Correlated-emission laser: Theory of the quantum-beat micromaser

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We show that if we inject three-level atoms, at a low rate, through a double cavity, with the two upper levels strongly coupled, and if, in the theoretical analysis, one makes the rotating-wave approximation, this problem is formally equivalent to the ordinary two-level micromaser. We also obtain the relevant master equation and the photon statistics.

I. INTRODUCTION

The possibility of having high-Q cavities has opened up the new field of cavity electrodynamics, where one can study the detailed behavior of one or very few atoms interacting with the electromagnetic field inside the cavity. In an ordinary micromaser, a monoenergetic beam of excited two-level atoms is injected into a high-Q resonator at a very low flux such that, at a given time, at most one atom interacts with the electromagnetic radiation. Such a system presents fascinating features such as the "Cummings collapse," multipeaked probability distributions for the number of photons, and antibunching.[1,2,3] In addition, we have the idea of correlated spontaneous emission,[4,5] where the relative phase noise of the two signals coexisting in a double cavity can be quenched and the relative phase-diffusion constant vanishes.

Here, we present a different physical scheme from the two-level micromaser where we have a double cavity through which we inject, at a low rate, an atomic beam of three-level atoms, with the two upper levels strongly pumped. We show in the present work that this system is formally equivalent to the two-level micromaser if we proceed as follows.

(a) We perform the rotating-wave approximation.

(b) We formulate the problem in such a way that the relevant field is a linear combination of the two fields generated as spontaneous emission from the two upper levels to the lower one.

If we perform this program, then the two problems are formally equivalent.

II. THE HAMILTONIAN

Consider a three-level atom, whose levels are denoted by a, b, and c, and assume that the two upper levels a and b are strongly pumped by an external classical field, characterized by a Rabi frequency Ω. Denoting by a_1 (frequency ν_1) and a_2 (frequency ν_2), the annihilation operators of the emissions resulting from transitions $a \to c$ and $b \to c$ respectively, we can write the Hamiltonian of the system as

$$H = H_0 + V \ , \tag{1}$$

where

$$\begin{aligned} H_0 &= \sum_{i=a,b,c} \hbar\omega_i \,|\, i \rangle\langle i \,|\, + \hbar\nu_1 a_1^\dagger a_1 + \hbar\nu_2 a_2^\dagger a_2 \ , \\ V &= \hbar g_1 (a_1 \,|\, a \rangle\langle c \,|\, + a_1^\dagger \,|\, c \rangle\langle a \,|\,) \\ &\quad + \hbar g_2 (a_2 \,|\, b \rangle\langle c \,|\, + a_2^\dagger \,|\, c \rangle\langle b \,|\,) \\ &\quad - \frac{\hbar\Omega}{2} (e^{-i\nu_s t} \,|\, a \rangle\langle b \,|\, + e^{i\nu_s t} \,|\, b \rangle\langle a \,|\,) \ . \end{aligned} \tag{2}$$

In Eq. (2), g_1 and g_2 are the two coupling constants between the a-c and b-c transitions and their respective radiation, and ν_3 is the external pump frequency.

It is convenient to go into the interaction picture. Thus we define

$$V' = e^{(i/h)H_0 t} V e^{-(i/h)H_0 t} \ . \tag{3}$$

It is simple to show that

$$V' = V_1 + V_2 \ , \tag{4}$$

where

$$\begin{aligned} V_1 &= \frac{\hbar\Omega}{2} \begin{pmatrix} 0 & -e^{it\Delta_3} & 0 \\ -e^{-it\Delta_3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \\ V_2 &= \hbar \begin{pmatrix} 0 & 0 & g_1 a_1 e^{it\Delta_1} \\ 0 & 0 & g_2 a_2 e^{it\Delta_2} \\ g_1 a_1^\dagger e^{-it\Delta_1} & g_2 a_2^\dagger e^{-it\Delta_2} & 0 \end{pmatrix} , \end{aligned} \tag{5}$$

and $\Delta_1 \equiv (\omega_a - \omega_c - \nu_1)$, $\Delta_2 \equiv (\omega_b - \omega_c - \nu_2)$, $\Delta_3 \equiv (\omega_a - \omega_b - \nu_3)$.

