Photon-number distributions for quantum fields generated in nonlinear optical processes

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We show that in a large class of nonlinear optical processes which are used to generate the squeezed states of the radiation field, the quantum state of the generated radiation is given by a Gaussian Wigner function. This is so even when *losses* in the medium are included. The photon-number distributions for such fields are evaluated. The number distributions exhibit oscillatory character for the range of parameters for which the field is squeezed.

Most processes in nonlinear optics can be described in terms of an effective Hamiltonian involving the interaction of either three modes or four modes. The three-wave interaction $\omega_c \rightleftarrows \omega_a + \omega_b$ can be written as

$$H = gc^{\dagger}ab + \text{H.c.} \;, \tag{1}$$

where the coupling constant g depends on the second-order nonlinearity $\chi^{(2)}$ of the medium. The four-wave interaction $\omega_c + \omega_d \rightleftarrows \omega_a + \omega_b$ is given by

$$H = gc^{\dagger}d^{\dagger}ab + \text{H.c.} \;, \tag{2}$$

where g is now proportional to the third-order susceptibility $\chi^{(3)}$ of the medium. All parametric and four-wave-mixing processes can be studied in terms of either (1) or (2). In most cases the pumps are quite intense and the pump modes can be treated as prescribed c-number fields. The Hamiltonians (1) and (2) can be approximated by

$$H = Gab + \text{H.c.} \tag{3}$$

The dynamics[1] of the modes a and b can be studied in terms of the solutions of (3). It turns out that if initially the modes a and b are in the vacuum state, then the state at time t can be described in terms of the Wigner distribution function $\Phi(z_a,z_b)$ of the form[1]

$$\Phi(z_a z_b) = \exp[P(z_a,z_b)] \;, \tag{4}$$

where $P(z_a,z_b)$ is a quadratic form involving the variables z_a, z_b, z_a^*, and z_b^*. In the considerations of the squeezed states[2-8] of the radiation field, one examines the dynamics of a mode γ, which is a linear combination of a and b,

$$\gamma = \mu a + \nu b \quad \text{with} \;\; |\mu|^2 + |\nu|^2 = 1 \;. \tag{5}$$

It is clear from (4) and (5) that the Wigner function corresponding to the mode γ will also be Gaussian, i.e.,

$$\Phi(z_\gamma) = \exp[P(z_\gamma)] \;, \tag{6}$$

where P now is a quadratic form involving z_γ and z_γ^*. In most of the recent works on squeezing, the modes a and b are generated starting from no input fields at the frequencies ω_a and ω_b.

It should be noted that in any realistic case the *losses* in the medium must be accounted for. The effective-Hamiltonian description is inadequate for this purpose. However, we can use the master-equation techniques.[4,9] Such methods show that the Wigner function for the mode γ is still given by[9] (6), where the coefficients in the quadratic form P depend upon losses. Note that such a Wigner function corresponds to a mixed state of the radiation field.

Thus from the foregoing it is clear that the state of the generated field in a large class of nonlinear optical phenomena can be characterized by a Wigner function of the form (6), which also includes as a special case the squeezed vacuum,[8] as well as the output of an arbitrary four-wave mixer. Even in a more complex problem of the behavior of *many atoms in a cavity*,[6,10] the fluctuation behavior of the transmitted field can be characterized in an approximate manner by a Gaussian Wigner function. Hence in this investigation we examine the number distributions of the fields characterized by a Gaussian Wigner function.

We thus consider a radiation field in the mode γ, characterized by the Wigner function given by[11]

$$\Phi(z) = \frac{1}{\pi(\tau^2 - 4|\mu|^2)^{1/2}} \times \exp\left\{ \frac{-[\mu z^2 + \mu^*(z^*)^2 + \tau|z|^2]}{\tau^2 - 4|\mu|^2} \right\} , \tag{7}$$

with

$$\tau > \mu + \mu^* \;. \tag{8}$$

The parameters μ and τ are related to the nonlinearities as well as the losses in the medium. For example, in cavity electrodynamics μ and τ will depend on the cavity losses, spontaneous emission, and the coupling between the atom and the field.[6,10] These parameters are related to the moment of γ,

$$\langle \gamma \rangle = 0, \quad \langle \gamma^2 \rangle = -2\mu^*, \quad \langle \gamma^\dagger \gamma \rangle = \tau - \tfrac{1}{2} \ . \tag{9}$$

