Exactly solvable model for thermal activation and quantum tunneling in Ohmic systems

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Combining Kramers's energy diffusion (at temperature T) with dissipative quantum tunneling (through a parabolic potential-energy barrier with curvature frequency ω_b and height U_b), a model for a localized metastable state (with harmonic frequency ω_0) is formulated and solved exactly for its quasiequilibrium distribution $\rho(E)$ and its decay rate Γ. It is shown that ρ remains locally extremely close to (although it essentially differs from) the Boltzmann distribution, unless the Ohmic friction coefficient λ is extremely small (of order Γ). Excluding this latter possibility, the ensuing decay rate is discussed for various temperature and friction regimes. It is shown to comprise several known results as special cases. *First*, an extended version of Bell's formula is shown to be valid in the strong-to-moderate friction regime. It involves the recently discovered crossover between thermal hopping and quantum tunneling at temperature $T_0=\hbar\kappa\omega_b/2\pi k_B$ (where κ is Kramers's viscosity correction factor). At high temperatures and very strong damping the result neatly reduces to Kramers's Smoluchovsky-limit formula. At zero temperature friction strongly suppresses the decay. *Second*, in the very weak friction regime Γ is shown to reduce to an extended version of Melnikov's formula. At high temperatures it equals Kramers's small-viscosity result, whereas at zero temperature and zero damping it attains the expected quantum value, the crossover again being at T_0. *Third*, in the classical limit ($\hbar\to 0$) an extended version of Büttiker, Harris, and Landauer's [Phys. Rev. B **28**, 1268 (1983)] result is recovered, now valid for any value of the damping. Finally, some remarks are made in relation to recent cryogenic measurements on metastable flux states.

I. INTRODUCTION

Metastable states seem to be the rule rather than the exception in nature. In any case, they can be said to be involved in a variety of interesting phenomena such as chemical reactions,[1-3] nuclear fission,[4,5] the dynamics of Josephson junctions,[6-9] and the possible decay of our own physical universe.[10,11] In many cases it will be possible to indicate one specific "reaction" coordinate, such that the decay process can be described as the motion of a particle in a one-dimensional potential $U(x)$. The particle is initially caught in a local potential hole which is separated from another lower-lying region by a high but finite potential-energy barrier, as shown in Fig. 1. The height of this barrier is typically taken to fulfill $U_b \gg \max(k_B T, \hbar\omega)$, where ω is a characteristic local frequency in the problem, e.g., the harmonic-oscillation frequency ω_0 in the potential hole. Under the—often implicit—assumption of a sufficiently strong dissipative coupling of the reaction coordinate to the environment (the thermal bath), the particle will essentially be caught in its local equilibrium Boltzmann distribution. However, this can, in fact, only be a quasiequilibrium state in view of the particle's finite chance to escape from the hole either by (classical) hopping over or by (quantum-mechanical) tunneling through the barrier.

Much effort has been spent on the escape phenomenon, from the early hours of modern notions about either classical stochastic processes[12] and statistical mechanics[13] or quantum mechanics,[1,4] until today.[14-25] The basic structure of the decay rate is that of a very small exponential factor (a Boltzmann factor in the thermal regime and a Gamov factor in the quantum regime) which sets the order of magnitude, multiplied by a prefactor (the so-called attempt frequency ω_a). The general dependence of the decay rate (and, in particular, of the prefactor) on such properties as the nature and strength of the dissipation, the temperature, and the properties of the potential is not

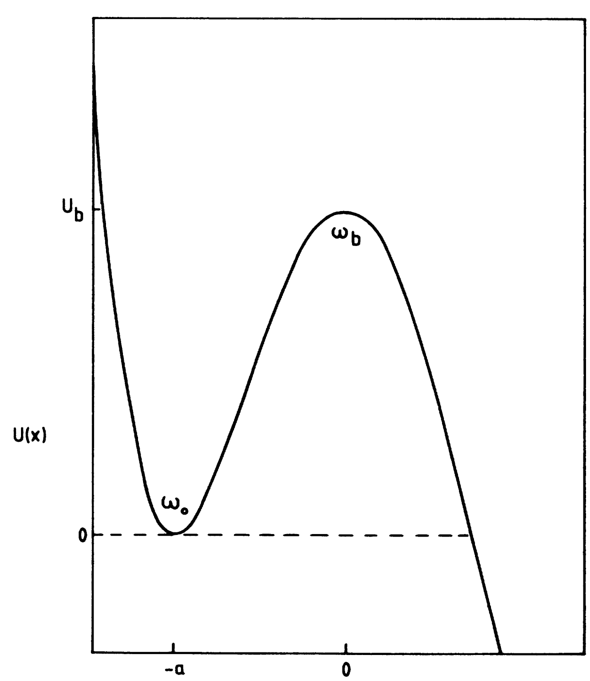

FIG. 1. Typical potential for a metastable system.

yet fully understood. And, despite some recent progress with metastable Josephson junction devices,[6,7,26–29] the experimental data often involve insufficiently known parameters. Evidently, a better theoretical framework could stimulate further experimental effort.

