Correlated-emission laser: Nonlinear theory of the quantum-beat laser

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A nonlinear quantum theory of the quantum-beat laser is developed using a "dressed-atom–dressed-mode" master-equation approach. The theory is valid under the detuning conditions which led to correlated spontaneous emission in a linear theory. It is shown that the quenching of the quantum noise remains valid in the nonlinear theory and this operation is stable above threshold.

I. INTRODUCTION

In the optical detection of small changes of a given physical quantity, the change is converted into a phase shift (passive scheme) or frequency shift (active scheme) of a laser field. This is accomplished by sending the laser light through or generating it in a cavity whose optical path length is sensitive to the physical effect to be measured. The shift is then detected by beating the output light with that from a reference laser. The typical examples, we are bearing in mind, are gravitational wave detection[1,2] and the laser gyroscope.[3]

In the active detection scheme, the limiting noise source is the fluctuation, caused by the independent spontaneous-emission events[4] in the relative phase between the two lasers. It was shown in a recent paper by one of us[5] that the linewidth and the associated uncertainty in the relative phase may be eliminated by preparing the laser medium in a coherent superposition of two upper states either via microwave coupling, as in quantum-beat experiments,[6,7] or by coherent pumping, as in the Hanle effect.[8] The arguments of Ref. 5 were of a very general nature. Explicit expressions for the various laser parameters (e.g., gain and cross coupling coefficients) of the quantum-beat laser were derived in a subsequent publication.[9] There the linear theory has been elaborated using the Fokker-Planck approach. In particular, an explicit expression for the diffusion constant of the relative phase of the two modes has been obtained, and physical conditions under which this diffusion constant vanishes have been given. The main results of that paper can be summarized as follows. Consider the system of Fig. 1. If the atoms are prepared in a coherent superposition of $|a\rangle$ and $|b\rangle$, then the difference (not necessarily the sum) of the corresponding phases $\phi_a-\phi_b$ is fixed.

The phase ϕ_c of the ground level $|c\rangle$ is, however, a true random variable. The spontaneously emitted fields in the *a-c* and *b-c* transitions therefore average to zero. However, from the beat signal of the two spontaneously emitted fields, the random phase ϕ_c cancels, leading to a nonfluctuating contribution to the beat note of the two lasing modes. This is the idea of noise quenching, and the physical condition under which it occurs is that the field detunings from the corresponding atomic lines are equal to half the Rabi frequency of the driving field that coherently mixes the upper levels, and they are much larger than the atomic decay constants.

In this paper we present the nonlinear theory of the quantum-beat laser which is valid under the above conditions of Ref. 9. The investigation of noise quenching in the Hanle-effect laser, as well as the study of less restrictive detuning and coupling requirements, are left to separate publications.[10] The key feature of our approach is that we take into account the strong coupling of the upper states to all orders and, in this way, instead of the strongly coupled upper states, we essentially have two uncoupled dressed states. Under the conditions of Ref. 9 this picture lends itself quite naturally to a rotating-wave approximation where only one of the two dressed upper levels contributes to the laser transitions. Since now both the upper and lower levels of the two laser transitions are the same, noise quenching in this dressed-atom picture turns out to be equivalent to one-mode operation.

The paper is organized as follows: In Sec. II we present the Hamiltonian model of our system and in Sec. III the solution of the corresponding Schrödinger equation. In Sec. IV we derive the master equation for the quantum-beat laser where the deterministic part is described by the model of Sec. II. We show that, in terms of the "dressed" modes, the steady-state photon statistics is that of a one-mode laser. In Sec. V we show that the beat signal between the two modes contains a part that

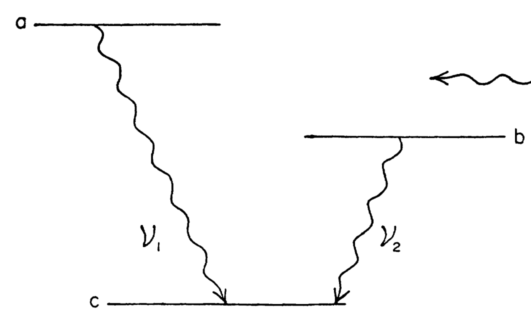

FIG. 1. In the three-level atom, the two upper levels *a* and *b* are coupled via a strong microwave field of frequency ν_3. The emissions from the (*b-c*) and (*a-c*) transitions are strongly correlated.

does not vanish in steady state, i.e., quantum noise due to spontaneous emission in the relative phase is quenched. Thus, in this more general framework of the nonlinear theory, the vanishing of the diffusion constant for the relative phase still persists. Finally, in Sec. VI, we briefly summarize the main points of the paper and discuss their connection with other coupled-two-mode laser theories, as well as further implications of the results.

II. THE MODEL

We consider the model of Ref. 9, namely, a system of three-level atoms, as shown in Fig. 1, which are being pumped into the state $|a\rangle$ at a rate r_a. The double cavity resonantly contains the two lasing modes at frequencies ν_1 and ν_2. The $|a\rangle$-$|c\rangle$ and $|b\rangle$-$|c\rangle$ transitions are assumed to be dipole allowed. The two upper levels are strongly coupled by a (classical) external field, characterized by a Rabi frequency Ω. The transitions at ν_1 and ν_2 are treated quantum mechanically, whereas the $|a\rangle$-$|b\rangle$ transition is treated semiclassically.

The Hamiltonian for the system is

$$H=H_0+V\ , \tag{1}$$

where

$$H_0=\sum_{i=a,b,c}\hbar w_i|i\rangle\langle i|+\hbar\nu_1 a_1^\dagger a_1+\hbar\nu_2 a_2^\dagger a_2\ , \tag{2}$$

and

$$\begin{aligned}V=&\hbar g_1(a_1|a\rangle\langle c|+a_1^\dagger|c\rangle\langle a|)\\&+\hbar g_2(a_2|b\rangle\langle c|+a_2^\dagger|c\rangle\langle b|)\\&-\frac{\hbar\Omega}{2}(e^{-i\nu_3t-i\phi}|a\rangle\langle b|+e^{i\nu_3t+i\phi}|b\rangle\langle a|)\ .\end{aligned} \tag{3}$$

Here $a_1,a_1^\dagger,a_2,a_2^\dagger$ are the annihilation and creation operators of photons in modes 1 and 2, g_1 and g_2 are the coupling constants for the transitions $|a\rangle$-$|c\rangle$ and $|b\rangle$-$|c\rangle$, and ν_3 is the frequency of the external field driving the $|a\rangle$-$|b\rangle$ transition.

