas the built-in nature of the absorber in dye molecules tends to reduce the linewidth in a dye laser.

II. LASER WITH A SATURABLE ABSORBER

We consider the interaction between a quantized electromagnetic field and a collection of two-level atoms inside a laser cavity. In a laser with a saturable absorber the atoms are of two different species; the first one (active atoms) is pumped in the upper state $|a\rangle$ at a rate r_a, and the second one (absorber) is pumped in the lower state $|d\rangle$ at a rate r_d, as shown in Fig. 1. The equation of motion for the reduced density matrix for the field is given by[5,6]

$$\dot{\rho}_{nn'} = -\left[\frac{N'_{nn'}A}{1+N_{nn'}(B/A)}\right]\rho_{nn'} + \left[\frac{\sqrt{nn'}\,A}{1+N_{n-1,n'-1}(B/A)}\right]\rho_{n-1,n'-1} - \left[\frac{M'_{nn'}D}{1+M_{nn'}(E/D)}\right]\rho_{nn'} + \left[\frac{\sqrt{(n+1)(n'+1)}}{1+M_{n+1,n'+1}(E/D)}\right]\rho_{n+1,n'+1} - \frac{C}{2}(n+n')\rho_{nn'} + C\sqrt{(n+1)(n'+1)}\,\rho_{n+1,n'+1} , \tag{1}$$

where A and B are the gain and saturation parameters for the active atoms, D and E are the absorption and saturation parameters for the absorber, C is the cavity loss parameter, and

$$N'_{nn'} = \tfrac{1}{2}(n+n'+2) + \frac{1}{8}\frac{B}{A}(n-n')^2 , \tag{2a}$$

$$N_{nn'} = \tfrac{1}{2}(n+n'+2) + \frac{1}{16}\frac{B}{A}(n-n')^2 , \tag{2b}$$

$$M'_{nn'} = \tfrac{1}{2}(n+n') + \frac{1}{8}\left[\frac{E}{D}\right](n-n')^2 , \tag{2c}$$

$$M_{nn'} = \tfrac{1}{2}(n+n') + \frac{1}{16}\left[\frac{E}{D}\right](n-n')^2 . \tag{2d}$$

The photon statistics in the steady state can be studied by considering the equation of motion for the diagonal elements of the reduced density matrix of the field. It follows from Eq. (1) that

$$\dot{\rho}_{nn} = -\frac{A(n+1)}{1+(B/A)(n+1)}\rho_{nn} + \frac{An}{1+(B/A)n}\rho_{n-1,n-1} + C(n+1)\rho_{n+1,n+1} - Cn\rho_{nn} + \frac{nD}{1+(E/D)n}\rho_{nn} + \frac{(n+1)D}{1+(E/D)(n+1)}\rho_{n+1,n+1} . \tag{3}$$

A steady-state solution ($\dot{\rho}_{nn}\equiv 0$) of the above equation is obtained in a straightforward manner using the detailed balance condition,

$$p(n) = p(0)\prod_{k=0}^{n}\left[\frac{A/C}{(1+(B/A)k)\left[1+\dfrac{D/C}{1+(E/D)k}\right]}\right] . \tag{4}$$

If the distribution function peaks at n_p, then the quantity in the very large parentheses approaches unity for $k=n_p$. For a sufficiently large value of the mean number of photons $\langle n\rangle$, the distribution is symmetric about n_p, so that $\langle n\rangle \simeq n_p$. The quantity $\langle n\rangle$ can then be evaluated from the following equation:

$$\frac{A/C}{(1+(B/A)\langle n\rangle)\left[1+\dfrac{D/C}{1+(E/D)\langle n\rangle}\right]}=1 . \quad (5)$$