The classical theory of noise-activated rate processes essentially dates back to the seminal 1940 paper[12] by Kramers, providing a dynamical framework for the original concepts of Arrhenius[2] for chemical reactions. Kramers studied the problem of Brownian motion in a metastable potential and calculated the escape rate (i.e., the reaction velocity) Γ from a stationary diffusion current at the top of the barrier. For a high barrier ($U_b \gg k_B T$) this typically yields a result of the form

$$\Gamma = \frac{\omega_a}{2\pi} e^{-\beta U_b} , \tag{1.1}$$

where $\beta = 1/k_B T$. The attempt frequency ω_a can be interpreted as the frequency with which the particle strikes the potential-energy barrier. For a smooth barrier, with a frequency ω_b defined by the parabolic nature of the peak region, Kramers found $\omega_a = \kappa\omega_0$ where

$$\kappa = [1 + (\lambda/\omega_b)^2]^{1/2} - \lambda/\omega_b ,$$

λ being the Ohmic friction coefficient.[8,20,30–32] For strong damping ($\lambda/\omega_b \gg 1$) this result neatly reduces to the correct Smoluchovsky-limit (easily obtained from an inverse friction expansion[16]) expression $\omega_a = \omega_0\omega_b/2\lambda$. On the other hand, in the zero damping limit one would end up with $\omega_a = \omega_0$, which is the value given by simple transition state theory[2,33,34] and which implies the—as noted earlier, *au fond* incompatible—assumption of full equilibrium even up to barrier peak energies.

Kramers solved this latter paradoxical situation by considering the very weak damping regime $\lambda/\omega_b \ll 1/\beta U_b \ll 1$ as a separate case. Studying the problem as Brownian motion along the energy coordinate he found $\omega_a = 2\lambda\omega_0\beta I_b$, where I_b is the classical action *in the potential well* at (or at least close to) the barrier peak energy. In fact, this result was reached only for a sharp-edged barrier (e.g., a sharply cut off perfectly parabolic potential hole). Nevertheless, Kramers's strongly suggested it to be a very reasonable result also for other cases even though, strictly speaking, near a smooth barrier's peak energy $E = U_b$ the classical oscillation frequency $\omega(E)$ through the potential hole becomes very small, so that even a slow rate of dissipation may have a significant influence. Fortunately, the result for the escape rate is practically insensitive to these details of $\omega(E)$, as will be clear from the model to be discussed in the present paper. Not only will the correctness of Kramers's intuition be confirmed, but his cautious treatment of energies approaching U_b will be seen to provide the insight for the mechanism connecting weak and strong damping regimes.

Perhaps Kramers's "anyhow somewhat less exact" and rather special treatment of the extremely weak damping case explains why the result was not appreciated[35,36] until it was rediscovered[37] as late as in the 1970s. An interesting variety of theories—and, hence, formulas for the decay rate—then surfaced in order to describe the intermediate friction range.[14,19,22,23,38–50] It is important to note that these theories are entirely classical and that for most of them it is by no means obvious how they can be generalized to include quantum effects. For that matter, here it suffices to mention only the theory of Büttiker, Harris, and Landauer, which does allow such an extension as will be shown in the sequel.

If the temperature is decreased (i.e., β increasing), the classical decay rate (1.1) rapidly decreases. On the other hand, quantum-mechanical tunneling will begin to play a role. Above a characteristic crossover temperature $T_0 = \hbar\kappa\omega_b/2\pi k_B$ the quantum effects mainly provide a correction to the exponential prefactor, the attempt frequency ω_a. Below T_0, however, the tunneling decay will be dominant and the escape tends to settle for its nonzero vacuum value. The result becomes typically of the form

$$\Gamma = \frac{\omega_a'}{2\pi} e^{-S_b/\hbar} , \tag{1.2}$$

where S_b is the mechanical action[51] *under the barrier* (i.e., in the classically forbidden region), and where, in general, the attempt frequency $\omega_a' \neq \omega_a$. The rate (1.2) is nonzero even at strictly zero damping $\lambda = 0$, although as mentioned earlier (see also Sec. III A 2) in that case the initial energy distribution inside the well requires careful consideration.[52,53]