Now, we assume that the pump is exactly resonant with the two upper levels and, furthermore, that the other two transitions have the same detunings, that is,

$$\Delta_3 = 0, \quad \Delta_1 = \Delta_2 \equiv \Delta \ . \tag{6}$$

Under the assumptions given by Eq. (6), we now perform a second transformation,

$$V'' = e^{iV_1 t} V_2 e^{-iV_1 t} \ . \tag{7}$$

A little algebraic work shows that

$$e^{iV_1 t}=\begin{bmatrix}\cos(\Omega t/2) & -i\sin(\Omega t/2) & 0\\ -i\sin(\Omega t/2) & \cos(\Omega t/2) & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{8}$$

and that

$$V''=\hbar\begin{bmatrix}0 & 0 & V''_{13}\\ 0 & 0 & V''_{23}\\ (V''_{13})^\dagger & (V''_{23})^\dagger & 0\end{bmatrix}, \tag{9}$$

with

$$\begin{aligned}V''_{13}&=\frac{e^{it[\Delta+(\Omega/2)]}}{2}(g_1a_1-g_2a_2)\\&\quad+\frac{e^{it[\Delta-(\Omega/2)]}}{2}(g_1a_1+g_2a_2),\\ V''_{23}&=-\frac{e^{it[\Delta+(\Omega/2)}}{2}(g_1a_1-g_2a_2)\\&\quad+\frac{e^{it[\Delta-(\Omega/2)]}}{2}(g_1a_1+g_2a_2).\end{aligned} \tag{10}$$

For the sake of simplicity, let us assume that $\Delta=\Omega/2$ and neglect the rapid time varying terms $e^{it[\Delta+(\Omega/2)]}$ as compared to the continuous wave ones (rotating-wave approximation). This approximation should be an excellent one for a strong classical pump. The final version of V'' is now

$$V''=\frac{\hbar(g_1^2+g_2^2)^{1/2}}{2}\begin{bmatrix}0 & 0 & A\\ 0 & 0 & A\\ A^\dagger & A^\dagger & 0\end{bmatrix}, \tag{11}$$

where A is defined as

$$A\equiv\frac{g_1a_1+g_2a_2}{(g_1^2+g_2^2)^{1/2}}, \tag{12}$$

in such a way that

$$[A,A^\dagger]=1. \tag{13}$$

III. MASTER-EQUATION–PHOTON STATISTICS

We are mainly concerned with the photon-density matrix. In the coarse-grain time approximation, one can write

$$\dot\rho_{\rm ph}(t)=r[{\rm Tr}_a\rho(t+T)-\rho_{\rm ph}(t)]+\mathcal{L}, \tag{14}$$

where ${\rm Tr}_a$ is the trace over the atoms, r is the atomic injection rate, and ρ and $\rho_{\rm ph}$ mean the total and photon-density operators, respectively. T is the interaction time between the three-level atom and the radiation fields and it can be interpreted as an atomic decay time or flight time, whichever is shorter. The specific form of the loss term $\mathcal{L}$ will be specified later [Eq. (19)]. Let us define

$$\alpha\equiv\begin{bmatrix}0 & 0 & A\\ 0 & 0 & A\\ A^\dagger & A^\dagger & 0\end{bmatrix},\quad g\equiv\frac{(g_1^2+g_2^2)^{1/2}}{2}. \tag{15}$$