It is convenient to work in the interaction picture, defined as

$$V_{\rm I}=e^{(i/\hbar)H_0t}Ve^{-(i/\hbar)H_0t}\ . \tag{4}$$

It is simple to show that

$$V_{\rm I}=V_1+V_2\ , \tag{5}$$

where, in an obvious matrix notation,

$$V_1=-\frac{\hbar\Omega}{2}\begin{pmatrix}0 & e^{-i\phi} & 0\\ e^{i\phi} & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{6a}$$

and

$$V_2=\hbar\begin{pmatrix}0 & 0 & g_1a_1e^{i\Delta_1t}\\ 0 & 0 & g_2a_2e^{i\Delta_2t}\\ g_1a_1^\dagger e^{-i\Delta_1t} & g_2a_2^\dagger e^{-i\Delta_2t} & 0\end{pmatrix}\ . \tag{6b}$$

Here we introduced the detunings as $\Delta_1\equiv\omega_a-\omega_c-\nu_1$ and $\Delta_2\equiv\omega_b-\omega_c-\nu_2$ and assumed that the driving field is resonant with the $|a\rangle$-$|b\rangle$ transition, i.e., $\nu_3=\omega_a-\omega_b$ or $\Delta_3=0$. Furthermore, we assume that $\Delta_1=\Delta_2\equiv\Delta$, as in Ref. 9. More general detuning conditions will be investigated in a subsequent paper.

The key point of our approach is that we define a second interaction picture, where V_1 is eliminated from the equation of motion, as

$$V_{\rm II}=e^{(i/\hbar)V_1t}V_2e^{-(i/\hbar)V_1t}\ . \tag{7}$$

Using the following property of V_1:

$$\begin{aligned}(V_1)^{2n}&=\left(\frac{\hbar\Omega}{2}\right)^{2n}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\ ,\\(V_1)^{2n+1}&=\left(\frac{\hbar\Omega}{2}\right)^{2n}(V_1)\ ,\end{aligned} \tag{8}$$

we can explicitly calculate the transformation operation. It is given by the expression

$$e^{\pm iV_1t/\hbar}=\begin{pmatrix}\cos(\Omega/2)t & \mp i\sin(\Omega/2)te^{-i\phi} & 0\\ \mp i\sin(\Omega/2)te^{i\phi} & \cos(\Omega/2)t & 0\\ 0 & 0 & 1\end{pmatrix}\ . \tag{9}$$

Using (9) in Eq. (7), we find that the interaction Hamiltonian $V_{\rm II}$ is of the form

$$V_{\rm II}=\hbar\begin{pmatrix}0&0&V_{ac}\\0&0&V_{bc}\\V_{ac}^\dagger&V_{bc}^\dagger&0\end{pmatrix}\ , \tag{10}$$

and the only nonvanishing matrix elements are given by the expressions

$$\begin{aligned}V_{ac}=&\tfrac{1}{2}[e^{i[\Delta+(\Omega/2)]t}(g_1a_1-g_2a_2e^{-i\phi})\\&+e^{i[\Delta-(\Omega/2)]t}(g_1a_1+g_2a_2e^{-i\phi})]\ ,\\V_{bc}=&\tfrac{1}{2}[-e^{i[\Delta+(\Omega/2)]t}(g_1a_1e^{i\phi}-g_2a_2)\\&+e^{i[\Delta-(\Omega/2)]t}t(g_1a_1e^{i\phi}+g_2a_2)]\ .\end{aligned} \tag{11}$$

The condition for correlated spontaneous emission was found in Ref. 9 to be $\Delta=\Omega/2$. In this case the first term in the outer square brackets in Eq. (11) is a rapidly varying one, while the second term represents a dc contribution. It is, therefore, appealing to introduce an effective rotating-wave approximation (RWA) at this point and retain the dc term only. The conditions for the validity of this RWA will be further discussed in Sec. III. Furthermore, we define the non-Hermitian operator for the composite mode as

$$A\equiv\frac{g_1a_1e^{i\phi/2}+g_2a_2e^{-i\phi/2}}{(g_1^2+g_2^2)^{1/2}}\ , \tag{12a}$$

in such a way that

$$[A,A^\dagger]=1 . \tag{12b}$$

If $g_1=g_2$ then A ($A^\dagger$) is the annihilation (creation) operator of the sum fields a_1+a_2. We also introduce the notation $g\equiv\frac{1}{2}(g_1^2+g_2^2)^{1/2}$. With this notation, and in the RWA, our interaction Hamiltonian reads as

$$V_{\rm II}=\hbar g\begin{bmatrix} 0 & 0 & Ae^{-i\phi/2} \\ 0 & 0 & Ae^{i\phi/2} \\ A^\dagger e^{i\phi/2} & A^\dagger e^{-i\phi/2} & 0 \end{bmatrix} . \tag{13}$$

In Sec. III we shall deal with the solution of the associated Schrödinger equation.