These moments can be calculated from the linearized Fokker-Planck equation for the Wigner function, which is eventually used to describe the dynamics starting from microscopic considerations. If we set

$$\mu = \tfrac{1}{4}\sinh(2|\rho|)e^{-i\theta}, \quad \tau = \tfrac{1}{2}\cosh(2|\rho|) \ , \tag{10}$$

then the Wigner function (7) corresponds to the squeezed vacuum state. We thus write

$$\mu = \tfrac{1}{4}Q\sinh(x)e^{-i\theta}, \quad \tau = \tfrac{1}{2}Q\cosh x \ . \tag{11}$$

The parameters Q and x must satisfy

$$Q\cosh x \geq 1, \quad \coth x > \cos\theta \ , \tag{12}$$

which follow from the positiveness of the photon number and (8). The condition for the existence of squeezing can be shown to be

$$Qe^{-x} < 1 \quad \text{if } \theta = 0 \ . \tag{13}$$

Note further that (7) must lead to a positive definite ρ. This condition implies that

$$(\tau^2 - 4|\mu|^2)^{1/2} - \tfrac{1}{2} \geq 0, \quad \text{i.e.,} \ Q \geq 1 \ . \tag{14}$$

With (14), the conditions (12) are automatically satisfied.

The photon-number distribution $p(n)$ is related to the Wigner function by

$$p(n) = \int d^2z \, \Phi(z) \frac{2}{n!}(-1)^n L_n(4|z|^2)\exp(-2|z|^2) \ , \tag{15}$$

where L_n is the Laguerre polynomial defined by

$$L_n(x) = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix} \frac{n!}{m!} x^m \ . \tag{16}$$

It should be remembered that for $x=0$ and $\mu=0$, (7) represents the usual thermal field with average occupation number $\frac{1}{2}(Q-1)$.

The photon-number fluctuations[11] can be obtained from the Gaussian property of Φ. Calculation shows that

$$(\Delta n)^2 \equiv \langle n^2 \rangle - \langle n \rangle^2 = \langle (\gamma^\dagger\gamma)^2 \rangle - \langle \gamma^\dagger \gamma \rangle^2 = \tfrac{1}{4}[Q^2\cosh(2x) - 1] \ . \tag{17}$$

Using (17) we find that the number fluctuations even exceed that for a thermal field, i.e.,

$$(\Delta n)^2 \geq \langle n \rangle (\langle n \rangle + 1) \ . \tag{18}$$

Thus the field can be squeezed under the condition (13) but it always shows super poissonian statistics as long as $\langle \gamma \rangle = 0$.

Note that if the field mode has nonzero $\langle \gamma \rangle$, then one can subtract the coherent part by mixing the field produced by a local oscillator so that the counting distributions of the fluctuating part can be measured.

We next sketch the derivation of the explicit form of $p(n)$. It is clear that the integral cannot depend on the phase θ. Writing $z = re^{i\varphi}$, the integral over φ can be done by expanding $\exp(\alpha\cos^2\varphi)$ in a Taylor series, with the result

$$p(n) = (-1)^n \frac{8}{n!Q} \sum_{s=0}^{\infty} \frac{1}{s!} \begin{bmatrix} -\frac{1}{2} \\ s \end{bmatrix} \left[\frac{4\sinh x}{Q} \right]^s \int_0^\infty dr \, r^{2s+1} L_n(4r^2) \exp\left[-2r^2 \left[1 + \frac{e^{-x}}{Q} \right] \right] . \tag{19}$$

The integral in (19) can be expressed in terms of hypergeometric functions as

$$p(n) = \frac{(-1)^n}{Q} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{v^{k+1}} F(\tfrac{1}{2}, k+1, 1, -(\sinh x)/vQ), \quad v = \tfrac{1}{2}(1 + e^{-x}/Q) \ . \tag{20}$$

An alternate expression for $p(n)$ can also be obtained by writing (19) as

$$p(n) = \frac{(-1)^n}{2n!} L_n \left[-\frac{\partial}{\partial v} \right] \left[\left[\frac{Q^2}{4} v + \frac{Q}{4}\sinh x \right]^{-1/2} v^{-1/2} \right]$$

$$= \frac{(-1)^n}{Q\sqrt{v(v + \sinh x/Q)}} \sum_{l=0}^{n} (-1)^l \begin{bmatrix} n \\ l \end{bmatrix} \frac{1}{l!} \left[\frac{1}{v} \right]^l \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix} (\tfrac{1}{2})_k (\tfrac{1}{2})_{l-k} \left[\frac{v}{v + \sinh x/Q} \right]^k , \tag{21}$$

