Deviations from exponential decay in the case of spontaneous emission from a two-level atom

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(Received 25 March 1987; revised manuscript received 21 December 1987)

A mathematically rigorous treatment of the Weisskopf-Wigner model of atomic spontaneous emission is presented. For the first time to the authors' knowledge, it is shown that in the correct asymptotic treatment with a cutoff frequency at which the dipole approximation breaks down, the main contribution to the long-time deviation from the exponential decay is of the order of $1/(t\ \ln^2 t)$. This contradicts the results of previous authors who have obtained a long-time behavior of the order of $1/t^2$ by nonrigorous mathematical treatment of the same model in the dipole approximation. However, we will show that the result $1/t^2$ can still be obtained if the retardation effects are taken into account, i.e., if no dipole approximation is made.

I. INTRODUCTION

Khalfin[1] has pointed out for the first time that the decay of an unstable quantum system whose energy spectrum is bounded from below cannot be purely exponential for large times (see also Ref. 2). By using a fundamental theorem on Fourier transforms due to Paley and Wiener, Khalfin has shown that there is a long-time deviation of the order of $1/t$ from the exponential form in the decay probability amplitude of a quasistationary state. By using different model systems other authors have also calculated deviations from exponential decay (see, e.g., Ref. 3).

Since in the Weisskopf-Wigner treatment[4] of spontaneous emission from a two-level atom the probability amplitude of the excited state decays exponentially, several attempts have been made to get long-time deviations from the exponential decay.[5,6] The Weisskopf-Wigner method, which implies the rotating-wave approximation (by way of neglecting the antiresonant terms in the interaction Hamiltonian) and the dipole approximation, yields an integro-differential equation for the probability amplitude associated with the excited state of a two-level atom. This equation can be treated by taking the Laplace transform and its inverse. However, in carrying out the inverse Laplace transformation mathematical difficulties (i.e., logarithmic singularities) appear. Therefore various approximation methods are applied. The approximation used most frequently is the so-called Weisskopf-Wigner pole approximation,[4] which is identical to the Markov approximation[7,8] where the memory effects in the equation of motion for the probability ampitude are ignored. This approximation leads to the formula for exponential decay.

A rigorous mathematical treatment of the problem is quite intricate and is usually avoided by physicists. To our knowledge, only a small number of papers has been published in which the problem has been treated adequately. Knight and Milonni,[5] for example, have obtained an asymptotic correction to the exponential decay formula of the probability amplitude in the order of magnitude of $1/t^2$; they accomplished this, without the use of the Weisskopf-Wigner approximation, by making use of other approximations and not taking into account logarithmic singularities. In their final result the frequency shift (Lamb shift) is also missing.

A mathematically adequate treatment of the problem was initiated by Davidovich in his thesis.[6] By applying the resolvent-operator formalism he obtains, for the probability amplitude, an integral with logarithmic singularities. As will be shown in the present paper, the same result can be obtained by the use of the simpler method of the Laplace transform and its inverse. In dealing with such integrals, it is appropriate to interpret the Laplace inversion as a complex integral and create an analytic continuation of the integrand in the infinitely sheeted Riemann surface. In this construction only two of the infinitely many branches of the Riemann surface will be used for further calculation, and the path of integration will be deformed adequately. In this way Davidovich proves the existence of two poles. One of them gives rise, in a direct way, to the exponential Weisskopf-Wigner decay. The asymptotic contribution of the other one is of the order of magnitude of 10^{-10^8} and is essentially negligible. Instead of carrying out an explicit calculation of the Weisskopf-Wigner pole, Davidovich uses a "wave-function renormalization constant" which, in our opinion, does not lead to any advantage in the model treated here. Furthermore, in his asymptotic treatment of the probability amplitude, Davidovich only obtains a term of the order of magnitude of $1/t^2$ (in agreement

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with Knight and Milonni[5]) without taking into account one integral whose coefficient depends upon the cutoff frequency. Inasmuch as the application of the dipole approximation requires the introduction of the cutoff frequency, the latter is inherent in the model. For this reason the asymptotic contribution of this term must not be neglected. A second, even more substantial reason is the fact, that the integral quoted here—whose proper estimation poses some technical difficulties—yields the asymptotically slowest term of the expansion of the order of magnitude of $1/(t\ln^2 t)$.

In the present paper we make an effort to present a rigorous yet mathematically simple approach to the above-mentioned Weisskopf-Wigner model of spontaneous emission. Starting from the integro-differential equation for the probability amplitude associated with the excited atomic state, we calculate the Laplace transform and its inverse. In this way we obtain an integral with logarithmic singularities which requires analytic continuation into the Riemann surface in order to be able to move the path of integration over singularities on the zeroth and first Riemann sheet.

Without using a renormalization constant (as done by Davidovich[6]), and by using Newton's method for finding zeros[9] and the argument principle,[10] we give a rather accurate estimate of the complex zero on the first Riemann sheet. It turns out that its real part gives precisely the Lamb shift[7,8] (which is not explicitly calculated in the works of Davidovich[6] and Knight and Milonni[5]). Its imaginary part corresponds to the Einstein coefficient for spontaneous emission. At the second pole on the zeroth Riemann sheet the integral is also evaluated by Newton's method.

The asymptotic expansion of the probability amplitude is presented in a rather complete fashion. As a new result, it is found that the main asymptotic contribution (for large time t) is of the order of $1/(t\ln^2 t)$. For extremely small times we give a Taylor expansion which shows significant deviations from exponential decay also.

Finally, we show that the result $1/t^2$ can be obtained only if no dipole approximation is used and the retardation effects are taken into account. This will be demonstrated in the special case of the Lyman-α radiation emission in an hydrogenic atom. The calculations are similar to those of Davidovich and Nussenzveig[11] but more simple and more explicit.