The earliest interest in (1.2) occurred in nuclear physics and Coulomb field scattering.[54,55] Reference should be made to a famous 1928 paper by Gamov.[4] In order for a quantity such as (1.2) to exist properly for an isolated (i.e., nonthermal nondissipative) quantum system, one needs running outgoing waves at the exit side of the barrier.[25,30,56] It is then easily understood how Γ appears as a small imaginary part $\Gamma = -(2/\hbar)\text{Im}E$ to the pertinent energy eigenvalue, usually for the local ground state (but see Refs. 21 and 57). An apparently natural extension of this notion to the quantum-statistical regime is the replacement of the mechanical energy E by the free energy $F = -(1/\beta)\ln Z$, where Z is the system's (quasiequilibrium) partition function. This intriguing idea, originally put forward by Langer in his important 1967 paper on classical nucleation,[13] paves a way to treating tunneling systems at elevated temperatures[18] and more recently including dissipation.[20,25,32,58–60] The method involves instanton-type theoretical techniques[10,11,17] to evaluate the path integral for the partition function.[16,61,62] It appears to be quite powerful, but above the crossover temperature T_0—in the classical limit— it yields Kramers's moderate damping result for Γ. That is, with $\lambda = 0$ it reduces to the incorrect transition state theory value.

In his 1981 Letter[18] Affleck pointed out that—within the omnipresent semiclassical approximation—the Γ as obtained following Langer's "imaginary part method" is essentially identical to a Boltzmann distribution average of the tunnel current $\Gamma(E)$. Affleck's text does not refer to any dissipation whatsoever, but from the involved exact equilibrium distribution (and the resulting transition state value at high T) it is clear that his original considerations in fact only apply in the moderate damping regime. Similar remarks apply to the work of Bell.[3,63]

Nevertheless, the notion of the decay rate as an appropriate energy average of a quantum-mechanical probability current across the barrier is physically at least as plausible as Langer's definition. Fully exploiting it, however, requires a self-consistent dynamical calculation of the appropriate deviations from the exact Boltzmann distribution as they arise *ipso facto* from the very existence of the decay process.

Fortunately, when viewed in this manner the problem becomes close to being the quantum generalization of Büttiker, Harris, and Landauer's mold[19] of Kramers's classical weak damping analysis. Apparently independent first steps in this direction were done by Melnikov,[22,42] who formulated an energy integral equation for the distribution function (see further Refs. 49 and 50), and by Rips and Jornter.[64] This has yielded the very weak damping result of Kramers plus tunneling corrections; however, it still fails in both the strong damping classical and quantum-mechanical low-temperature limits.

In the present paper an exactly solvable model will be presented which does extend the Büttiker, Harris, and Landauer theory into these two directions. That is, apart from containing Melnikov's result as an approximate special case, it (i) correctly describes the Kramers-Smoluchovsky strong damping limit plus quantum corrections and (ii) properly includes the pure quantum limit. By virtue of the self-consistent dynamical nature of the model it also allows us to show that—with quantum effects being there (i.e., $\hbar \neq 0$)—letting the dissipation go to zero ultimately (if $\lambda \ll$ the quantum decay rate Γ_0) prevents the system from reaching or maintaining a local Boltzmann-like distribution even for energies at the bottom of the metastable well. Of course, the theory also has its limitations , e.g., (i) it cannot yet accommodate dissipative cases other than Ohmic ones and (ii) the exact solvability is restricted to a perfectly parabolic barrier.

In Sec. II the model will be formulated, in Sec. III it will be solved both for its quasiequilibrium energy distribution and for the decay rate, and in Sec. IV it will be discussed.

II. THE MODEL

The model basically consists of three interrelated dynamical ingredients: *First*, energy diffusion inside the potential well; *second*, deterministic classical barrier peak dynamics; *third*, dissipative quantum-mechanical transmission.

A. Energy diffusion

If the (Ohmic) damping rate λ is sufficiently small such that the phase-space density $\rho(x,p)$—x being the particle's position, p its momentum— is practically constant along trajectories of given energy, then the leading-order effect of (thermal) Brownian motion will be diffusion along the energy coordinate. By a sufficiently small λ is meant here that the energy loss per (semi)classical round trip through the potential minimum should be much smaller than the energy itself, i.e., $2\lambda I(E) \ll E$ with $I(E)$ being the classical action integral.[65] For instance, for a perfectly parabolic well one has $I=2\pi E/\omega_0$, which would imply $4\pi\lambda/\omega_0 \ll 1$. Fortunately, as far as the decay rate is concerned it will be seen that this condition can be relaxed considerably.

Then, following Kramers[12] (see also Refs. 19, 22, 44, and 64), transforming the classical bivariate Fokker-Planck equation for $\rho(x,p)$ to x and E coordinates and averaging along the x coordinate over one oscillation period, one obtains

$$\frac{\partial \rho}{\partial t} = 2\lambda \nu(E) I_b \left[\frac{\partial \rho}{\partial E} + k_B T \frac{\partial^2 \rho}{\partial E^2} \right] - \Gamma(E)\rho(E) , \qquad (2.1)$$

where $\nu(E) = dE/dI$, where advantage has been taken of the property that (at least for a parabolic barrier) $I \approx I_b$ is almost constant in the classically important energy range near $E = U_b$, and where the as yet unspecified escape rate $\Gamma(E)$ represents the metastable character of the system.