We now derive the equation for the coherent-state representation of the field $P(\alpha)$, which is defined by

$$\rho=\int P(\alpha)\,|\alpha\rangle\langle\alpha|\,d^2\alpha . \quad (6)$$

Here $|\alpha\rangle$ is a coherent state. We can translate Eq. (1) for $\rho_{nn'}$ into the following equivalent equation for $P(\alpha)$. The resulting equation is

$$\dot{P}(\alpha)=-\frac{A}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*-2\frac{\partial^2}{\partial\alpha^*\partial\alpha}+\frac{B}{4A}\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]M$$
$$+\frac{D}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*-\frac{E}{4D}\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]L+\frac{C}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]P , \quad (7)$$

where

$$M=\left[1-\frac{B}{2A}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*-2(1+|\alpha|^2)\right]+\frac{B^2}{16A^2}\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]^{-1}P , \quad (8a)$$

$$L=\left[1-\frac{E}{2D}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*-2|\alpha|^2\right]+\frac{E^2}{16D^2}\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]^{-1}P . \quad (8b)$$

In terms of new variables (r,θ) defined by $\alpha=r\exp(i\theta)$, Eqs. (7), (8a), and (8b) can be written as

$$\dot{p}=-\frac{A}{2}\left[\frac{1}{r}\frac{\partial}{\partial r}r^2-\frac{1}{2r^2}\left[r\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial\theta^2}\right]-\frac{B}{4A}\frac{\partial^2}{\partial\theta^2}\right]M+\frac{C}{2}\left[\frac{1}{r}\frac{\partial}{\partial r}r^2\right]p+\frac{D}{2}\left[\left[\frac{1}{r}\frac{\partial}{\partial r}r^2\right]+\frac{E}{4D}\frac{\partial^2}{\partial\theta^2}\right]L , \quad (9)$$

where

$$M=\left[1-\frac{B}{2A}\left[\frac{1}{r}\frac{\partial}{\partial r}r^2-2(1+r^2)\right]-\frac{B}{16A^2}\frac{\partial^2}{\partial\theta^2}\right]^{-1}P , \quad (10a)$$

$$L=\left[1-\frac{E}{2D}\left[\frac{1}{r}\frac{\partial}{\partial r}r^2-2r^2\right]-\frac{E^2}{16D^2}\frac{\partial^2}{\partial\theta^2}\right]^{-1}P . \quad (10b)$$

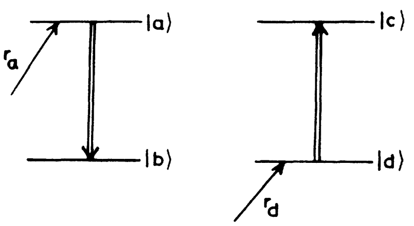

FIG. 1. Energy-level diagram of the active medium and the saturable absorber.

For a laser operating far above threshold the changes in P along the radial coordinate are restricted by the steady-state operating conditions. Hence these changes can be neglected. For $\langle n\rangle \gg 1$ we can replace r^2 by $\langle n\rangle$ in Eqs. (9), (10a), and (10b). Since $(B/A)\ll 1$ and $(E/D)\ll 1$, therefore we can ignore the (B^2/A^2) and (E^2/D^2) terms in Eqs. (9a) and (9b). After making these approximations, Eqs. (9), (10a), and (10b) can be rewritten in the following simple forms:

$$\dot{P}(\theta)=\frac{A}{2}\left[\frac{1}{2\langle n\rangle}+\frac{B}{4A}\right]\frac{\partial^2}{\partial\theta^2}M+\frac{E}{8}\frac{\partial^2}{\partial\theta^2}L , \quad (11)$$

where

$$M=\left[1+\frac{B}{A}\langle n\rangle\right]^{-1}P , \quad (12a)$$

$$L=\left[1+\frac{E}{D}\langle n\rangle\right]^{-1}P . \quad (12b)$$

On substituting for M and L from Eqs. (12a) and (12b) into Eq. (11), we obtain

$$P(\theta)=D_\theta\frac{\partial^2 P}{\partial\theta^2} , \quad (13)$$

where we have the diffusion constant,

$$D_\theta=\left[\frac{A}{4\langle n\rangle}+\frac{B}{8}\right]\frac{1}{[1+(B/A)\langle n\rangle]}+\frac{E}{8[1+(E/D)\langle n\rangle]}\ . \tag{14}$$

This expression for D_θ can be simplified considerably by using Eq. (5). After some straightforward calculations we obtain

$$D_\theta=\frac{1}{8\langle n\rangle}(A+C+D)\ . \tag{15}$$

This expression clearly shows that the role of absorber atoms is to enhance the laser linewidth for a given value of $\langle n\rangle$. All the gain and loss mechanism in a laser contribute equally to the natural linewidth.