We would like to stress that to our knowledge, it will be shown here for the first time that the dipole approximation used by many authors (see, e.g., Refs. 4–8 and 12) leads to a totally different asymptotic behavior than the exact calculation including retardation effects.

The paper is organized as follows. In Sec. II we give a description of the model treated here and derive an integro-differential equation for the probability amplitude. In Sec. III this equation is treated analytically and asymptotic deviations from the Weisskopf-Wigner exponential decay as well as short-time corrections are obtained. In Sec. IV we treat the model without dipole approximation. In Sec. V we draw a conclusion. In Appendixes A–C we derive various mathematical expressions which were used in preceding sections.

II. DESCRIPTION OF THE MODEL

The Hamiltonian for a single two-level atom (system A) interacting with the radiation field in the rotating-wave approximation (where the antiresonant terms are neglected) and in the dipole approximation (where the spatial extension of the atom is neglected for frequencies $\omega \ll c/a_0$) is given by[7,8]

$$\begin{aligned} H &= H_0 + H_{AR} \\ &= \omega_0 S^z \otimes I_R + \sum_{\mathbf{k},s} \omega a^\dagger_{\mathbf{k}s} a_{\mathbf{k}s} \otimes I_A \\ &\quad + \sum_{\mathbf{k},s} (g_{\mathbf{k}s} S^+ \otimes a_{\mathbf{k}s} + g^*_{\mathbf{k}s} S^- \otimes a^\dagger_{\mathbf{k}s}) \quad (\hbar = 1)\,, \end{aligned} \tag{2.1}$$

where $S^z = \frac{1}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|)$ is the population inversion operator, $S^+ = |1\rangle\langle 2|$, $S^- = |2\rangle\langle 1|$ are the dipole-moment operators with $|1\rangle$, $|2\rangle$ being the atomic excited and ground state, respectively. $a^\dagger_{\mathbf{k}s}$ and $a_{\mathbf{k}s}$ are the photon creation and annihilation operators for the mode $\mathbf{k}s$, respectively, and

$$g_{\mathbf{k}s} = -i\omega_0 (2\pi/\omega L^3)^{1/2} (\mathbf{D}_{12} \cdot \mathbf{e}_{\mathbf{k}s})$$

is the coupling constant with $\mathbf{D}_{12}$ as the dipole matrix element, $\mathbf{e}_{\mathbf{k}s}$ as the polarization vector (s is the polarization index), and L^3 as the volume of the field. Furthermore, ω_0 is the energy separation of the two atomic levels, $\omega = kc$, and I_A and I_R are the unit operators in the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_R$ of systems A and R.

At initial time $t=0$, we assume that the atom is in the excited state and the radiation field in the vacuum state

$$|\psi(0)\rangle = |1\rangle \otimes |\{0\}\rangle\,. \tag{2.2}$$

Since the rotating-wave approximation is made in the Hamiltonian (2.1), the time evolution of the system $A+R$ is restricted to the subspace spanned by the state vectors $|1\rangle \otimes |\{0\}\rangle$, $|2\rangle \otimes |\{1_{\mathbf{k}s}\}\rangle$:

$$\begin{aligned} |\psi(t)\rangle &= b_1(t)\,|1\rangle \otimes |\{0\}\rangle e^{-i\omega_0 t/2} \\ &\quad + \sum_{\mathbf{k},s} b_{2,\{1_{\mathbf{k}s}\}}(t)\,|2\rangle \otimes |\{1_{\mathbf{k}s}\}\rangle e^{i(\omega_0/2-\omega)t}\,, \end{aligned} \tag{2.3}$$

where $b_1(t)$ is the probability amplitude for finding the atom in the excited state ($|1\rangle$) and no photons in the radiation field ($|\{0\}\rangle$), and $b_{2,\{1_{\mathbf{k}s}\}}(t)$ is the probability for finding the atom in the ground state ($|2\rangle$) and one photon in the mode $\mathbf{k}s$ of the radiation field ($|\{1_{\mathbf{k}s}\}\rangle$).

By inserting Eq. (2.3) into the Schrödinger equation we obtain a set of coupled equations of motion for the probability amplitudes:

$$i\frac{db_1(t)}{dt} = i\dot{b}_1(t) = \sum_{\mathbf{k},s} g_{\mathbf{k}s} e^{i(\omega_0-\omega)t} b_{2,\{1_{\mathbf{k}s}\}}(t)\,, \tag{2.4}$$

$$i\dot{b}_{2,\{1_{\mathbf{k}s}\}}(t) = g^*_{\mathbf{k}s} e^{-i(\omega_0-\omega)t} b_1(t)\,. \tag{2.5}$$

By using Eq. (2.5) in Eq. (2.4) we get a closed integro-differential equation for the probability amplitude $b_1(t)$,

$$\dot{b}_1(t) = -\sum_{\mathbf{k},s} |g_{\mathbf{k}s}|^2 \int_0^t d\tau\, e^{i(\omega_0-\omega)\tau} b_1(t-\tau) . \quad (2.6)$$

By replacing the summation over $\mathbf{k}s$ by an integral over the continuum of modes, we obtain the final form of the integro-differential equation:

$$\dot{b}_1(t) = -\frac{\gamma}{2\pi\omega_0} \int_0^{\tilde{\omega}} d\omega\, \omega \int_0^t d\tau\, e^{i(\omega_0-\omega)\tau} b_1(t-\tau) , \quad (2.7)$$

where $\gamma = \frac{4}{3} |\mathbf{D}_{12}|^2 \omega_0^3 / c^3$ is the Einstein coefficient for spontaneous emission, and $\tilde{\omega} \approx c/a_0$ is the cutoff frequency at which the dipole approximation inherent in the Hamiltonian (2.1) breaks down.