It may be noted that the attempt rate $\nu(E)$ goes to zero at the barrier peak energy—at least for a smooth barrier—but using its above-given definition the explicit occurrence of $\nu(E)$ in (2.1) can easily be avoided by considering the diffusion current along the action coordinate instead of the energy coordinate. The subsequent analysis for $\rho(E)$ must, however, be done entirely in terms of the energy variable. Actually, in Sec. II B it will become clear that the steady-state energy density is essentially independent of $\nu(E)$. See also Sec. II B.

Another comment on (2.1) concerns the undetermined factor α multiplying the outgoing flux $\Gamma(E)\rho(E)$, introduced by Büttker *et al.*[19] (see also Refs. 45, 49, and 50) in order to account for the fact that the density in phase space near the barrier peak differs in general from the density averaged along an orbit in the well. Although such a factor should indeed exist, it is not included here since—even in the classical domain—it will be a rather complicated function (not only of the friction λ and the temperature but also of the energy E) which cannot be determined within the context of the present simple model. Therefore, actually in line with the global findings of Ref. 19 (see also Ref. 46 and 48), the current value of this factor α will be set equal to 1.

In the weak damping limit the main role of the diffusion terms in (2.1) is to provide the correct classical particle current for the escape process. On the other hand, in the strong damping limit their principal consequence will only be to enforce the local equilibrium Boltzmann distribution

$$\rho_{\rm eq}(E) = \beta\hbar\omega_0 e^{-\beta E} \qquad (2.2)$$

with increasing precision. In the heavy damping regime the actually important dynamical properties will—in Sec. II B—be seen to be those of the dissipative ballistic barrier dynamics rather than any details of the energy diffusion. In fact, in the high friction limit the result for the decay rate will not depend any more on I_b. It is these features which allow the extended use of (2.1) far into the Smoluchovsky regime.

The prefactor in (2.2) arises from the (semi)classical normalization $\oint dx \int dp\, \rho(E) = 1$ taken in the local harmonic approximation, so that $\oint dx(dp/dE)=\tau(E)$—representing the classical oscillation period—equals $\tau(E)=2\pi/\omega_0$. See further Sec. III B, where it will be shown that the above replacement of the actual $\sum_n$ by the (semi)classical energy integral with measure $\tau(E)dE$ is particularly fit for the averaging of $\Gamma(E)$, since $\Gamma(E)$ is proportional to $\nu(E)$ according to (2.3), while always $\nu(E)\tau(E)=1$. In (2.2)— and also in Sec. III—one easily reinserts, however, the true partition function expression, i.e., $Z_0^{-1}=2\sinh(\frac{1}{2}\beta\hbar\omega_0)$ in the harmonic-well approximation, in lieu of its semiclassical limit $\beta\hbar\omega_0$. See also Refs. 18 and 21. Of course, any semiclassical analysis fails if $\hbar\omega_0/U_b$ is not small (see, e.g., Ref. 3, especially Sec. 3.6.3 and Ref. 66, especially Sec. I).

B. Classical barrier dynamics

In the classical (semiclassical) case the escape process takes place mainly within a relatively narrow energy range above (around) $E=U_b$, roughly of the order of $|\delta E|\le \max(k_BT,\hbar\omega_b)$, which in terms of distance away from the barrier peak amounts to a

$$\delta x \le a\max(\sqrt{k_BT/U_b},\sqrt{\hbar\omega_b/U_b}) ,$$

where a is typically the spatial distance between the top of the barrier and the potential minimum, while throughout $k_BT/U_b \ll 1$ and $\hbar\omega_b/U_b\ll 1$. It is this small energy range $|\delta E|$ into which the diffusion process—as described by the first contribution on the right-hand side (rhs) of (2.1)—should properly supply particles coming from the potential minimum. However, as already noticed by Kramers, close to the barrier maximum the assumption of weak damping implied in (2.1) becomes moot for a smooth barrier peak. For instance, for a perfectly parabolic barrier one easily calculates $\omega(E)=2\pi\nu(E)$ to be $\omega(E)\approx\omega_b/\ln(U_b/|\delta E|)$ when $\delta E\to 0$, so that the requirement $\lambda/\omega(E)\ll 1$ leads to $|\delta E|\gg U_b\exp(-\omega_b/\lambda)$. Clearly, if $\lambda/\omega_b\gtrsim 1$ the energy diffusion supply into the above-mentioned barrier peak region has to be reconsidered.