Then it is a simple matter to prove that

$$\alpha^{2n}=\begin{bmatrix}2^{n-1}(AA^\dagger)^n & 2^{n-1}(AA^\dagger)^n & 0\\ 2^{n-1}(AA^\dagger)^n & 2^{n-1}(AA^\dagger)^n & 0\\ 0 & 0 & (2A^\dagger A)^n\end{bmatrix},\quad n=1,2,\ldots,$$

$$\alpha^{2n+1}=\begin{bmatrix}0 & 0 & 2^nA(A^\dagger A)^n\\ 0 & 0 & 2^nA(A^\dagger A)^n\\ 2^nA^\dagger(AA^\dagger)^n & 2^nA^\dagger(AA^\dagger)^n & 0\end{bmatrix},\quad n=0,1,2,\ldots. \tag{16}$$

Now, since

$$\begin{aligned}\rho_{\rm ph}(t+T)&={\rm Tr}_a U(T)\rho(t)U^{-1}(T)\\&={\rm Tr}_a e^{-igT\alpha}\rho(t)e^{igT\alpha},\end{aligned} \tag{17}$$

$\rho(t)$ is the initial density matrix and is taken to be

$$\rho(t)=\begin{bmatrix}\rho & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{18}$$

that is, the atom is initially in its upper state.

If we expand the right-hand side of Eq. (17) and use Eqs. (16), (18), and (14), we get the master equation for the photons,

$$\dot{\rho}_{\rm ph}=\frac{r}{2}[\cos(\sqrt{2AA^\dagger}gT)\rho_{\rm ph}\cos(\sqrt{2AA^\dagger}gT)-\rho_{\rm ph}+A^\dagger(AA^\dagger)^{-1/2}\sin(\sqrt{2AA^\dagger}gT)\rho_{\rm ph}A(A^\dagger A)^{-1/2}\sin(\sqrt{2A^\dagger A}gT)]$$
$$-\frac{\nu_1}{Q_1}(a_1^\dagger a_1\rho_{\rm ph}+\rho_{\rm ph}a_1^\dagger a_1-2a_1\rho_{\rm ph}a_1^\dagger)-\frac{\nu_2}{Q_2}(a_2^\dagger a_2\rho_{\rm ph}+\rho_{\rm ph}a_2^\dagger a_2-2a_2\rho_{\rm ph}a_2^\dagger)\ , \tag{19}$$

where we have also introduced the losses in the usual manner and Q_1 and Q_2 are the cavity Q factor at the two frequencies of the double cavity.

Now we introduce a new mode B,

$$B\equiv\frac{g_2a_1-g_1a_2}{(g_1^2+g_2^2)^{1/2}}\ , \tag{20}$$

such that

$$[A,B^\dagger]=[A,B]=[A^\dagger,B]=[A^\dagger,B^\dagger]=0\ , \tag{21}$$

so that the modes A and B are independent. In terms of the A and B modes, one can write the master equation as

$$\dot{\rho}_{\rm ph}=\frac{r}{2}[\cos(\sqrt{2AA^\dagger}gT)\rho_{\rm ph}\cos(\sqrt{2AA^\dagger}gT)-\rho_{\rm ph}+A^\dagger(AA^\dagger)^{-1/2}\sin(\sqrt{2AA^\dagger}gT)\rho_{\rm ph}A(A^\dagger A)^{-1/2}\sin(\sqrt{2A^\dagger A}gT)]$$
$$-C[(A^\dagger A\rho_{\rm ph}+\rho_{\rm ph}A^\dagger A+B^\dagger B\rho_{\rm ph}+\rho_{\rm ph}B^\dagger B-2A\rho_{\rm ph}A^\dagger-2B\rho_{\rm ph}B^\dagger)]\ , \tag{22}$$