III. ANALYTIC TREATMENT OF EQUATION (2.7)

Basically we look for solutions of Eq. (2.7) which are bounded by some real exponential function for $t \geq 0$. Thus one can calculate the Laplace transform and, for $\hat{b}_1(z) \equiv \int_0^\infty b_1(t) e^{-tz} dt$, we obtain the algebraic equation

$$z\hat{b}_1(z) - b_1(0) = -\frac{\lambda}{2\pi} \hat{b}_1(z) \int_0^{\tilde{\omega}} \frac{\omega\, d\omega}{z - i(\omega_0-\omega)} , \quad \lambda = \frac{\gamma}{\omega_0} . \quad (3.1)$$

Equation (3.1) has the solution

$$\hat{b}_1(z) = \frac{b_1(0)}{z - \frac{i\lambda}{2\pi}\tilde{\omega} + \frac{\lambda}{2\pi}(z - i\omega_0)\{\mathrm{Log}^\uparrow[z + i(\tilde{\omega}-\omega_0)] - \mathrm{Log}^\uparrow(z - i\omega_0)\}} . \quad (3.2)$$

Here, $\mathrm{Log}^\uparrow(z) \equiv \ln|z| + i\,\mathrm{Arg}^\uparrow(z)$ denotes that special branch of $\log z$ which satisfies $\pi/2 \leq \mathrm{Arg}^\uparrow(z) < 5\pi/2$ (the cut is made vertically rather than horizontally). In Eq. (3.2) we used the fact that

$$\int_0^{\tilde{\omega}} \frac{d\omega}{\omega - \omega_0 - iz} = \mathrm{Log}^\uparrow[z + i(\tilde{\omega}-\omega_0)] - \mathrm{Log}^\uparrow(z - i\omega_0) \quad (3.3)$$

for $\mathrm{Re}z \neq 0$. Here, we use the convention $\log z$ for the multiple-valued natural logarithm of a complex variable z and $\mathrm{Log}z$ for the principal value of $\log z$.

In order to be able to use the more common branch of $\log z$, we introduce a new variable $u = iz + \omega_0$ and a new function

$$\hat{B}_1(u) \equiv -i\hat{b}_1(-iu + i\omega_0) . \quad (3.4)$$

After simple calculations we find

$$\hat{B}_1(u) = \frac{b_1(0)}{u - \omega_0 + \frac{\lambda\tilde{\omega}}{2\pi} + \frac{\lambda}{2\pi} u[\mathrm{Log}(u-\tilde{\omega}) - \mathrm{Log}(u)]} . \quad (3.5)$$

Using the Laplace inversion formula

$$b_1(t) = \frac{1}{2\pi i} \int_C e^{tz} \hat{b}_1(z) dz , \quad (3.6)$$

where C is as in Fig. 1(a), we find

$$b_1(t) = \frac{e^{i\omega_0 t}}{2\pi i} \int_C e^{-itu} \hat{B}_1(u) du , \quad (3.7)$$

with C as in Fig. 1(b). In the following we shall make use of the following constants:

$$\tilde{\omega} \approx 10^{19}, \quad \alpha \equiv \omega_0 - \frac{\lambda\tilde{\omega}}{2\pi} \approx 10^{16} , \quad (3.8)$$

$$\beta \equiv \frac{\lambda}{2\pi} \approx 10^{-8}, \quad \gamma \equiv \omega_0\lambda \approx 10^8 .$$

We also use a complex function

$$N(u) \equiv u - \alpha + \beta u[\log(u-\tilde{\omega}) - \log u] , \quad (3.9)$$

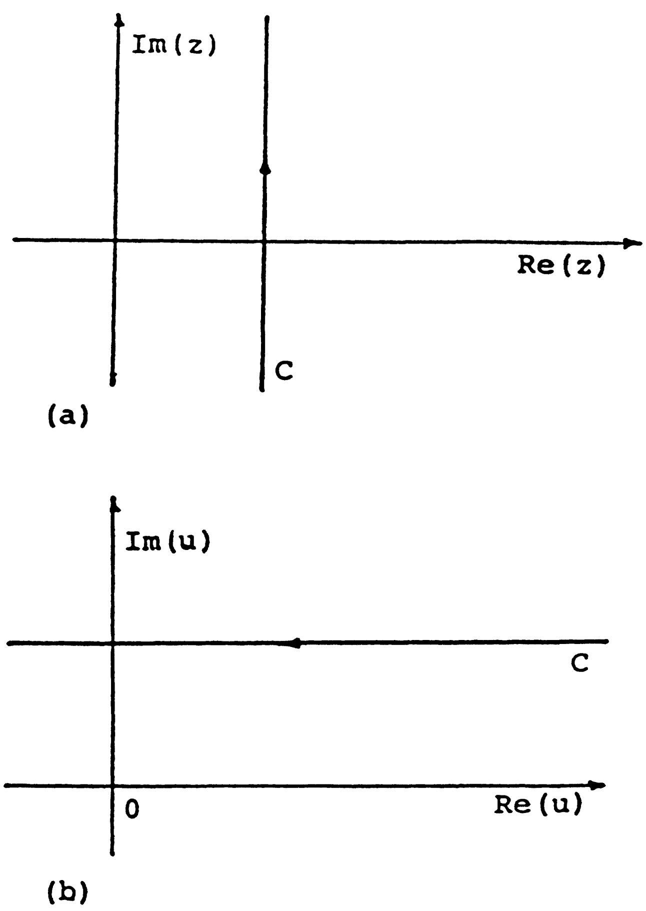


FIG. 1. Path of integration (a) in Eq. (3.6) and (b) in Eq. (3.7).