On the other hand, it will be shown self-consistently (*en passant* confirming recent findings from Grabert and co-workers[32,59]) that (i) in the strong damping limit the classical escape regime extends ever deeper into the quantum range of low temperatures, while (ii) the classical strong damping escape rate is basically determined by the Boltzmann distribution at $E=U_b$ together with deterministic—but essentially dissipative—dynamics in the parabolic barrier peak region. Fortunately the primary consequence of the Brownian motion in (2.1) in the large friction limit precisely is the enforcement with sufficient accuracy of the required equilibrium distribution up to and including the energy range near $E=U_b$. Therefore it makes sense to keep it as it stands even in the heavy damping limit, on the premise that the barrier dynamics is properly incorporated in the escape rate $\Gamma(E)$.

The (semi)classical escape rate $\Gamma(E)$ can—at least if the damping is sufficiently small—be taken to equal the attempt rate $\nu(E)$ multiplied by the escape probability $P(E)$ at each attempt. Obviously, in the classical limit—as originally studied in Ref. 19—$P(E)=0$ if $E<U_b$ and $P(E)=1$ if $E>U_b$, i.e., $p(E)$ is the unit step function in that case. Let us more generally set

$$\Gamma(E)=\kappa\nu(E)P(E) , \tag{2.3}$$

where κ is an as yet unknown coefficient which incorporates the intended dissipative (classical) correction to the escape current. Since the escape current is mainly determined by a very small range of momenta (of order $\sqrt{k_BT}$) near $E=U_b$, let us—in order to learn about κ—consider the ballistics of a particle in the barrier peak region where $U(x)\approx U_b-\frac{1}{2}\omega_b^2x^2$, for convenience taking $x_b=0$ here. Consequently, for the exact inverted parabola the following analysis is exact.

The separatrix $\mathcal{S}$ is defined as the curve in phase space which separates the domains of attraction of the spatial regions on the left- and right-hand sides of the barrier.[40,46,48,67] Using Hamilton's equations of motion with $U'=-\omega_b^2x$, $\mathcal{S}$ is easily seen to follow from the ordinary differential equation

$$dp/dx=-2\lambda+\omega_b^2x/p , \tag{2.4}$$

where $p=p(x|\mathcal{S})$ denotes the value along $\mathcal{S}$, and where $p(0|\mathcal{S})=0$. This yields

$$p(x|\mathcal{S})+[(\omega_b^2+\lambda^2)^{1/2}+\lambda]x=0 . \tag{2.5}$$

Obviously, this result for $\mathcal{S}$ tells us what momentum is required, for a given value of x near the barrier peak, in order to precisely wind up at the top. The second solution arising from (2.4) defines another curve $\mathcal{T}$, perpendicular to $\mathcal{S}$. Namely,

$$p(x|\mathcal{T})-[(\omega_b^2+\lambda^2)^{1/2}-\lambda]x=0 , \tag{2.6}$$

which tells us what momentum will be gained by a particle starting infinitesimally close to the top of the barrier with an infinitesimally small momentum. Since it is precisely these momenta (taken to the exit side of the barrier) which are involved in the escape process in the weak noise (or deterministic) limit, the leading-order effect of dissipation on the exiting current will be to decrease it by the ratio

$$\kappa=p(x|\mathcal{T},\lambda)/p(x|\mathcal{T},0) ,$$

so that

$$\kappa=(1+\lambda^2/\omega_b^2)^{1/2}-\lambda/\omega_b , \tag{2.7}$$

which equals Kramers's correction factor for large viscosity. In the weak damping limit its influence on the classical decay rate Γ will be seen (in Sec. IV C) to disappear completely in a somewhat subtle manner. It is further interesting to note that the above-given ballistic weak noise arguments leading to Kramers's correction factor κ in the decay rate (2.3) also give the correct answer for the diffusive motion in the heavy damping limit. The understanding of this feature hinges on (i) the linearity of the motion in the parabolic barrier region to the effect that the propagator will be a Gaussian centered at the classical ballistic trajectory (see, e.g., Refs. 44 and

68, and Ref. 69, especially Sec. IV); (ii) the definition of Γ as the expectation value of the momentum for a value of x at or beyond the barrier peak (see, e.g., Ref. 12) which essentially makes Γ proportional to $p_{\rm cl}$; and (iii) the instability of the pertinent motion which—e.g., following a particle, starting near the barrier peak with some x_0,p_0, down the exit hill—is easily shown to imply that $p_{\rm cl}$ tends towards the value $\kappa\omega_b x_{\rm cl}$ for any x_0,p_0. It may further be remarked that κ should also be expected to play its role in the quantum regime as it is basically the linearity of the dynamics which maps the average quantum motion precisely onto the classical limit. Finally, notice (and see further in Sec. II C) that for the marginally exiting classical particles the barrier appears as having an effective dissipatively renormalized curvature frequency $\kappa\omega_b$.[69–71]