III. SOLUTION OF THE MODEL

The starting point of the nonlinear theory of the quantum-beat laser is the interaction Hamiltonian, Eq. (13). First we investigate the deterministic time evolution of the coupled three-level–two-mode system as described by this interaction Hamiltonian. The following consideration will then help to fix the initial condition and to carry out the averaging with respect to the atomic variables in the master equation for the field-density matrix in the derivation of Sec. IV. The Schrödinger equation in the second interaction picture can be written as

$$i\hbar\dot\psi=V_{\rm II}\psi . \tag{14}$$

Here ψ is a column vector with components ψ_a, ψ_b, and ψ_c. Besides the atomic variables, each component is still a function of A and $A^\dagger$, i.e., an operator acting on the mode variables. When written in components, Eq. (14) represents three coupled equations,

$$\begin{aligned} i\dot\psi_a&=gA\psi_c-i\frac{\gamma}{2}\psi_a , \\ i\dot\psi_b&=gA\psi_c-i\frac{\gamma}{2}\psi_b , \\ i\dot\psi_c&=gA^\dagger\psi_a+gA^\dagger\psi_b-i\frac{\gamma}{2}\psi_c . \end{aligned} \tag{15}$$

In Eq. (15), the phase ϕ has been eliminated by the simple transformation $\psi_a e^{i\phi/2}\to\psi_a, \psi_b e^{-i\phi/2}\to\psi_b$. Here we have introduced the phenomenological decay constant γ for the levels a, b, and c (for simplicity we have taken them to be equal). If we make the substitution

$$\psi=\exp\left[-\frac{\gamma}{2}(t-t_0)\right]\tilde\psi ,$$

then the components of $\tilde\psi$ satisfy an equation similar to (15), only the decay terms on the right-hand side (rhs) are missing. Introducing the components $\tilde\psi_\pm=(\tilde\psi_a\pm\tilde\psi_b)/\sqrt{2}$, Eq. (15) can be written as

$$\begin{aligned} i\dot{\tilde\psi}_-&=0 , \\ i\dot{\tilde\psi}_+&=g'A\tilde\psi_c , \\ i\dot{\tilde\psi}_c&=g'A^\dagger\tilde\psi_+ . \end{aligned} \tag{16}$$

Here $g'=\sqrt{2}g$. The equation for $\tilde\psi_-$ is not coupled to the other two equations, and its solution is $\tilde\psi_-=$const. In the following, we set this constant equal to zero, implying that the strong coupling of the upper two levels equalizes the amplitudes of these two states on a time scale much shorter than γ^{-1}. Thus we just found that the validity condition of our RWA is

$$\Omega\gg\gamma . \tag{17}$$

Now, it is easy to solve Eq. (15). The solution, satisfying the initial condition that the atom is injected into the excited level $|a\rangle$ at time t_0 and $\psi_a(t_0)=\hat\psi_F(t_0)$ is a function of the field variables only, is given by

$$\begin{aligned} \psi_a=\psi_b&=\frac{1}{\sqrt{2}}e^{-(\gamma/2)(t-t_0)}\cos[\hat\omega_A(t-t_0)]\hat\psi_F(t_0) , \\ \psi_c&=-ie^{-(\gamma/2)(t-t_0)}A^{-1}(AA^\dagger)^{1/2} \\ &\quad\times\sin[\hat\omega_A(t-t_0)]\hat\psi_F(t_0) . \end{aligned} \tag{18}$$

Here $\hat\omega_A\equiv g'(AA^\dagger)^{1/2}$.

In other words, the dynamics of the system is very similar to the dynamics of a simple two-level system coupled to one quantized mode of the radiation field. In Sec. IV we shall use Eq. (18) to obtain the master equation for the field-density operator and to determine the resulting photon statistics.

IV. MASTER EQUATION AND PHOTON STATISTICS

The density matrix of the coupled system three-level atom plus two-mode field satisfies the following equation of motion in the second interaction picture introduced above:

$$\dot\rho=-\frac{i}{\hbar}[V_{\rm II},\rho] , \tag{19}$$

where the bracket stands for the commutator.

The reduced density operator ρ_F for the field itself is defined as the trace of ρ over the atoms

$$\rho_F={\rm Tr}_a\rho . \tag{20}$$

Using the expression Eq. (13) for $V_{\rm II}$ in (19) and carrying out the trace operation, we find that ρ_F satisfies the following equation of motion:

$$\dot\rho_F=-ig\{[A,(\rho_{ca}+\rho_{cb})]+[A^\dagger,(\rho_{ac}+\rho_{bc})]\}+\mathcal{L} , \tag{21}$$

where $\mathcal{L}$ is a loss term, which we shall specify later. To proceed further we need an expression for $\rho_{ac}+\rho_{bc}$ and its Hermitian conjugate. We adopt the following procedure[4] to obtain this expression. We first calculate the contribution of one atom injected at time t_0 into the upper level $|a\rangle$ and then sum the contribution of all atoms which are injected at random times $t-\gamma^{-1}<t_0<t$ at a rate r_a. In this way one finds that

$$\rho_{ac}+\rho_{bc}=r_a\int_{t-\gamma^{-1}}^{t}dt_0[\psi_a(t,t_0)+\psi_b(t,t_0)]\psi_c^\dagger(t,t_0) . \tag{22}$$

We now substitute the expression for ψ_a, ψ_b, and ψ_c from Eq. (18) which yields

$$\rho_{ac}+\rho_{bc}=\sqrt{2}ir_a\int_{t-\gamma^{-1}}^{t}dt_0 e^{-\gamma(t-t_0)}\cos[\hat{\omega}_A(t-t_0)] \times\rho_F(t_0)\sin[\hat{\omega}_A(t-t_0)] \times(AA^\dagger)^{1/2}(A^\dagger)^{-1} . \quad (23)$$

In this expression we can approximate $\rho_F(t_0)$ by $\rho_F(t)$, since the dynamics of the photon field is governed by the cavity lifetime γ_c^{-1}, which is much longer than γ^{-1}, and thus during the integration time ρ_F does not change appreciably. Then we can extend the lower limit of the integration to $-\infty$, since due to the exponential damping factor in the integrand the contribution from $t_0<t-\gamma^{-1}$ is negligible. Finally, when acting on the right $(A^\dagger)^{-1}$ can be replaced by $(AA^\dagger)^{-1}A$. After performing these steps and taking the nn' matrix element of Eq. (23), it is easy to carry out the time integration, giving

$$(\rho_{ac}+\rho_{bc})_{n,n'}=ir_a g(R^+_{n+1,n'}-R^-_{n+1,n'})\rho_{F_{n,n'-1}} , \quad (24a)$$

where

$$R^\pm_{n,n'}=\frac{(\sqrt{n}\pm\sqrt{n'})^2}{\gamma^2+2g^2(\sqrt{n}\pm\sqrt{n'})^2} . \quad (24b)$$