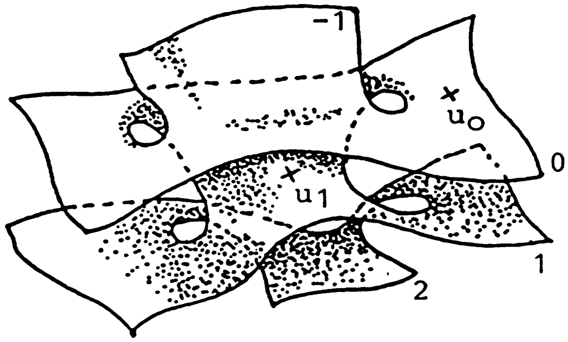


FIG. 2. Two Riemann sheets of the infinitely branched Riemann surface of $\log(u-\tilde{\omega})-\log(u)$. The branch points have been removed for reasons of better visibility.

which is the denominator of $\hat{B}_1(u)$. In order to be able to discuss asymptotic properties of $b_1(t)$ we shall deform the path of the integration in Eq. (3.7) suitably.

As will be explained in Appendix A, N is a multivalued function of an infinitely sheeted Riemann surface. The latter consists of planes having a cut (Fig. 2) which are pasted together in a suitable way. Let $N_l(z)$ denote that branch of $N(z)$ which is defined on the lth sheet. Then

$$N_l(u)\equiv N_0(u)+2\pi l i\beta u \tag{3.10}$$

holds, where $l=0,\pm1,\pm2,\ldots$.

Our first deformation of C takes place on the zeroth Riemann sheet, where $\hat{B}_1(u)$, and therefore $N(u)$, originally "lives." We use a new path of integration C_0 (Fig. 3). However, in moving from C to C_0, according to Appendix A, lemma 2, we cross a real zero:

$$u_0\approx\tilde{\omega}\left[1+\exp\left[-\frac{\tilde{\omega}-\alpha}{\tilde{\omega}\beta}\right]\right] . \tag{3.11}$$

Using a suitable parametrization of C_0 and the residue theorem—$\hat{B}_1(u)$ has the simple pole u_0—we find

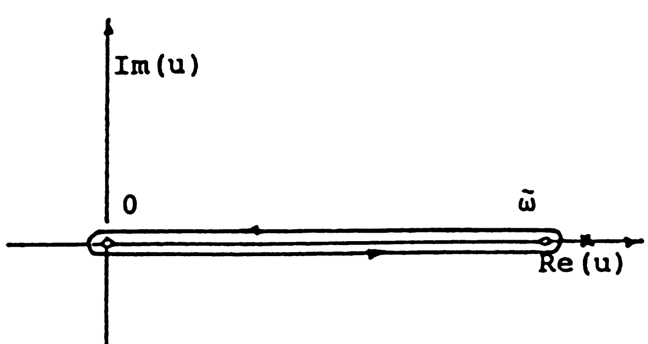


FIG. 3. Path of integration C_0 used in Eq. (3.12).

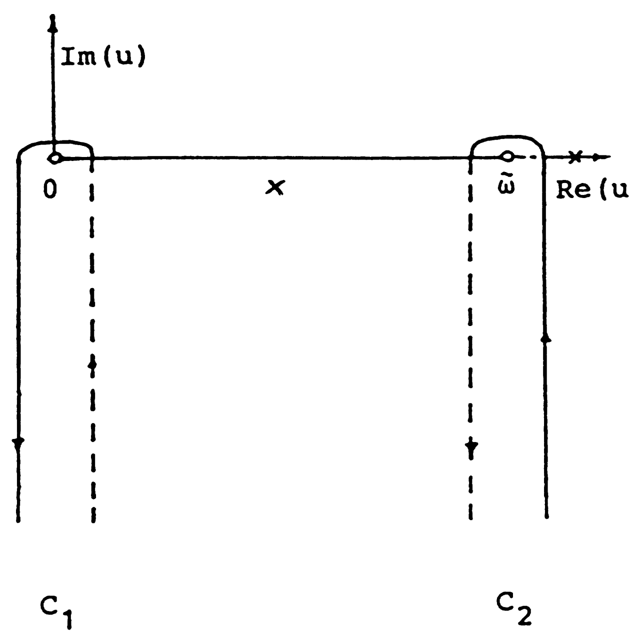
$$b_1(t)=\beta b_1(0)e^{it\omega_0}\times\int_0^{\tilde{\omega}}\frac{e^{-itx}x\,dx}{\left[x-\alpha+\beta x\ln\left[\frac{\tilde{\omega}-x}{x}\right]\right]^2+\beta^2\pi^2x^2}+A_0e^{it(\omega_0-u_0)} , \tag{3.12}$$

where

$$A_0\approx\frac{1}{\beta}\exp\left[-\frac{\tilde{\omega}-\alpha}{\tilde{\omega}\beta}\right]\approx10^{-10^8} , \tag{3.13}$$

by Appendix A, lemma 2.

The pole u_0 is a consequence of introducing a cutoff frequency $\tilde{\omega}$ and its amplitude A_0 is so small that it is not of any physical relevance. Next using the lower sheet of the zeroth leaf, namely, leaf number 1, we may deform C_0, the path of integration, again. Our new path of in-



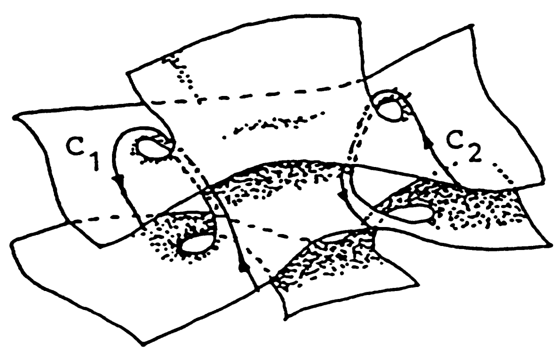


FIG. 4. Paths of integration C_1 and C_2 in Eqs. (3.15) and (3.16). The poles u_0 and u_1 of $\hat{B}_1(u)$ are given by Eqs. (3.11) and (3.14).