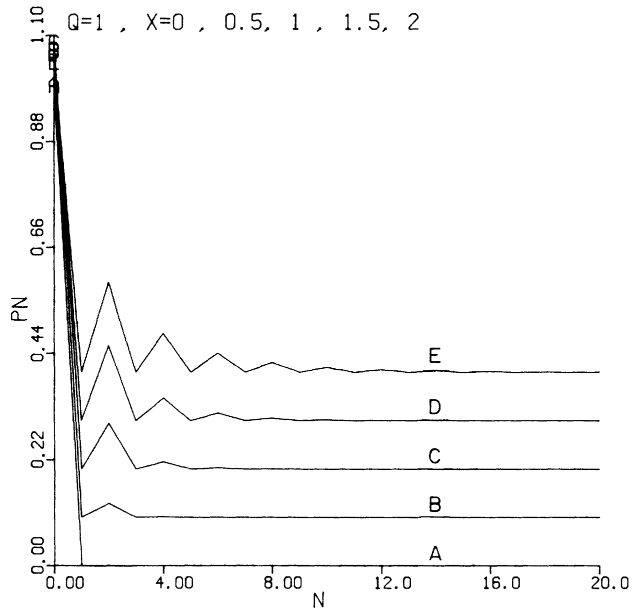

FIG. 1. The photon-number distribution $p(N) \equiv PN$ as a function of N for a radiation field in a mixed state characterized by a Gaussian Wigner function. The curves labeled A, B, C, D, and E are for $x=0$, 0.5, 1, 1.5, and 2, respectively. The actual quantity plotted on the y axis is $p(N)+0.2x$. This is done for the clarity of the curves.

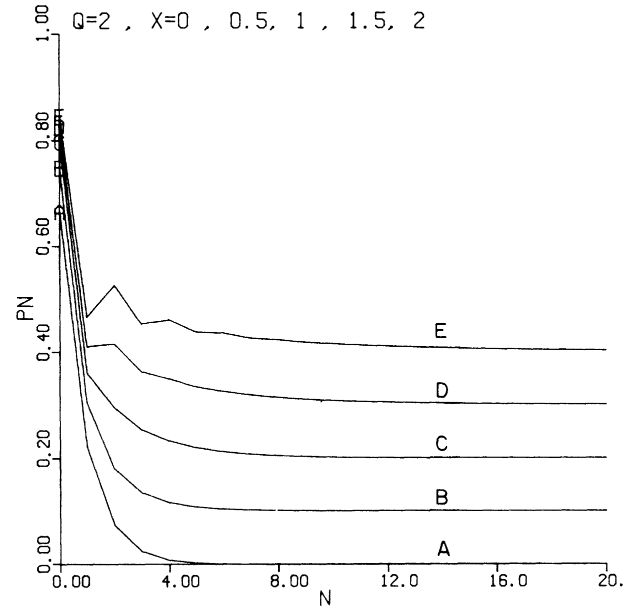

FIG. 3. Same as in Fig. 1 with $Q=2.0$.

where the Pockhammer symbol

$$(\tfrac{1}{2})_k = \tfrac{1}{2} \times \tfrac{3}{2} \times \tfrac{5}{2} \times \cdots \times \frac{(2k-1)}{2} . \tag{22}$$

For $x=0$, it is easily verified that (20) reduces to the Bose-Einstein distribution, as it should. We show in Figs. 1–4 the behavior of $p(n)$ for different values of the parameters Q and x. Figure 1 gives the photon number distribution for the squeezed[12] vacuum ($Q=1$). The vacuum becomes more and more squeezed as x increases. We see that the increased squeezing in the field results in a more and more oscillatory character of $p(n)$. With an increase in Q the average occupation number in the field increases. When there is no squeezing in the field, i.e., $Qe^{-x} > 1$, the number distribution is nonoscillatory. However, when squeezing starts occuring then the distribution starts acquiring oscillatory character. This is evident from Figs. 2–4.