III. DYE LASER

The dye laser is also an interesting system to study. Due to its energy-level structure, with singlet electronic states and corresponding triplets, it behaves as if it has a built-in absorber in each molecule. We consider a dye-laser model as shown in Fig. 2. Molecules are injected in level $|a\rangle$ at a rate r_a. The nonradiative decays from levels a,b,c,f are denoted by γ_a, γ_m, γ_b, γ_c, and γ_f, as shown in Fig. 2. Lasing action takes place between levels $|a\rangle$ and $|b\rangle$.

For the dye laser we also start with the equation of motion for the reduced density matrix for the field

$$\begin{aligned}\dot\rho_{nn'}=&\left[\frac{-A_1N'_{nn'}}{1+(B_1/A_1)N_{nn'}}\right]\rho_{nn'}+\left[\frac{\sqrt{nn'}A_1}{1+(B_1/A_1)N_{n-1,n'-1}}\right]\rho_{n-1,n'-1}\\&+\frac{\sqrt{(n+1)(n'+1)}R_2}{[1+(E_1/D_1)M_{n+1,n'+1}]}\rho_{n+1,n'+1}-\frac{R_2M'_{nn'}}{[1+(E_1/D_1)M_{nn'}]}\rho_{nn'}+\frac{N'_{nn'}M'_{nn'}R_1}{[1+(B_1/A_1)N_{nn'}][1+(E_1/D_1)M_{nn'}]}\rho_{nn'}\\&-\frac{R_2\sqrt{(n+1)(n'+1)}}{[1+(B_1/A_1)N_{nn'}][1+(E_1/D_1)M_{nn'}]}N'_{n+1,n'+1}\rho_{n+1,n'+1}-\frac{C_1}{2}(n+n')\rho_{nn'}+C\sqrt{(n+1)(n'+1)}\rho_{n+1,n'+1}\ ,\end{aligned} \tag{16}$$

where

$$A_1=\frac{4g_1^2r_a}{(\gamma_a+\gamma_m)(\gamma_a+\gamma_b+\gamma_m)}\ , \tag{17a}$$

$$B_1=\frac{16g_1^4r_a}{\gamma_b(\gamma_a+\gamma_m)^2(\gamma_a+\gamma_b+\gamma_m)}\ , \tag{17b}$$

$$D_1=\frac{4g_2^2r_a}{\gamma_c(\gamma_c+\gamma_f+\gamma_a)}\ , \tag{17c}$$

$$E_1=\frac{16g_2^2r_a(\gamma_c+\gamma_f)}{\gamma_f\gamma_c^2(\gamma_c+\gamma_f+\gamma_a)}\ , \tag{17d}$$

$$N'_{nn'}=\tfrac{1}{2}(n+n'+2)+S_1(n-n')^2\ , \tag{17e}$$

$$N_{nn'}-\tfrac{1}{2}(n+n'+2)+S_2(n-n')^2\ , \tag{17f}$$

$$M'_{nn'}=\tfrac{1}{2}(n+n')+T_1(n-n')^2\ , \tag{17g}$$

$$M_{nn'}=\tfrac{1}{2}(n+n')+T_2(n-n')^2\ , \tag{17h}$$

$$R_1=\frac{1}{r_a}\left[\frac{\gamma_a}{\gamma_a+\gamma_m}\right]A_1D_1\ , \tag{17i}$$