tegration actually consists of two new paths C_1 and C_2 which partially belong to leaf number 0 while the remainder of each of them is on the lower sheet (Fig. 4). According to Appendix A, lemma 2, we cross a zero of $N_1(u)$,

$$u_1 \approx \omega_0 - \frac{\lambda\tilde{\omega}}{2\pi} - \frac{\lambda\omega_0}{2\pi}\ln\left[\frac{\tilde{\omega}}{\omega_0}\right] - i\frac{\lambda\omega_0}{2} , \tag{3.14}$$

on the lower sheet, which, on the basis of the residue theorem, gives rise to

$$b_1(t) = I_1(t) + I_2(t) + (A_1 e^{-iu_1 t} + A_0 e^{-iu_0 t})e^{i\omega_0 t} , \tag{3.15}$$

where

$$I_j(t) \equiv \frac{e^{i\omega_0 t}}{2\pi i}\int_{C_j} e^{-itu}\hat{B}_1(u)du, \quad j=1,2 \tag{3.16}$$

and

$$A_1 \approx 1 \tag{3.17}$$

according to Appendix A, lemma 2.

The term $-\lambda\tilde{\omega}/2\pi$ in Eq. (3.14) which diverges linearly for $\tilde{\omega} \to \infty$ will be discarded since it gives no contribution if the mass renormalization of the electron is carried out or if counter-rotating terms are taken into account (cf. Refs. 7,8). Moreover, the frequency shift $-\lambda/2\pi\ln(\tilde{\omega}/\omega_0)$ is just half of the Lamb shift (cf. Refs. 7,8) since it includes only the energy shift of the excited state while that one of the ground state is missing.

As is pointed out in Appendix C, I_1 yields, as the main asymptotic contribution for large t, the term

$$I_1(t) = \frac{-\lambda}{2\pi\omega_0^2}\frac{e^{i\omega_0 t}}{t^2} + o(1/t^2) , \tag{3.18}$$

where $b_1(0)=1$ was used and $o(1/t^2)$ denotes terms which decrease faster than the main contribution. This is in accordance with the results of Knight and Milonni and Davidovich. Similarly, I_2 yields the term

$$I_2(t) = \frac{2\pi i}{\lambda\tilde{\omega}}\frac{e^{i(\omega_0-\tilde{\omega})t}}{t\ln^2 t} + o(1/t\ln^2 t)) . \tag{3.19}$$

Altogether we find out that Eq. (3.19) describes the main asymptotic contribution of $b_1(t)$. Thus one may write

$$b_1(t) = \left[A_1 e^{-iu_1 t} + A_0 e^{-iu_0 t} - \frac{\lambda}{2\pi\omega_0^2}\frac{1}{t^2}\right]e^{i\omega_0 t} + \frac{2\pi i}{\lambda\tilde{\omega}}\frac{e^{i(\omega_0-\tilde{\omega})t}}{t\ln^2 t} + o\left[\frac{1}{t\ln^2 t}\right] . \tag{3.20}$$

The nondecaying term in Eq. (3.20) has a coefficient $A_0 \approx 10^{-10^8}$ and physically does not play any role. Therefore asymptotically only the term $1/(t\ln^2 t)$ plays a role.

For extremely small times t, we may use a Taylor expansion of $|b_1(t)|$ around $t=0$, and we find that

$$|b_1(t)| - e^{-\gamma t/2} = \gamma t/2 - \left[\frac{\lambda\tilde{\omega}^2}{8\pi} + \frac{\gamma^2}{4}\right]t^2 + O(t^3) . \tag{3.21}$$

where $O(t^3)$ denotes terms of the third order. Neglecting terms of the third order we obtain

$$|b_1(t_0)| = e^{-\gamma t_0/2}$$

$$\text{for } t_0 \approx \frac{4\pi\gamma}{\lambda\tilde{\omega}^2 + 2\gamma^2\pi} \approx \frac{4\pi\gamma}{\lambda\tilde{\omega}^2} \approx 10^{-21} . \tag{3.22}$$

This shows that the deviation of $|b_1(t)|$ from the exponential decay is significant for extremely small times $t < t_0$.

IV. RESULTS WITHOUT DIPOLE APPROXIMATION

In order to compare our results obtained in the dipole approximation (with a sharp cutoff frequency) with those results where no dipole approximation is made, we specialize ourselves to the case of Lyman-α transition in a hydrogenic atom.[11] For this special case Moses[13] has calculated the exact matrix elements of the interaction Hamiltonian by taking into account all the retardation effects. Then the interaction Hamiltonian in Eq. (2.1) takes the form

$$H_{AR} = \int_0^\infty d\omega[g(\omega)a(\omega)\otimes S^+ + g^*(\omega)a^\dagger(\omega)\otimes S^-] , \tag{4.1}$$

$$g(\omega) = \left[\frac{\gamma}{2\pi\omega_0}\right]^{1/2}\frac{(-i)\omega^{1/2}}{[1+(\omega/\Omega)^2]^2}, \quad \Omega \equiv \frac{3}{2}\frac{c}{a_0} , \tag{4.2}$$

and Eq. (2.7) reads as

$$\dot{b}_1(t) = -\frac{\lambda}{2\pi}\int_0^\infty d\omega f(\omega)\int_0^t d\tau e^{i(\omega_0-\omega)\tau}b_1(t-\tau) , \tag{4.3}$$

with the natural smooth cutoff function

$$f(\omega) = \frac{\omega\Omega^8}{(\Omega^2+\omega^2)^4} \tag{4.4}$$

stemming from the coupling constant $g(\omega)$ in Eq. (4.2).