where, for simplicity, we have assumed that

$$C=\frac{\nu_1}{Q_1}=\frac{\nu_2}{Q_2}\ . \tag{23}$$

Taking the matrix elements of Eq. (22), one obtains

$$\dot{\rho}_{\substack{n_A,n_B\\ n'_A,n'_B}}=-\frac{r}{2}\{1-\cos[\sqrt{2(n_A+1)}gT]\cos[\sqrt{2(n'_A+1)}gT\}\rho_{\substack{n_A,n_B\\ n'_A,n'_B}}+\frac{r}{2}\sin(\sqrt{2n_A}gT)\sin(\sqrt{2n'_A}gT)\rho_{\substack{n_A-1,n_B\\ n'_A-1,n'_B}}$$
$$-C\left[(n_A+n'_A)\rho_{\substack{n_A,n_B\\ n'_A,n'_B}}+(n_B+n'_B)\rho_{\substack{n_A,n_B\\ n'_A,n'_B}}-2\sqrt{(n_A+1)(n'_A+1)}\rho_{\substack{n_A+1,n_B\\ n'_A+1,n'_B}}-2\sqrt{(n_B+1)(n'_B+1)}\rho_{\substack{n_A,n_B+1\\ n'_A,n'_B+1}}\right]\ . \tag{24}$$

Since the two modes A and B commute, one could try

$$\rho_{\substack{n_A,n_B\\ n_A,n_B}}=\rho_{n,n}^{(A)}\rho_{n,n}^{(B)} \tag{25}$$

as a solution for the steady-state and diagonal version of Eq. (24). Using Eq. (25), one gets

$$2C(n_B+1)\rho_{n+1,n+1}^{(B)}-2Cn_B\rho_{n,n}^{(B)}=K\rho_{n,n}^{(B)}\ , \tag{26a}$$

$$\left[-\frac{r}{2}\sin^2[\sqrt{2(nn_A+1)}gT]-2Cn_A\right]\rho_{n,n}^{(A)}+\frac{r}{2}\sin^2(\sqrt{2n_A}gT)\rho_{n-1,n-1}^{(A)}+2C(n_A+1)\rho_{n+1,n+1}^{(A)}=-K\rho_{n,n}^{(A)}\ , \tag{26b}$$

where K is a separation constant. From Eq. (26a), one concludes that the only way that $\rho_{n,n}^{(B)}$ is normalizable is if

$$\rho_{n,n}^{(B)}=\delta_{n,0}\ \text{and}\ K=0\ , \tag{27}$$

which is to be expected since the mode B only damps and has no gain.[6]

Therefore, for the A mode, we have

$$\rho_{n,n}^{(A)}\left[-\frac{r}{2}\sin^2[\sqrt{2(n_A+1)}gT]-2Cn_A\right]+\frac{r}{2}\sin^2(\sqrt{2n_A}gT)\rho_{n-1,n-1}^{(A)}+2C(n_A+1)\rho_{n+1,n+1}^{(A)}=0\ , \tag{28}$$

which, in the case of exact balance, reduces to

$$\rho_{n,n}=\frac{r}{4Cn}\sin^2(\sqrt{2n}gT)\rho_{n-1,n-1}\ . \tag{29}$$

In Eq. (29), we have dropped the index A, since from now on we will refer only to this mode. This result is identical to the two-level micromaser, in resonance and with no thermal photons.[3]

From Eq. (29), one readily gets

$$\rho_{n,n}=\rho_{0,0}\frac{X^n}{n!}\prod_{j=1}^{n}\sin^2(\sqrt{2j}\,gT)\ , \tag{30}$$

where

$$X\equiv\frac{r}{4C}\ .$$

One can also write Eq. (30) in the form

$$\frac{\rho_{n,n}}{\rho_{0,0}}=\frac{X^n}{n!}\exp\left[\sum_{j=1}^{n}\ln[\sin^2(\sqrt{2j}\,gT)]\right]$$
$$\approx\frac{X^n}{n!}\exp\left[2\int_1^n\ln[\sin(\sqrt{2j}\,gT)]dj\right]\ . \tag{31}$$