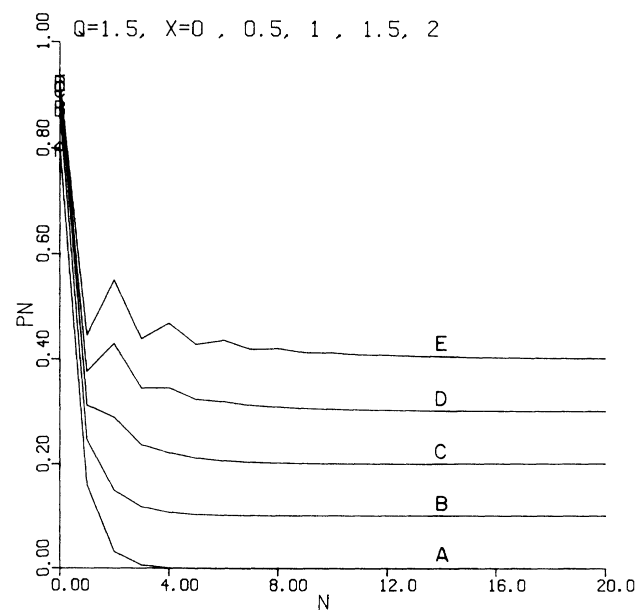

FIG. 2. Same as in Fig. 1, but now $Q=1.5$.

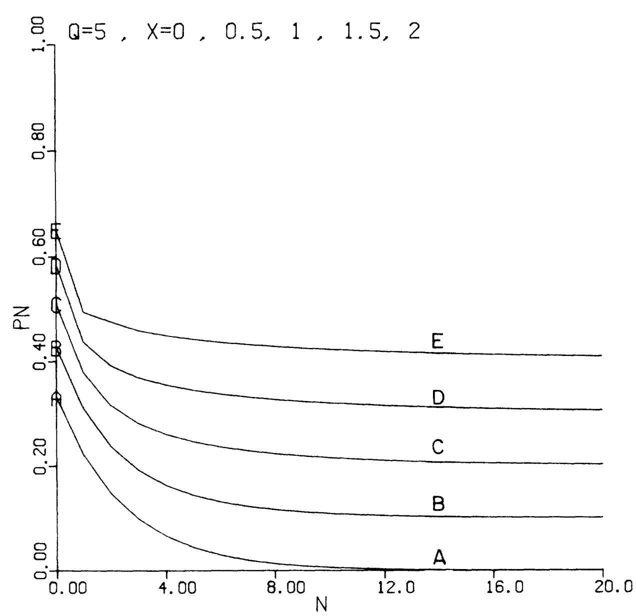

FIG. 4. Same as in Fig. 1 with $Q=5.0$.

In conclusion, we would like to emphasize that our results for photon-number distributions have very wide applicability since the fluctuations in most of the problems in quantum optics, particularly in the context of the squeezing studies, are studied using linearized Fokker-Planck equations for a suitable quasiprobability distribution such as the Wigner function. The parameters Q and x will depend on the system at hand.

Note added in proof. Since this paper was submitted for publication several other works have been submitted for publication: W. Schleich, D. F. Walls, and J. A. Wheeler (unpublished) have discussed the problem of interference in phase space using Wigner distribution functions; A. Vourdas and R. M. Weiner [Phys. Rev. A **36**, 5866 (1987)] and G. J. Milburn and D. F. Walls (unpublished) have also examined the effects of dissipation on interference in phase space; and we have succeeded in getting analytical and numerical results for the case when the mode γ [Eq. (5)] has a coherent part, i.e., $\langle\gamma\rangle\neq 0$. We have also found that Eq. (21) can be written in a much simplified form as

$$\frac{2}{(Q^2+2Q\cosh x+1)^{1/2}}\left[\frac{Qe^x-l}{Qe^x+l}\right]^n \times F\left[-n,\tfrac{1}{2},1;\frac{4Q\sinh x}{Q^2+2Q\sinh x-1}\right].$$

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[11]A previous paper [G. S. Agarwal, J. Mod. Optics **34**, 909 (1987)] discusses the importance of Gaussian Wigner functions in the quantum theory of interferometers.
[12]The photon number distributions have been evaluated for the two-photon coherent state by Yuen. The oscillations in the number distributions have been studied by W. Schleich and J. A. Wheeler [Nature **326**, 574 (1987)] for nonzero $\langle\gamma\rangle$. In contrast, we study the case when $\langle\gamma\rangle=0$ and when the squeezing occurs in the presence of *losses* and thus we are dealing with *mixed* squeezed states of the radiation field. These authors have given a very interesting interpretation of these oscillations in terms of the interference in phase space.