Quite analogously to Sec. III we look for solutions of Eq. (4.3) which are bounded by some real exponential function for $t \geq 0$. The application of the Laplace transformation to Eq. (4.3) then gives

$$\hat{b}_1(z) = \frac{b_1(0)}{z + \frac{\lambda}{2\pi}\int_0^\infty \frac{d\omega f(\omega)}{z - i(\omega_0-\omega)}} . \tag{4.5}$$

By substitution $u = iz + \omega_0$ we obtain a new function

$$\hat{B}_1(u) = -i\hat{b}_1(-iu + i\omega_0) = \frac{b_1(0)}{u - \omega_0 + \frac{\lambda}{2\pi}I(u)} , \tag{4.6}$$

with

$$I(u) = \int_0^\infty \frac{d\omega f(\omega)}{\omega - u} = -\frac{C_1(u)}{\Omega^6} + \frac{C_2(u)\pi}{2\Omega} + f(u)(-\log u + \ln\Omega + i\pi) , \tag{4.7}$$

$$C_1(u)=\frac{f(u)}{12}(11\Omega^6+18\Omega^4u^2+9\Omega^2u^4+2u^6) \quad (4.8)$$

$$C_2(u)=\frac{f(u)}{16\Omega^4 u}(5\Omega^6-15\Omega^4u^2-5\Omega^2u^4-u^6)\ . \quad (4.9)$$

Further, analogously to Sec. III, the application of the Laplace inversion formula leads to

$$b_1(t)=\frac{e^{i\omega_0 t}}{2\pi i}\int_C e^{-itu}\hat{B}_1(u)du\ , \quad (4.10)$$

where C is as in Fig. 1(b). Similarly as in Sec. III, by using the fact that $\hat{B}_1(u)$ is defined on the Riemann surface of $\log u$ and applying the technique of deforming the path of integration, and afterwards employing Abel's asymptotic, we obtain finally

$$b_1(t)=\left[e^{-iu_1t}-\frac{\lambda}{2\pi\omega_0^2}\frac{1}{t^2}\right]e^{i\omega_0 t}+o(1/t^2),\quad t\uparrow\infty \quad (4.11)$$

whereby we used the argument principle (see Appendix A) which tells us that there is only one relevant zero lying on the lower Riemann sheet whose estimation follows by iteration:

$$u_1=\omega_0+\frac{11\lambda\omega_0}{24\pi}-\frac{5}{64}\lambda\Omega-\frac{\lambda\omega_0}{2\pi}\ln\left[\frac{\Omega}{\omega_0}\right]-i\frac{\omega_0\lambda}{2}\ . \quad (4.12)$$

V. CONCLUSION

In the present paper we have tried to give a rigorous mathematical treatment for the Weisskopf-Wigner model of spontaneous emission from a two-level atom and to throw some light on the long-time deviations from exponential decay. In the dipole approximation we have explicitly calculated the Weisskopf-Wigner pole without using Davidovich's complicated renormalization constant procedure.[6] This pole gives rise to the exponential decay. By a simple method we were able to determine the second pole on the real axis which gives rise to a rapidly oscillating extremely small nondecaying contribution to the probability amplitude for finding the atom in the initial state. This pole is a consequence of the cutoff frequency. Its contribution is of the order of 10^{-10^8} and, physically, deviations of this order do not play any role.

We have also shown, for the first time, that in a correct asymptotic treatment of the Weisskopf-Wigner model with a cutoff frequency at which the dipole approximation breaks down, the main contribution to the long-time deviation from the exponential decay is of the order of $1/(t\ln^2 t)$, rather than $1/t^2$, as is usually accepted in the literature. For extremely small times deviations from exponential decay have also been calculated.

Furthermore, without making the dipole approximation by taking into account the retardation effects, we have derived the asymptotic result $1/t^2$. Thus we have shown, to our knowledge for the first time, that there is a significant difference in the asymptotic behavior between the exact model and that in the dipole approximation. We have not investigated the accuracy of the dipole approximation in finite time intervals, where, according to the literature and existing experiments, this approximation should be applicable.

ACKNOWLEDGMENT

One of us (J.S.) would like to acknowledge the financial support by the Fonds zur Förderung der wissenschaftlichen Forschung in Österreich (Vienna, Austria) under Grant No. P6690P.

APPENDIX A

We discuss the complex function

$$N(u)=u-\alpha+\beta u[\log(u-\tilde{\omega})-\log u]\ , \quad (A1)$$

which is defined in the u plane, from which the positive real axis has been removed. In order to be able to move the path of integration for the purpose of deriving (3.12) and (3.15) we would like to extend the domain of definition of N.

In this way one is led to consider the Riemann surface of

$$F(u)=\log(u-\tilde{\omega})-\log(u)\ . \quad (A2)$$

$F(u)$ has branch points at $u=0$ and at $u=\tilde{\omega}$. If we define $F_l(u)\equiv F(u)+2\pi il$, F_l can be seen to be continuous for $\mathrm{Re}(u)>\tilde{\omega}$ and $\mathrm{Re}(u)<0$. For $0<\mathrm{Re}(u)<\tilde{\omega}$ one easily finds that

$$F_l(x+io)=F_{l+1}(x-io)\ . \quad (A3)$$

Equation (A3) easily yields a picture of the Riemann surface of F.

Take two Riemann surfaces of $\log(u)$ (Fig. 5) and let one of them wind in the opposite sense of orientation. On each of them cut away a half plane on the side of the slot. Now move the two surfaces together and paste the edges together. What we obtain is a surface which has something in common with a garage where one moves from one level to the next one only in the slot between 0 and $\tilde{\omega}$: moving up coming from the south and down coming from the north. It is this Riemann surface on which N