Now, we specify the loss term in (21) in the usual manner[4] as

$$\mathcal{L}=-\frac{\nu_1}{2Q_1}(a_1^\dagger a_1\rho_F+\rho_F a_1^\dagger a_1-2a_1\rho_F a_1^\dagger)-(1\rightarrow 2) , \quad (25)$$

where Q_i is the Q factor of the double cavity at frequency $\nu_i (i=1,2)$. It is convenient, at this point, to introduce the difference mode as

$$B\equiv\frac{g_2a_1-g_1a_2}{(g_1^2+g_2^2)^{1/2}} \quad (26a)$$

which has the properties

$$[B,B^\dagger]=1, \quad [A,B]=[A,B^\dagger]=0 , \quad (26b)$$

i.e., the modes A and B are independent. One can now express a_1 and a_2 in terms of A and B and use this expression in (25). Finally, upon inserting (24) and its Hermitian conjugate and (25) into (21), we obtain the following master equation for the matrix elements of the field-density operator (for the sake of simplicity the subscript F will henceforth be omitted from ρ_F):

$$\dot{\rho}_{\substack{n_A,n_B\\ n'_A,n'_B}}=\left[\frac{\sqrt{n_An'_A}\mathcal{A}}{1+\mathcal{N}_{n_A-1,n'_A-1}\mathcal{B}/\mathcal{A}}\rho_{\substack{n_A-1,n_B\\ n'_A-1,n'_B}}-\frac{\mathcal{N}'_{n_A,n'_A}\mathcal{A}}{1+\mathcal{N}_{n_A,n'_A}\mathcal{B}/\mathcal{A}}\rho_{\substack{n_A,n_B\\ n'_A,n'_B}}\right]$$
$$-\frac{\gamma_c}{2}\left[(n_A+n'_A+n_B+n'_B)\rho_{\substack{n_A,n_B\\ n'_A,n'_B}}-2\sqrt{(n_A+1)(n'_A+1)}\rho_{\substack{n_A+1,n_B\\ n'_A+1,n'_B}}-2\sqrt{(n_B+1)(n'_B+1)}\rho_{\substack{n_A,n_B+1\\ n'_A,n'_B+1}}\right] . \quad (27)$$

Here, for simplicity, we have assumed that

$$\frac{\nu_1}{Q_1}=\frac{\nu_2}{Q_2}\equiv\gamma_c , \quad (28a)$$

i.e., the cavity lifetime for both modes is the same, and introduced the notations

$$\mathcal{A}\equiv 4r_a\left[\frac{g}{\gamma}\right]^2, \quad \mathcal{B}\equiv 8\left[\frac{g}{\gamma}\right]^2\mathcal{A} ,$$
$$\mathcal{N}_{n,n'}=\tfrac{1}{2}(n+1+n'+1)+\tfrac{1}{16}(n-n')^2\mathcal{B}/\mathcal{A} , \quad (28b)$$
$$\mathcal{N}'_{n,n'}\equiv\tfrac{1}{2}(n+1+n'+1)+\tfrac{1}{8}(n-n')^2\mathcal{B}/\mathcal{A} .$$

As usual, $\mathcal{A}$ has the meaning of the linear gain and $\mathcal{B}$ is the self-saturation coefficient.

The central result of our paper is represented by Eq. (27). The equation, in general, separates into two independent equations corresponding to the A and B modes. When written in the above form it is easy to see that the part describing the A mode is identical with the master equation of a one-mode laser with a two-level active medium[4] [see Eqs. (31) and (34a) below]. In fact, if one has $g_1=g_2\equiv g_0$ then $g^2=g_0^2/2$ and the identity is complete in terms of g_0.

First we briefly consider the question of photon statistics resulting from this master equation. As is well known,[4] one obtains the steady-state photon statistics by taking the diagonal elements $(n_A=n'_A,\ n_B=n'_B)$ and setting $\partial/\partial t=0$ in Eq. (27). Since A and B are independent modes, by virtue of Eq. (26b), and there is no coupling between them, the general solution must be separable, i.e.,

$$\rho_{\substack{n_A,n_B\\ n_A,n_B}}=\rho^{(A)}_{n_A,n_A}\rho^{(B)}_{n_B,n_B} . \quad (29)$$

When substituted into the diagonal and steady version of Eq. (27) this ansatz gives two separate equations for $\rho^{(A)}$ and $\rho^{(B)}$. From the condition of normalizability of $\rho^{(B)}$ we find that the separation constant must equal zero, and the solution for $\rho^{(B)}$ is

$$\rho^{(B)}_{n,n}=\delta_{n,0} . \quad (30)$$

This result could be anticipated, since there is no gain in mode B and it is coupled to a loss reservoir only.

The equation for the steady-state photon distribution $\rho^{(A)}_{n,n}$ in mode A is then

$$\frac{n\mathcal{A}}{1+\mathcal{N}_{n-1,n-1}\mathcal{B}/\mathcal{A}}\rho^{(A)}_{n-1,n-1}-\frac{(n+1)\mathcal{A}}{1+\mathcal{N}_{n,n}\mathcal{B}/\mathcal{A}}\rho^{(A)}_{n,n}-\gamma_c[n\rho^{(A)}_{n,n}-(n+1)\rho^{(A)}_{n+1,n+1}]=0 . \quad (31)$$

This equation is identical with the one that describes the photon statistics of a one-mode laser with a two-level active medium. Therefore, in terms of the composite mode A, the quantum-beat laser exhibits the same type of behavior (photon statistics, threshold, saturation) as the one-mode laser. The detailed analysis of this equation can be found in the literature (see, e.g., Ref. 4) and we do not pursue it here any further. We shall in Sec. VI, however, return to the discussion and interpretation of this result. In Sec. V we shall elaborate on another consequence of Eq. (27), namely, the diffusion constant of and noise quenching from the relative phase between modes a_1 and a_2.