$$R_2=\left[\frac{\gamma_a}{\gamma_a+\gamma_m}\right]D_1\ , \tag{17j}$$

$$S_1=\frac{B_1}{4A_1}\frac{\gamma_a+\gamma_m}{(\gamma_a+\gamma_m+\gamma_b)}\ , \tag{17k}$$

$$S_2=\frac{B_1}{4A_1}\frac{\gamma_a+\gamma_m}{(\gamma_a+\gamma_m+\gamma_b)^2}\gamma_b\ , \tag{17l}$$

$$T_1=\frac{E_1}{4D_1}\frac{\gamma_c}{(\gamma_c+\gamma_f)}\ , \tag{17m}$$

$$T_2=\frac{E_1}{4D_1}\frac{\gamma_c\gamma_f}{(\gamma_c+\gamma_f)^2}\ . \tag{17n}$$

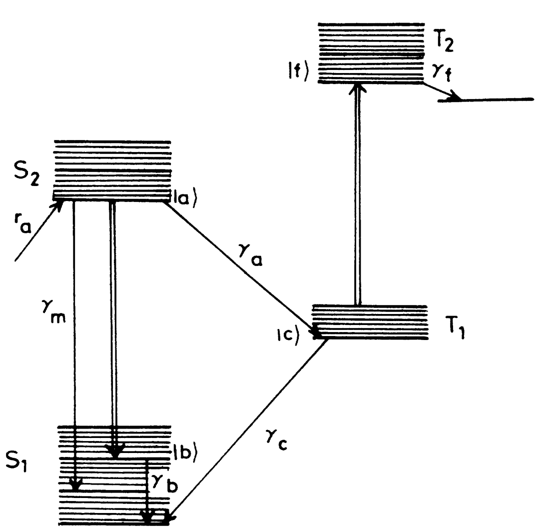

FIG. 2. Energy-level diagram of a dye molecule.

In Eqs. (17a)–(17d), g_1 and g_2 are the coupling constants of the field with a-b and c-f transitions, respectively.

We derived an equation for $\langle n \rangle$ in the case of a laser with a saturable absorber in Sec. II. Proceeding along the same lines, we can derive the following approximate equation for $\langle n \rangle$ in a dye laser:

$$\frac{A_1}{(1+(B_1/A_1)\langle n \rangle)}\left[C+\left[\frac{\gamma a}{\gamma_a+\gamma_m}\right]\frac{D_1}{[1+(E_1/D_1)\langle n \rangle]} - \frac{1}{r_a}\left[\frac{\gamma_a}{\gamma_a+\gamma_m}\right]\frac{A_1 D_1(\langle n \rangle+1)}{1+(B_1/A_1)(\langle n \rangle+1)[1+(E_1/D_1)\langle n \rangle]}\right]^{-1}=1 . \tag{18}$$

The equation of motion for the coherent-state representation corresponding to Eq. (17) for the reduced density matrix for the field is given by

$$\begin{aligned}\dot{p}=&-\frac{A_1}{2}\left[\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]+2S_1\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2-\frac{2\partial^2}{\partial\alpha\,\partial\alpha^*}\right]M_1\\
&-\frac{R_2}{2}\left[-\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*+2T_1\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]L_1+\frac{C}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]p\\
&+R_1\left[-\frac{1}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]|\alpha|^2+\frac{1}{4}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2-\frac{1}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]\right]N_1\\
&-R_1S_1\left[\frac{1}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]N_1\\
&-R_1T_1\left[\frac{1}{2}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2-\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2|\alpha|^2-\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]\right]N_1\\
&+R_1S_1T_1\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^4 N_1 ,\end{aligned} \tag{19}$$

where

$$M_1=\left[1-\frac{B_1}{2A_1}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*-2(1+|\alpha|^2)\right]+\frac{B_1^2}{4A_1^2}\frac{\gamma_a+\gamma_m}{(\gamma_a+\gamma_m+\gamma_b)^2}\gamma_b\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]^{-1}p , \tag{20a}$$