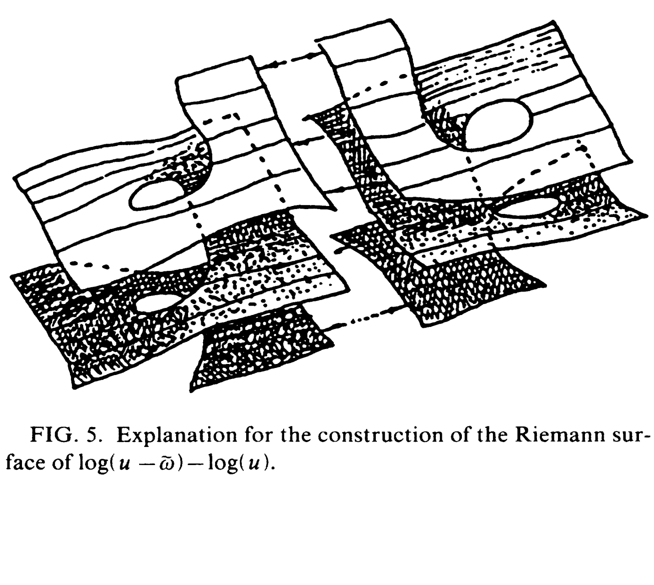


FIG. 5. Explanation for the construction of the Riemann surface of $\log(u-\tilde{\omega})-\log(u)$.

can be regarded as a single-valued complex analytic function with branch points at 0 and $\tilde{\omega}$. We define

$$N_l(u) \equiv N(u) + 2l\beta\pi i u \qquad \text{(A4)}$$

so that $N_0(u) \equiv N(u)$ and $l = 0, \pm 1, \pm 2 \ldots$. N_l will be referred to as the lth branch of N and it is defined on the lth sheet of the Riemann surface.

Lemma 1. $N_l(u)$ has exactly one zero and no poles.

Proof. There are no poles, as can be seen from Eqs. (A1) and (A4). In order to count the number of zeros we use the argument principle in a region $R_l(\rho)$ lying between the curves L_1 and $L_{2\rho}$, where $L_{2\rho}$ is a circle with a large radius ρ (Fig. 6). Let $\nu_l(\rho)$ denote the number of zeros of N_l in $R_l(\rho)$; then by theorem 2.21, p. 48, in Ref. 10, we find that

$$\nu_l(\rho) = \frac{1}{2\pi}\{\Delta_{L_{2\rho}}\text{Arg}[N_l(u)] - \Delta_{L_1}\text{Arg}[N_l(u)]\} \ . \qquad \text{(A5)}$$

As is pointed out in Ref. 10 on p. 48,

$$\frac{1}{2\pi}\Delta_{L_{2\rho}}\text{Arg}[N_l(u)]$$

is precisely the winding number of the normalized vector $N_l(u)/|N_l(u)|$ while the variable point u describes the closed curve $L_{2\rho}$. If ρ is large enough, $N_l(u) = u + o(u)$, where $o(u)$ denotes terms which can be neglected in comparison with u. Then it follows easily that $\Delta_{L_{2\rho}}\text{Arg}[N_l(u)] = \Delta_{L_{2\rho}}\text{Arg}u = 2\pi$. For the calculation of $\Delta_{L_1}\text{Arg}[N_l(u)]$ we used the fact that

$$N_l(x + io^+) = x - \alpha + \beta x \left[\ln\frac{\tilde{\omega} - x}{x} + (2l+1)\pi i\right] , \qquad \text{(A6)}$$

$$N_l(x - io^+) = x - \alpha + \beta x \left[\ln\frac{\tilde{\omega} - x}{x} + (2l-1)\pi i\right] . \qquad \text{(A7)}$$

[Here, x is real, satisfies $0 < x < \tilde{\omega}$, and parametrizes L_1 (Fig. 6).] An elementary discussion finally gives us $\Delta_{L_1}\text{Arg}N_l(u) = 0$ and therefore $\nu_l(\rho) = 1$ for ρ large enough. If ρ tends to infinity the result follows. Q.E.D.

Lemma 2. N_0 has a single zero,

$$u_0 \approx \tilde{\omega}\left[1 + \exp\left[-\frac{\tilde{\omega} - \alpha}{\tilde{\omega}\beta}\right]\right] ,$$

and the residue of $1/N_0(u)$ at the pole u_0 is

$$A_0 \approx 1 \Big/ \left[\beta \exp\left[\frac{\tilde{\omega} - \alpha}{\beta\tilde{\omega}}\right]\right] .$$

N_1 has a single zero,

$$u_1 \approx \omega_0 - \frac{\lambda\tilde{\omega}}{2\pi} - \frac{\lambda\omega_0}{2\pi}\ln\left[\frac{\tilde{\omega}}{\omega_0}\right] - i\lambda\omega_0/2 ,$$

and the residue of $1/N_1(u)$ at the pole u_1 is $A_1 \approx 1$.

Proof. Since $N_0(x)$ is real, for real $x > \tilde{\omega}$, and since $N_0(\tilde{\omega}) = -\infty$ and $N_0(\tilde{\omega}+1) > 0$, respectively, $N_0(x)$ must have a real zero in the open interval $(\tilde{\omega}, \tilde{\omega}+1)$. In order to find it we put $x = \tilde{\omega}(1+e^{\mu})$ and $\phi(\mu) = N_0[\tilde{\omega}(1+e^{\mu})]/\tilde{\omega}$. More concretely, we have

$$\phi(\mu) = 1 + e^{\mu} - \frac{\alpha}{\tilde{\omega}} + \beta(1+e^{\mu})[\mu - \ln(1+e^{\mu})] \ . \qquad \text{(A8)}$$