V. VANISHING OF DIFFUSION CONSTANT FOR RELATIVE PHASE

The beat signal between two modes of an ordinary laser or between two independent lasers decays to zero in time because, due to the independent spontaneous-emission events, the relative phase fluctuates freely. The decay rate for the beat signal is, in principle, given by the Schawlow-Townes linewidth.[4] Below we show that in the quantum-beat laser the beat signal has a nonvanishing part in the steady state, which means that the quantum noise due to spontaneous emission is quenched from the relative phase.

Our starting point is the observation that, by using the inverse of (12a) and (26a), the beat signal can be expressed as the real part of

$$e^{i(\nu_1-\nu_2)t}\mathrm{Tr}(a_1^\dagger a_2\rho) = \tfrac{1}{2}e^{i\nu_3 t}\mathrm{Tr}[(A^\dagger A - B^\dagger B + AB^\dagger - A^\dagger B)\rho] , \quad (32)$$

where, for simplicity, we assumed that $g_1=g_2$ and ρ satisfies equation (27). The master equation (27) can be factorized even in the time dependent case by the substitution

$$\rho_{n_A,n_B \atop n_A',n_B'}(t)=\rho^{(A)}_{n_A,n_A'}(t)\rho^{(B)}_{n_B,n_B'}(t) , \quad (33)$$

leading to two separate equations for $\rho^{(A)}$ and $\rho^{(B)}$,

$$\dot\rho^{(A)}_{n,n'}=\left[\frac{\sqrt{nn'}\mathcal{A}}{1+\mathcal{N}_{n-1,n'-1}\mathcal{B}/\mathcal{A}}\rho^{(A)}_{n-1,n'-1}-\frac{\mathcal{N}'_{nn'}\mathcal{A}}{1+\mathcal{N}_{nn'}\mathcal{B}/\mathcal{A}}\rho^{(A)}_{n,n'}\right]-\frac{\gamma_c}{2}[(n+n')\rho^{(A)}_{n,n'}-2\sqrt{(n+1)(n'+1)}\rho^{(A)}_{n+1,n'+1}] , \quad (34a)$$

$$\dot\rho^{(B)}_{n,n'}=-\frac{\gamma_c}{2}[(n+n')\rho^{(B)}_{n,n'}-2\sqrt{(n+1)(n'+1)}\rho^{(B)}_{n+1,n'+1}] . \quad (34b)$$

Looking for time-dependent solutions of Eqs. (34a) and (34b) one can show[4] that the general solution is of the form

$$\rho_{n,n}(t)=\sum_{j=0}^{\infty}\phi_j(n)e^{-\mu_j t} \quad (35a)$$

for the diagonal elements and

$$\rho_{n,n+p}(t)=\sum_{j=0}^{\infty}\phi_j(n,p)e^{-\mu_j^{(P)}t} \quad (35b)$$

for the off-diagonal elements. Furthermore, $\nu_j \geq 0$ and the lowest eigenvalue is $\mu_0=0$, allowing for a nonvanishing stationary solution for the diagonal elements. Also, $\mu_j^{(p)}>0$ for $p\neq 0$, so that the off-diagonal elements decay to zero for large times. In the case of an ordinary two-mode laser (OL) or two separate lasers, the density matrix factorizes in terms of the original a_1 and a_2 modes,

$$\rho^{(1.2)}=\rho^{(1)}\rho^{(2)} , \quad (36)$$

and the beat signal defined in (32) will be proportional to the product of the off-diagonal elements of the type

$$\mathrm{Re}\langle a_1^\dagger a_2\rangle_{\mathrm{OL}}\approx\frac{1}{2}\left[\sum_{n,n'}\rho^{(1)}_{n,n+1}(t)\rho^{(2)}_{n',n'-1}(t)\sqrt{(n+1)(n')}+\mathrm{c.c.}\right] . \quad (37)$$

According to Eq. (35b) these matrix elements will decay at a rate $\mu_0^{(1)}$ (neglecting the more rapidly decaying terms). This defines the phase-diffusion coefficient D as $\mu_0^{(1)}\equiv D/2$, where from Eq. (34a) it can be shown that

$$D=\frac{\gamma_c}{2\langle n\rangle} , \quad (38)$$

which is the Schawlow-Townes linewidth.

The crucial difference between the quantum-beat laser (QBL) and the ordinary two-mode laser is that the beat signal contains a part that is diagonal in the "true" eigenmodes A and B of the system,

$$\langle a_1^\dagger a_2\rangle_{\mathrm{QBL}}\sim\sum_n n\rho^{(A)}_{n,n}(t)+\cdots . \quad (39)$$

According to (35a) these matrix elements will decay at a rate $\mu_0=0$, i.e., there is always a nonvanishing part of the beat signal. In fact, the beat signal is a measure of the true eigenmodes of the system. If we maintain the definition of the diffusion coefficient as twice the lowest decay rate in (32), then

$$D = 0 \quad (40)$$

for the quantum-beat laser and the beat signal will be given by

$$\mathrm{Re}\langle a_a^\dagger a_2 \rangle(t) = \cos(\nu_3 t)\bar{n}_A \ . \quad (41)$$

We can interpret this result in the following way. The beat signal is nothing else but the beating of the signal in the true eigenmodes of the system with itself. This explains why the fluctuating phase cancels and we obtain a nonvanishing steady-state contribution.