$$L_1=\left[1-\frac{E_1}{2D_1}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*-2|\alpha|^2\right]+\frac{E_1^2}{4D_1^2}\frac{\gamma_c\gamma_f}{(\gamma_c+\gamma_f)^2}\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]^{-1}p , \tag{20b}$$

$$\begin{aligned}N_1=&\left[1-\frac{B_1}{2A_1}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*-2(1+|\alpha|^2)\right]+\frac{B_1^2}{4A_1^2}\frac{\gamma_a+\gamma_m}{(\gamma_a+\gamma_m+\gamma_b)^2}\gamma_b\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]^{-1}\\
&\times\left[1-\frac{E_1}{D_1}\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*-2|\alpha|^2\right]+\frac{E_1^2}{4D_1^2}\left[\frac{\gamma_c\gamma_f}{(\gamma_c+\gamma_f)^2}\right]\left[\frac{\partial}{\partial\alpha}\alpha-\frac{\partial}{\partial\alpha^*}\alpha^*\right]^2\right]^{-1}p .\end{aligned} \tag{20c}$$

Using the same conditions and approximations as those used in obtaining Eq. (14), the following expression for the diffusion constant for the dye laser is obtained:

$$\begin{aligned}D_\theta=&\frac{1}{8\langle n \rangle}\left[2A_1\left[\frac{\gamma_a+\gamma_m}{\gamma_a+\gamma_m+\gamma_b}\right]+2C\left[\frac{\gamma_b}{\gamma_a+\gamma_m+\gamma_b}\right]+2D_1\left[\frac{\gamma_a\gamma_b}{(\gamma_a+\gamma_m)(\gamma_a+\gamma_m+\gamma_b)}\right]\right]\\
&-\frac{E_1}{4}\frac{\gamma_a\gamma_b}{(\gamma_a+\gamma_m+\gamma_b)(\gamma_a+\gamma_m)}\left[\frac{1+\dfrac{D_1^2}{E_1 r_a}\left[\dfrac{\gamma_a}{\gamma_a+\gamma_m}\right]-\dfrac{(\gamma_a+\gamma_m+\gamma_b)\gamma_c}{\gamma_b(\gamma_c+\gamma_f)}\left[1-\dfrac{\langle n \rangle c}{r_a}\right]}{1+\dfrac{E_1}{D_1}\langle n \rangle+\dfrac{D_1}{r_a}\left[\dfrac{\gamma_a}{\gamma_a+\gamma_m}\right]\langle n \rangle}\right] .\end{aligned} \tag{21}$$

This expression for D_θ is rather complicated. However, it simplifies considerably under the realistic assumptions that $\gamma_m=\gamma_b=\gamma_c=\gamma_f\equiv\gamma$ and $\gamma\gg\gamma_a$. We then obtain

$$D_\theta=\frac{1}{8\langle n\rangle}(A_1+C)+\frac{D_1}{8\langle n\rangle}\left[\frac{\gamma_a}{\gamma_a+\gamma_m}\right]\left[1-\left[\frac{1}{r_a}\frac{\left[\frac{D_1^2}{E_1}\right]\left[\frac{\gamma_a}{\gamma_a+\gamma_m}\right]+C\langle n\rangle}{1+\frac{D_1}{E_1\langle n\rangle}+\frac{D_1^2}{E_1 r_a}\left[\frac{\gamma_a}{\gamma_a+\gamma_m}\right]}\right]\right]. \tag{22}$$

The first two terms give the usual contribution for an ordinary laser. The term

$$D_1\gamma_a/8\langle n\rangle(\gamma_a+\gamma_m)$$

can be interpreted as being due to the absorber part of the dye molecules in a manner similar to the D term in Eq. (15), provided that we define the pumping rate r_a in Eq. (17c) for D_1 by $r_d=r_a[\gamma_a/(\gamma_a+\gamma_m)]$. The term in brackets is due to the built-in nature of the absorber and its effect is to reduce the natural linewidth as compared to the linewidth of a laser with a saturable absorber.

ACKNOWLEDGMENT

This research was supported, in part, by the Pakistan Science Foundation.

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