We make use of a Newton iteration[9]

$$\mu_{n+1} = \mu_n - \phi(\mu_n)/\phi'(\mu_n) \qquad \text{(A9)}$$

giving successive approximations of the zero $\bar{\mu}$. It is conceivable to use $\mu_0 = [-(\tilde{\omega} - \alpha)/\beta\tilde{\omega}]$ as a starting point. An elementary estimate shows that $|\mu_1 - \mu_0| < 2e^{\mu_0}$, so that in fact the next correction term of $\mu_0 \approx -10^8$ is of the order of 10^{-10^8}, which is small enough to be neglected. So, finally,

$$u_0 \approx \tilde{\omega}\left[1 + \exp\left[-\frac{\tilde{\omega} - \alpha}{\beta\tilde{\omega}}\right]\right]$$

holds and

$$A_0 = 1/N_0'(u_0) \approx \exp\left[-\frac{\tilde{\omega} - \alpha}{\tilde{\omega}\beta}\right]/\beta$$

follows as well. In order to find the second zero u_1 we start a Newton-iteration procedure with $\phi(u) = N_1(u)$ and $\xi_0 \equiv \alpha$. Then the first step gives us

$$\xi_1 = \alpha - \alpha\beta\frac{\{\ln[(\tilde{\omega} - \alpha)/\alpha] + i\pi\}}{1 + \beta\left[\{\ln[(\tilde{\omega} - \alpha)/\alpha] + i\pi\} - \dfrac{\alpha}{\alpha - \tilde{\omega}} - 1\right]} , \qquad \text{(A10)}$$

which is an accurate approximation because $|\xi_2 - \xi_1| < 10^3$. Thus we obtain

$$u_1 \approx \xi_1 \approx \omega_0 - \frac{\lambda\tilde{\omega}}{2\pi} - \frac{\lambda\omega_0}{2\pi}\ln\left[\frac{\tilde{\omega}}{\omega_0}\right] - i\frac{\omega_0\lambda}{2} \ . \qquad \text{(A11)}$$

It should be mentioned that in all error estimations we

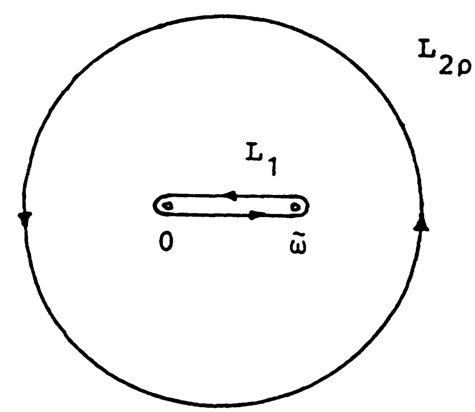


FIG. 6. Contours used in Appendix A, lemma 1.

have used the constants of Eq. (3.8). The calculation of A_1 works in the same way as the one for A_0. Q.E.D.

APPENDIX B

We show that

$$J(t)=\int_0^\infty \frac{e^{-st}ds}{(\ln s+ik)^2}=\frac{1}{t\ln^2 t}+o\left[\frac{1}{t\ln^2 t}\right] \quad \text{(B1)}$$

holds for real $k>0$ and $t\uparrow\infty$. Here, $o(1/t\ln^2 t)$ denotes a term which decreases to zero faster than $1/t\ln^2 t$ as t tends to infinity, i.e., $1/t\ln^2 t$ is the main contribution of the integral.

Defining $J(t)=J_1(t)+J_2(t)$, where $J_1(t)=\int_0^\delta e^{-st}/(\ln s+ik)^2 ds$ with $\delta<e^{-k}$, one easily obtains

$$J_1(t)=\int_0^\delta \frac{e^{-st}ds}{(\ln s)^2}+o\left[\int_0^\delta \frac{e^{-st}ds}{(\ln s)^2}\right] . \quad \text{(B2)}$$

Using lemma 3 of Ref. 14, p. 12, we find

$$\int_0^\delta \frac{e^{-st}ds}{(\ln s)^2}\frac{1}{t\ln^2 t}+o\left[\frac{1}{t\ln^2 t}\right] \quad \text{(B3)}$$

by standard arguments. So we have

$$J_1(t)=1/(t\ln^2 t)+o\left[\frac{1}{t\ln^2 t}\right] . \quad \text{(B4)}$$

Finally, a standard estimate yields

$$|J_2(t)|\le\int_\delta^\infty \frac{e^{-st}ds}{|\ln s+ik|^2}\le\frac{1}{k^2}\int_\delta^\infty e^{-st}ds=\frac{e^{-\delta t}}{tk^2} . \quad \text{(B5)}$$

Therefore, $J_2(t)=o(1/t\ln^2 t)$ since $e^{-\delta t}/t$ decreases to 0 for $t\uparrow\infty$ faster than $1/(t\ln^2 t)$. Thus (B4) and (B5) together give (B1).

APPENDIX C

We proceed to discuss the integrals I_1 and I_2 defined in Eqs. (3.15) and (3.16), respectively. Using suitable parametrizations of C_1 and C_2, respectively, we find that

$$I_1(t)=\frac{-b_1(0)e^{i\omega_0 t}}{2\pi}\int_0^\infty \left[\frac{1}{N_0(-is)}-\frac{1}{N_1(-is)}\right]e^{-st}ds \quad \text{(C1)}$$

and

$$I_2(t)=\frac{-b_1(0)e^{i(\omega_0-\bar\omega)t}}{2\pi}\times\int_0^\infty \left[\frac{1}{N_1(\bar\omega-is)}-\frac{1}{N_0(\bar\omega-is)}\right]e^{-st}ds . \quad \text{(C2)}$$

Since the integrands in $I_1(t)$ and $I_2(t)$ are bounded for $s\le\delta>0$, a standard argument allows us to replace the integrals $\int_0^\infty$ through $\int_0^\delta$ for a small δ. Now a Taylor expansion around $s=0$ yields

$$\frac{1}{N_0(-is)}-\frac{1}{N_1(-is)}=\frac{2\pi\beta s}{\alpha^2}+o(s) \quad \text{(C3)}$$

and

$$\frac{1}{N_1(\bar\omega-is)}-\frac{1}{N_0(\bar\omega-is)}=-\frac{2\pi i}{\beta\bar\omega\left[\ln s+\frac{3i\pi}{2}\right]^2}+O\left[\frac{1}{(\ln s)^3}\right] . \quad \text{(C4)}$$

Finally, using $\int_0^\infty se^{-st}ds=(1/t^2)$ and (B1), we deduce Eqs. (3.18) and (3.19), respectively.

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