Performing the variable change $j=\sqrt{2j}\,gT$, in terms of the new variables, Eq. (31) becomes

$$\frac{\rho_{n,n}}{\rho_{0,0}}\approx\frac{X^n}{n!}\exp\left[\frac{2}{(gT)^2}\int_{\sqrt{2}gT}^{\sqrt{2n}gT}y\ln(\sin y)dy\right]$$
$$=\frac{X^n}{n!}\frac{\sin^2(\sqrt{2n}\,gT)}{\sin^2(\sqrt{2}gT)}$$
$$\times\exp\left[\sum_{k=0}^{\infty}\frac{(-1)^k2^{k+1}B_{2k}}{(2+2k)(2k)!}(\sqrt{2}gT)^{2k}\times(n^{k+1}-1)\right]\ , \tag{32}$$

where B_{2k} are the Bernoulli numbers. Considering only the first two terms of the expansion in k, one gets

$$\frac{\rho_{n,n}}{\rho'_{0,0}}\approx\frac{[x\sin^2(\sqrt{2n}\,gT)]^n}{n!}\times\exp\left[-\left[(n-1)-\frac{g^2T^2}{6}(n^2-1)\right]\right]\ , \tag{33}$$

where $\rho'_{0,0}$ is a normalization factor that included $\sin^2(\sqrt{2}gT)$.

Now if we define

$$\theta\equiv\sqrt{2x}\,gT\ , \tag{34}$$

we can finally write

$$\frac{\rho_{n,n}}{\rho'_{0,0}}=\frac{n^n}{n!}\left[\theta^2\left[\frac{\sin(\sqrt{2n}\,gT)}{\sqrt{2n}\,gT}\right]^2\right]^n\times\exp\left[-\left[(n-1)-\frac{g^2T^2}{6}(n^2-1)\right]\right]\ . \tag{35}$$

The first laser threshold can be easily obtained, if one writes Eq. (30) in the following form:

$$\rho_{n,n}=\rho_{0,0}\prod_{k=1}^{n}\theta^2\left[\frac{\sin[(k/x)^{1/2}\theta]}{(k/x)^{1/2}\theta}\right]\ . \tag{36}$$

From Eq. (36), it is obvious that $\theta=1$ is the first threshold. If we assume that we are well below that threshold, that is, $\theta\ll1$, then

$$\rho_{n,n}\approx\rho_{0,0}\theta^{2n}\ , \tag{37}$$

which, when normalized, becomes

$$\rho_{n,n}=\frac{(\bar{n})^n}{(1+\bar{n})^{n+1}}\quad\text{with }\bar{n}=\frac{\theta^2}{1-\theta^2}\ . \tag{38}$$

Notice that the result given in Eq. (38) corresponds to a Bose photon statistics.

Above the first threshold, we will use the result given by Eq. (35). A little algebraic work[7] shows that the photon distribution has only a single maximum, above the first threshold, if $1\le\theta\le\pi/2$. For θ slightly above $\pi/2$ the distribution will have a second peak, for $\theta\approx3\pi/2$ a third peak, etc. In general,

$$\theta\approx(2n+1)\frac{\pi}{2},\quad n=0,1,2,\ldots \tag{39}$$

maximizes $\rho_{n,n}$. Therefore, for large values of θ, $\rho_{n,n}$ is a multipeaked distribution.

IV. DISCUSSION

We have formally proven that, if we inject three-level atoms at a low rate, through a double cavity, under the conditions specified above, the photon statistics of the A mode is equivalent to that of the two-level micromaser. However, we have to remember, that the A field is a linear superposition of the two fields a_1 and a_2, weighted with two coupling constants g_1 and g_2. So a novel feature here will be, for example, mode competition and of course, the quenching of the relative phase noise of the two modes. Also, the experimental setup here is different from the usual micromaser. A detailed analysis of the new features in the three-level micromaser is planned to be the subject of a forthcoming publication.

Finally, we would like to remark that in Ref. 3, a stochastic average over the random spacing of the atoms is performed in order to obtain an averaged photon number distribution. In the three-level micromaser, the same results are obtained for the A mode without such an averaging procedure. A possible explanation of this difference is that the atoms are injected, because of the strong pump, in a coherent superposition of the two upper levels, and this coherence is transferred to the fields, making the stochastic averaging unnecessary.

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