VI. DISCUSSION AND SUMMARY

Starting from a Hamiltonian model of a three-level atomic system, where there is a strong classical coupling between the upper two levels and these levels are in turn coupled to the lower level via interacting with two modes of the quantized radiation field (see Fig. 1), we have developed the nonlinear quantum theory of the quantum-beat laser. The key feature of our theory is the transformation given by Eq. (7). In this way, instead of the two strongly coupled upper levels, we introduced two, virtually uncoupled, "dressed" states. Under the special detuning conditions for correlated spontaneous emission, only one of these modes is coupled resonantly to the ground state, while the other dressed state is coupled via antiresonant terms only. This suggests a rotating-wave approximation that reduces the problem to a two-level system coupled resonantly to one quantized radiation mode. This quantized mode is, however, a linear combination of the initial "bare" modes. The combination $\psi_+ = \psi_a + \psi_b$ of the dressed states is coupled to the ground state and $\psi_- = \psi_a - \psi_b$ is decoupled from the system. Thus our approach is a "dressed-atom–dressed-mode" description. The emerging physical picture is shown in Fig. 2. It is worthwhile to mention at this point that the condition $\Delta = \Omega/2$, found in Ref. 9 and elaborated here in detail, selects $\psi_+ \to \psi_c$ as the laser transition and the combination $a_1 + a_2 \sim A$ as the lasing mode. The condition $\Delta = -\Omega/2$ would select the $\psi_- \to \psi_c$ transition and the combination $a_1 - a_2 \sim B$, and we still get the nonvanishing beat signal in steady state. Summarizing the conditions under which our approach is valid, we find that the Rabi frequency of the coupling between the upper levels, the atomic decay rate, and the cavity loss rate have to satisfy the following inequality:

$$\Omega \gg \gamma \gg \gamma_c \ , \quad (42)$$

a condition which is well fulfilled in most cases. If we solve the equation $\Delta_1 = \Delta_2 = \Omega/2$ for the frequencies we find that

$$\nu_1 - \nu_2 = \nu_3 \ , \quad (43a)$$

which is the usual beat condition, and

$$\tfrac{1}{2}(\nu_1 + \nu_2) = \frac{\omega_a + \omega_b}{2} + \frac{\Omega}{2} - \omega_c \ . \quad (43b)$$

This second condition states that the frequency of the lasing mode A (the average of the mode frequencies ν_1 and ν_2) is resonant with the transition frequency of the $\psi_+ \to \psi_c$ transition. This justifies the dressed-atom–dressed-mode picture of Fig. 2. One should also notice that the beat signal between modes 1 and 2 is directly proportional to the intensity in the dressed mode A, thus providing a direct way of measuring properties of the dressed mode. Finally, normally a strong microwave pump is a very classical object, the reason being that if we compare, at equal powers, the input microwave field and the output optical correlated-emission laser (CEL) signal, the microwave photon number is four or five orders of magnitude larger than the CEL photon number at optical frequencies. Therefore the Schawlow-Townes linewidth of the microwave pump, $(a/2\bar{n})$, is very small and we need not be concerned about noisy input pumps.

For the sake of completeness, we have included a derivation of the full nonlinear Fokker-Planck equation in the Appendix.

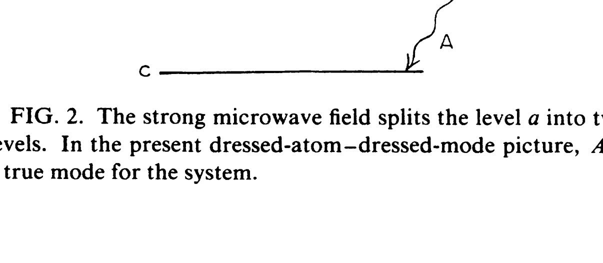

FIG. 2. The strong microwave field splits the level a into two levels. In the present dressed-atom–dressed-mode picture, A is a true mode for the system.

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APPENDIX

Equation (21) can be transformed into the Fokker-Planck representation, if one writes

$$\begin{aligned} &a_1\rho \leftrightarrow \alpha_1 P, \quad \rho a_1^\dagger \leftrightarrow \alpha_1^* P \ , \\ &a_1^\dagger \rho \leftrightarrow \left[\alpha_1^* - \frac{\partial}{\partial \alpha_1}\right] P, \quad \rho a_1 \leftrightarrow \left[\alpha_1 - \frac{\partial}{\partial \alpha_1^*}\right] P \ , \end{aligned} \quad (A1)$$

where P is the usual Glauber representation. If we replace a_1 and a_2 by the corresponding differential operators in Eq. (23), and then perform the integral, we obtain

$$\rho_{ac}+\rho_{bc}=2ir_a\left[\frac{1}{\alpha_1^*+\alpha_2^*}\right]g\beta^*\frac{[\gamma^2+g^2(\beta^{*2}-\beta^2)]}{[\gamma^2+g^2(\beta^{*2}+\beta^2)]^2-4g^4(\beta\beta^*)^2}\ , \tag{A2}$$

where

$$\beta\equiv\left[|\alpha_1+\alpha_2|^2-(\alpha_1+\alpha_2)\left[\frac{\partial}{\partial\alpha_1}+\frac{\partial}{\partial\alpha_2}\right]\right]^{1/2}. \tag{A3}$$

The Fokker-Planck equation then reads

$$\frac{\partial P(\alpha_1,\alpha_1^*,\alpha_2,\alpha_2^*)}{\partial t}=-\frac{2g^2r_a}{\gamma^2}\left[\left[\frac{\partial}{\partial\alpha_1}+\frac{\partial}{\partial\alpha_2}\right](\alpha_1+\alpha_2)-\left[\frac{\partial}{\partial\alpha_1}+\frac{\partial}{\partial\alpha_2}\right]\left[\frac{\partial}{\partial\alpha_1^*}+\frac{\partial}{\partial\alpha_2^*}\right]\right]$$
$$\times\left\{1+\left[\frac{g}{\gamma}\right]^2\left[(\alpha_1+\alpha_2)\left[\frac{\partial}{\partial\alpha_1}+\frac{\partial}{\partial\alpha_2}\right]-(\alpha_1^*+\alpha_2^*)\left[\frac{\partial}{\partial\alpha_1^*}+\frac{\partial}{\partial\alpha_2^*}\right]\right]\right\}\frac{1}{D}+\text{c.c.}\ , \tag{A4}$$

where

$$D=1+\left[\frac{g}{\gamma}\right]^2\left[4|\alpha_1+\alpha_2|^2-2(\alpha_1+\alpha_2)\left[\frac{\partial}{\partial\alpha_1}+\frac{\partial}{\partial\alpha_2}\right]-2(\alpha_1^*+\alpha_2^*)\left[\frac{\partial}{\partial\alpha_1^*}+\frac{\partial}{\partial\alpha_2^*}\right]\right]+O\left[\frac{g}{\gamma}\right]^4. \tag{A5}$$