C. Quantum-mechanical tunneling

In the quantum-mechanical regime the nonzero value of Planck's constant $\hbar$ basically modifies the escape probability $P(E)$ per attempt. In the semiclassical (WKB) approximation the general expression for $P(E)$ may be written as[3,5,18,22,54,57,72,73]

$$P(E)=1/(1+e^{W(E)/\hbar}) , \tag{2.8}$$

where $W(E)$ is the "classical" (Euclidean) action integral under the (inverted) potential-energy barrier.[74] In what follows an expression for $W(E)$ will be obtained for the dissipative, exactly parabolic barrier. The derivation is based on the low-temperature Ohmic version of Langer's theory.[13] Following Refs. 20, 25, 32, 58–60, 75, and 76 the system's Euclidean action (i.e., the nonextremal time integral of its thermodynamic action) reads

$$S\{x(\tau)\}=\int_0^\theta d\tau[\tfrac{1}{2}\dot{x}^2+U(x)] +\tfrac{1}{2}\int_0^\theta d\tau\int_0^\theta d\tau' k(\tau-\tau')x(\tau)x(\tau') , \tag{2.9}$$

where the dissipative influence kernel may be given as

$$k(\tau)=\sum_m K(\nu_m)e^{i\nu_m\tau} , \tag{2.10}$$

with m running from $-\infty$ to $+\infty$ and with $\nu_m=2\pi m/\theta$ representing the Matsubara frequencies conjugate to the thermal period θ. For a perfect Ohmic environment $K(\nu_m)=(2\lambda/\theta)|\nu_m|$, with the same classical friction coefficient as before. For the parabolic barrier $U(x)=U_b-\frac{1}{2}\omega_b^2x^2$, and with

$$x(\tau)=\sum_m x(\nu_m)e^{i\nu_m\tau} , \tag{2.11}$$

(2.9) is transformed into

$$S\{x(\nu_m)\}=\theta U_b+\theta\sum_m(-\tfrac{1}{2}\omega_b^2+\tfrac{1}{2}\nu_m^2+\lambda|\nu_m|) \times x(\nu_m)x(\nu_{-m}) . \tag{2.12}$$

For energies below the barrier peak (temperatures below the crossover T_0) the pertinent extremal—saddle-point—trajectory starts at $\tau=0$ from a position in the metastable well corresponding to energy E, such that $\dot{x}_b(0)=0$ and $x_b(0)=\sqrt{2(U_b-E)}/\omega_b$. Some time later it reaches its exit point on the other side of the barrier, from where it bounces back so as to return to its initial position and velocity again at $\tau=\theta$. Clearly, in view of (2.11) one has

$$x_b(\nu_m)=\frac{\sqrt{2(U_b-E)}}{\omega_b}\frac{1}{\nu_m^2-\omega_b^2+2\lambda|\nu_m|}\Big/\sum_m\frac{1}{\nu_m^2-\omega_b^2+2\lambda|\nu_m|} . \tag{2.13}$$

It may be worth noticing that the actual physical energy E is defined in terms of the "true" barrier—namely, $E-U_b=\frac{1}{2}\dot{x}_b^2(0)-\frac{1}{2}\omega_b^2x_b^2(0)$—whereas the tunneling motion, in fact, takes place in the "upside-down" barrier. It is then easily seen that for energies above the barrier peak there exists another valid extremal trajectory, namely, with $x_b(0)=0$ instead of $\dot{x}_b(0)=0$. Formally in that case one can just analytically continue $(U_b-E)^{1/2}$ as $\pm i(E-U_b)^{1/2}$ in (2.13) for the Fourier amplitudes [in addition, just in order to achieve the proper value of $\dot{x}_b(0)$, one divides by ν_m rather than ω_b]. The remainder of the analysis will not be changed.

Inserting then (2.13) into (2.12) the ensuing extremal action S_b takes on the significance of Hamilton's principal function. Defining $W(E)$ as the associated characteristic function,[51,57,65,72–74,77] i.e., $W(E)=S_b-E\theta$, one obtains

$$W(E)=(U_b-E)\theta\left[1+1\Big/\sum_m\frac{\omega_b^2}{\nu_m^2-\omega_b^2+2\lambda|\nu_m|^2}\right]. \tag{2.14}$$

Finally, the appropriate period $\theta(E)$ can be found self-consistently now from the usual requirement that $\theta(E)=-\partial W/\partial E$, which by (2.14) yields

$$1\Big/\sum_m(\nu_m^2-\omega_b^2+2\lambda|\nu_m|)^{-1}=0 . \tag{2.15}$$

Recalling that $\nu_m=2\pi m/\theta(E)$, this condition can only be satisfies if $\theta(E)$ is such that $\nu_m^2-\omega_b^2+2\lambda|\nu_m|=0$ for some $m=0,\pm1,\ldots$. Since at $\lambda=0$ one should recover the nondissipative value $\theta(E)=2\pi/\omega_b$, it is readily clear that $m=1$ is the correct choice. Hence