Defining $\alpha_1=\rho_1e^{i\theta_1}$, $\alpha_2=\rho_2e^{i\theta_2}$, $\theta=\theta_1-\theta_2$, $\mu=(\theta_1+\theta_2)/2$, then we have

$$\frac{\partial}{\partial\alpha_1}=\tfrac{1}{2}e^{-i\theta_1}\frac{\partial}{\partial r_1}+\frac{e^{-i\theta_1}}{2ir_1}\frac{\partial}{\partial\theta_1}\ , \tag{A6}$$

$$\frac{\partial}{\partial\alpha_2}=\tfrac{1}{2}e^{-i\theta_2}\frac{\partial}{\partial r_2}+\frac{e^{-i\theta_2}}{2ir_1}\frac{\partial}{\partial\theta_2}\ , \tag{A7}$$

$$\frac{\partial}{\partial\theta_1}=\frac{1}{2}\frac{\partial}{\partial\mu}+\frac{\partial}{\partial\theta}\ , \tag{A8}$$

$$\frac{\partial}{\partial\theta_2}=\frac{1}{2}\frac{\partial}{\partial\mu}-\frac{\partial}{\partial\theta}\ . \tag{A9}$$

If we expand the Fokker-Planck equation to fourth order in g and use the relations (A6) to (A9), after a rather tedious calculation, we obtain the full Fokker-Planck equation,

$$\dot{P}=d_0P+d(\rho_1)\frac{\partial P}{\partial\rho_1}+d(\rho_2)\frac{\partial P}{\partial\rho_2}+d(\mu)\frac{\partial P}{\partial\mu}+d(\theta)\frac{\partial P}{\partial\theta}+D(\theta)\frac{\partial^2P}{\partial\theta^2}+D(\mu)\frac{\partial^2P}{\partial\mu^2}+D(\rho_1)\frac{\partial^2P}{\partial\rho_1^2}+D(\rho_2)\frac{\partial^2P}{\partial\rho_2^2}$$
$$+D(\theta,\mu)\frac{\partial^2P}{\partial\theta\,\partial\mu}+D(\rho_1,\rho_2)\frac{\partial^2P}{\partial\rho_1\partial\rho_2}+D(\rho_1,\mu)\frac{\partial^2P}{\partial\rho_1\partial\mu}+D(\rho_2,\mu)\frac{\partial^2P}{\partial\rho_2\partial\mu}+D(\rho_1,\theta)\frac{\partial^2P}{\partial\rho_1\partial\theta}+D(\rho_2,\theta)\frac{\partial^2P}{\partial\rho_2,\partial\theta}\ , \tag{A10}$$

where (with $g_1=g_2=g_0$),

$$d_0=-\frac{g_0^2r_a}{\gamma^2}-32\frac{g_0^4}{\gamma^4}r_a(1-\rho_1^2-\rho_2^2-2\rho_1\rho_2\cos\theta)\ , \tag{A11}$$

$$d(\rho_1)=-\frac{g_0^2r_a}{4\gamma^2}\left[\left[\rho_1-\frac{1}{2\rho_1}\right]+\rho_2\cos\theta\right]$$
$$-\frac{g_0^4r_a}{\gamma^4}\left[(30-4\alpha\alpha^*)(\rho_1+\rho_2\cos\theta)-\frac{3}{2\rho_1}[\rho_1^2+2\rho_1\rho_2\cos\theta+\rho_2^2\cos(2\theta)]+\frac{5\alpha\alpha^*-8}{2\rho_1}\right]\ , \tag{A12}$$

$$d(\rho_2)=-\frac{g_0^2r_a}{4\gamma^2}\left[\left[\rho_2-\frac{1}{2\rho_2}\right]+\rho_1\cos\theta\right]$$
$$-\frac{g_0^4r_a}{\gamma^4}\left[(30-4\alpha\alpha^*)(\rho_1\cos\theta+\rho_2)-\frac{3}{2\rho_2}[\rho_2^2+\rho_1^2\cos(2\theta)+2\rho_1\rho_2\cos\theta]+\frac{5\alpha\alpha^*-8}{2\rho_2}\right]\ , \tag{A13}$$

$$d(\mu)=-\frac{g_0^2r_a}{8\gamma^2}\left[\frac{\rho_2}{\rho_1}-\frac{\rho_1}{\rho_2}\right]\sin\theta-\frac{g_0^4r_a}{\gamma^4}\left[\frac{(30-4\alpha\alpha^*)}{2}\left[\frac{\rho_1}{\rho_2}-\frac{\rho_2}{\rho_1}\right]\sin\theta-\frac{3}{2}\left\{\left[\left[\frac{\rho_1}{\rho_2}\right]^2-\left[\frac{\rho_2}{\rho_1}\right]^2\right]\sin(2\theta)+2\left[\frac{\rho_1}{\rho_2}-\frac{\rho_2}{\rho_1}\right]\sin\theta\right\}\right], \tag{A14}$$

$$d(\theta)=-\frac{g_0^2r_a}{4\gamma^2}\left[\frac{\rho_2}{\rho_1}-\frac{\rho_1}{\rho_2}\right]\sin\theta-\frac{g_0^4r_a}{\gamma^4}\left[-(30-4\alpha\alpha^*)\left[\frac{\rho_1}{\rho_2}+\frac{\rho_2}{\rho_1}\right]\sin\theta+6\left[\frac{\rho_1}{\rho_2}+\frac{\rho_2}{\rho_1}\right]\sin\theta+3\left[\frac{\rho_1^2}{\rho_2^2}+\frac{\rho_2^2}{\rho_1^2}\right]\sin(2\theta)\right], \tag{A15}$$

$$D(\theta)=\frac{g_0^2r_a}{8\gamma^2}\left[\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}-\frac{2}{\rho_1\rho_2}\cos\theta\right]-\frac{g_0^4r_a}{\gamma^4}\left[-\frac{3}{2}\left[\frac{\rho_1^2+2\rho_1\rho_2\cos\theta+\rho_2^2\cos(2\theta)}{\rho_1^2}+\frac{\rho_1^2\cos(2\theta)+2\rho_1\rho_2\cos\theta+\rho_2^2}{\rho_2^2}-\frac{2(\rho_1^2\cos\theta+2\rho_1\rho_2+\rho_2^2\cos\theta)}{\rho_1\rho_2}\right]+\frac{5\alpha\alpha^*-8}{2}\left[\frac{1}{\rho_1}-\frac{1}{\rho_2}\right]^2\right], \tag{A16}$$

$$D(\mu)=\frac{g_0^2r_a}{32\gamma^2}\left[\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}+\frac{2}{\rho_1\rho_2}\cos\theta\right]-\frac{g_0^4r_a}{\gamma^2}\left[-\frac{3}{8}\left[\frac{\rho_1^2+2\rho_1\rho_2\cos\theta+\rho_2^2\cos(2\theta)}{\rho_1^2}+\frac{\rho_1^2\cos(2\theta)+2\rho_1\rho_2\cos\theta+\rho_2^2}{\rho_2^2}+\frac{2(\rho_1^2\cos\theta+2\rho_1\rho_2+\rho_2^2\cos\theta)}{\rho_1\rho_2}\right]+\frac{5\alpha\alpha^*-8}{8}\left[\frac{1}{\rho_1}+\frac{1}{\rho_2}\right]^2\right], \tag{A17}$$

$$D(\rho_1)=\frac{g_0^2r_a}{8\gamma^2}-\frac{g_0^4r_a}{\gamma^4}\left[\tfrac{3}{2}[\rho_1^2+2\rho_1\rho_2\cos\theta+\rho_2^2\cos(2\theta)]+\frac{5\alpha\alpha^*-8}{2}\right], \tag{A18}$$

$$D(\rho_2)=\frac{g_0^2r_a}{8\gamma^2}-\frac{g_0^4r_a}{\gamma^4}\left[\tfrac{3}{2}[\rho_1^2\cos(2\theta)+2\rho_1\rho_2\cos\theta+\rho_2^2]+\frac{5\alpha\alpha^*-8}{2}\right], \tag{A19}$$

$$D(\rho_1,\mu)=-\frac{g_0^2r_a}{8\gamma^2\rho_2}\sin\theta-\frac{g_0^4r_a}{\gamma^4}\left[-\frac{3}{2\rho_1}[\rho_2^2\sin(2\theta)+2\rho_1\rho_2\sin\theta]+\frac{3}{2\rho_2}(\rho_1^2-\rho_2^2)\sin\theta+\frac{5\alpha\alpha^*-8}{2\rho_2}\sin\theta\right], \tag{A20}$$

$$D(\rho_2,\mu)=-\frac{g_0^2r_a}{8\gamma^2\rho_1}\sin\theta-\frac{g_0^4r_a}{\gamma^4}\left[\frac{3}{2\rho_2}[\rho_1^2\sin(2\theta)+2\rho_1\rho_2\sin\theta]+\frac{3}{2\rho_1}(\rho_1^2-\rho_2^2)\sin\theta-\frac{5\alpha\alpha^*-8}{2\rho_1}\sin\theta\right], \tag{A21}$$

$$D(\rho_1,\theta)=\frac{g_0^2r_a}{4\gamma^2\rho_2}\sin\theta-\frac{g_0^4r_a}{\gamma^4}\left[-\frac{3}{\rho_1}[2\rho_1\rho_2\sin\theta+\rho_2^2\sin(2\theta)]-\frac{3}{\rho_2}(\rho_1^2-\rho_2^2)\sin\theta-\frac{5\alpha\alpha^*-8}{\rho_2}\sin\theta\right], \tag{A22}$$

$$D(\rho_2,\theta)=\frac{g_0^2r_a}{4\gamma^2\rho_1}\sin\theta-\frac{g_0^4r_a}{\gamma^4}\left[\frac{3}{\rho_1}(\rho_1^2-\rho_2^2)\sin\theta-\frac{3}{\rho_2}[\rho_1^2\sin(2\theta)+2\rho_1\rho_2\sin\theta]-\frac{5\alpha\alpha^*-8}{\rho_1}\sin\theta\right], \tag{A23}$$

$$D(\rho_1,\rho_2)=\frac{g_0^2r_a}{8\gamma^2}\cos\theta-\frac{g_0^4r_a}{\gamma^4}[3(\rho_1^2\cos\theta+2\rho_1\rho_2+\rho_2^2\cos\theta)+(5\alpha\alpha^*-8)\cos\theta], \tag{A24}$$

$$D(\theta,\mu)=\frac{g_0^2r_a}{8\gamma^2}\left[\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2}\right]-\frac{g_0^4r_a}{\gamma^4}\left[+\frac{3}{2\rho_2^2}[\rho_1^2\cos(2\theta)+2\rho_1\rho_2\cos\theta+\rho_2^2]-\frac{3}{2\rho_1^2}[\rho_1^2+2\rho_1\rho_2\cos\theta+\rho_2^2\cos(2\theta)]+\frac{5\alpha\alpha^*-8}{2}\left[\frac{1}{\rho_1^2}-\frac{1}{\rho_2^2}\right]\right], \tag{A25}$$

where

$$\alpha\alpha^*\equiv\rho_1^2+\rho_2^2+2\rho_1\rho_2\cos\theta\ . \tag{A26}$$

It is interesting to notice that if $\rho_1=\rho_2$ ($\bar{n}_1=\bar{n}_2=\bar{n}$) then

$$D(\theta)=\frac{g_0^2 r_a}{4\gamma^2\bar{n}}(1-\cos\theta)-\frac{3g_0^4 r_a}{\gamma^4}[1-\cos(2\theta)]\ , \tag{A27}$$

and obviously $D(\theta)=0$ for $\theta=0$, so that the quenching of the relative phase noise is still true for higher-order nonlinear terms.

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