$$\theta(E)=2\pi/[(\omega_b^2+\lambda^2)^{1/2}-\lambda] . \tag{2.16}$$

Recalling the definition (2.7) of κ, this can also be written as $\theta(E)=2\pi/\kappa\omega_b$. It is, of course, reassuring that this confirms the conclusion of Sec. II B. With (2.16) one thus has

$$W(E)=\frac{2\pi}{\kappa\omega_b}(U_b-E) . \tag{2.17}$$

Our model is now complete. It is defined by the energy diffusion equation (2.1), with the escape rate being given by (2.3). The dissipative factor κ can be found in (2.7), while the escape probability $P(E)$ is specified by (2.8) and (2.17). In the steady state $\partial\rho/\partial t=0$ (2.1) reduces to

$$2\lambda I_b\left[\frac{\partial\rho}{\partial E}+k_BT\frac{\partial^2\rho}{\partial E^2}\right]=\kappa P(E)\rho(E)\;, \tag{2.18}$$

with

$$P(E)=1/\{1+\exp[2\pi(U_b-E)/\hbar\kappa\omega_b]\}\;. \tag{2.19}$$

The feature $P(U_b)=\frac{1}{2}$ arises (i) because of the parabolic nature of the barrier and (ii) since Ohmic friction is a linear damping mechanism. If, e.g., the barrier were not perfectly parabolic, then this feature would only be valid within the WKB approximation (see Ref. 3, especially p. 33).

III. EXACT SOLUTION

A. The quasiequilibrium distribution

Let us consider the steady state $\partial\rho/\partial t=0$. Defining the new variable $y=2\pi(E-U_b)/\hbar\kappa\omega_b$, and the parameters $\epsilon=2\lambda I_b/\kappa$ and $\mu=\beta\hbar\kappa\omega_b/2\pi$, one has

$$\frac{\beta\epsilon}{\mu}\left[\frac{1}{\mu}\rho''+\rho'\right]=\rho/(1+e^{-y})\;, \tag{3.1}$$

where a prime denotes differentiation with respect to y and which must be solved subject to the boundary condition $\rho(y\to\infty)\to 0$ sufficiently fast at least to ensure normalization. Next setting $z=-\exp y$, introducing

$$a=\tfrac{1}{2}\mu(1+\sqrt{1+4/\beta\epsilon})\;,\qquad b=\tfrac{1}{2}\mu(1-\sqrt{1+4/\beta\epsilon})\;, \tag{3.2}$$

and putting $\rho(z)=\rho(y(z))$, (3.1) becomes

$$z(1-z)\rho''+(1+\mu)(1-z)\rho'-ab\rho=0\;. \tag{3.3}$$

The general solution of this Gauss hypergeometric equation reads[78-80]

$$\rho=c_1F(a,b;1+\mu;z)+c_2(-z)^{-\mu}F(-b,-a;1-\mu;z)\;. \tag{3.4}$$

Near the bottom of the well one has $E\approx 0$ so that $-y_0=2\pi U_b/\hbar\kappa\omega_b\gg 1$, which implies

$$-z_0=\exp(-2\pi U_b/\hbar\kappa\omega_b)\ll 1\;.$$

In order to study the behavior of ρ at large energies $E\to\infty$, i.e., $|z|\to\infty$, one invokes the pertinent linear transformation formula (e.g., Ref. 79, formula 15.3.7). The two linearly independent solutions in (3.4) then combine into two new linearly independent solutions near $z=-\infty$, one of which is seen to diverge ultimately like $(-z)^{-b}$—having noticed from (3.2) that always $b<0$. By an appropriate proportionality relation between c_1 and c_2 the contribution from this exploding solution can be set identically equal to zero. Redefining the remaining constant as c_0, one is left with

$$\rho=c_0(-z)^{-a}F(a,-b;1+a-b;1/z)\;, \tag{3.5}$$

which at high energies always vanishes faster than the Boltzmann distribution. The coefficient c_0 is determined by normalization and can be expressed in a formally exact way in terms of a generalized hypergeometric function (Ref. 80, formula 7.527.1), namely,

$${}_3F_2(a,-b,a;1+a-b,1+a;1/z_0)\;.$$

However, for all practical purposes it suffices to require that $\rho(E)$ be equal to the local equilibrium Boltzmann distribution (2.2) near $E=0$, apart from exponentially small corrections [i.e., of order $\exp(-2\pi U_b/\hbar\kappa\omega_b)$]. The prefactor $\beta\hbar\omega_0$ in (2.2) is recalled to arise from semiclassical normalization, e.g., letting $\sum_n\rho(E_n)=1$, replacing the sum by an integral and noticing that following WKB theory[54,55,65] $dn/dE\approx(2\pi\hbar)^{-1}dI/dE=\tau(E)/2\pi\hbar$ with $\tau(E)=2\pi/\omega_0$ in the parabolic-well approximation. Finally, transforming (3.5) back to the low-energy region $z\approx 0$, one obtains

$$\rho(E)=\beta\hbar\omega_0\left[(z/z_0)^{-\mu}F(-b,-a;1-\mu;z)+(-z_0)^{\mu}\frac{\Gamma(a)\Gamma(1+a)\Gamma(-\mu)}{\Gamma(\mu)\Gamma(1-b)\Gamma(-b)}F(a,b;1+\mu;z)\right]\Big/F(-b,-a;1-\mu;z_0)\;. \tag{3.6}$$

1. Zero-temperature limit

From the occurrence of the quantity $\beta\epsilon$ in (3.2) for a and b it is clear that the zero-temperature limit ($\beta\to\infty$) and the zero-damping limit ($\epsilon\to 0$) are opposing one another. Since, however, it is generally suggested that quantum-mechanical tunnel rates exist at zero temperature even for zero damping (i.e., for an isolated system), it is clearly of interest to briefly investigate these two limiting regimes from (3.6).

Let $\beta\to\infty$, keeping all other parameters in the system fixed at nonzero finite values, so that $a\approx\mu+\nu$ while $b\approx-\nu$, where $\nu=\hbar\omega_b/2\pi\epsilon$. This requires at least $\beta\epsilon\gg 4$, i.e., $k_BT\ll(\lambda/2\kappa)I_b$. Since $\mu=\beta\hbar\kappa\omega_b/2\pi$, if $\beta\to\infty$ then $a\to\infty$ but b remains finite. Invoking Stirling's asymptotic formula for the pertinent gamma functions[78-80] one then finds from (3.6) that

$$\rho(0)\approx\beta\hbar\omega_0\left[1+(-z_0)^{\mu}\frac{\pi}{\sin\pi\mu}\frac{1}{\mu\nu\Gamma^2(\nu)}\frac{F(\mu+\nu,-\nu;1+\mu;z_0)}{F(\nu,-\mu-\nu;1-\mu;z_0)}\right]\;. \tag{3.7}$$

If $\mu \to \infty$ through a sequence of values excluding $\mu = 1,2,\ldots$ and upon using the property

$$F(\pm\mu, \mp\nu; \pm\mu; z_0) = (1-z_0)^{\pm\nu} \approx 1 ,$$

since $|z_0| \ll 1$, one easily concludes that the nonequilibrium contribution near the bottom of the metastable well always remains small of relative order $(-z_0)^\mu/\mu$, i.e.,

$$(2\pi/\beta\hbar\kappa\omega_b)\exp(-\beta U_b)$$

if $\beta \to \infty$. In other words, local thermal equilibrium is guaranteed down to zero temperature on the premise that there exists at least some dissipation (i.e., $\epsilon \neq 0$).

The excluded cases where μ is an integer require a more subtle analysis, but it is readily seen that the zeros of $\sin(\pi\mu)$ will be compensated for by poles in

$$F(\nu, -\mu-\nu; 1-\mu; z_0) .$$

This again produces a finite result, typically applying in exponentially small bands of μ values of the order of $(-z_0)^m$ around $\mu = m$, and which therefore clearly represents a subset of measure zero. Rather than saddling the analysis with the full details of these exceptional cases (apart from the case $\mu = 1$ in some places), we assume the pedestrian attitude (see also Ref. 3) that in making actual calculations it will be more practical to avoid values of μ which are exponentially close to an integer (note that in realistic cases $2\pi U_b/\hbar\omega_b \gtrsim 20$ so that $|\mu - m| \lesssim 10^{-9}$) and to simply smoothly interpolate the results in these regions.

2. Zero-damping limit

If $\lambda \to 0$, keeping all other parameters fixed at nonzero finite values, then $a \approx \frac{1}{2}\mu + \eta$ and $b \approx \frac{1}{2}\mu - \eta$ with $\eta = \mu/\sqrt{\beta\epsilon} \to \infty$. To begin with this requires that $\beta\epsilon \ll 4$, i.e., $\lambda \ll 2k_B T/I_b$. The pertinent asymptotic formula for the hypergeometric function has been given by Watson [Ref. 78, formula 2.3.2(17)]. The evaluation is slightly laborious but elementary. All occurring gamma functions precisely cancel and the result reads

$$\rho(0) \approx \beta\hbar\omega_0 \left[1 - \frac{1-\mu/2\eta}{1+\mu/2\eta} \right] , \tag{3.8}$$

which is written as such in order to still recognize the unit contribution from the genuine Boltzmann term and the usually small corrections to it. Clearly, since with $\beta\epsilon \ll 1$ one has $\mu/2\eta \ll 1$, the correction terms now scale up to the same order of magnitude as the Boltzmann contribution and precisely cancel it. What will be left is easily found to be a distribution proportional to $(-z)^{-1}$ rather than to $(-z)^{-\mu}$. However, strictly speaking Watson's formula applies in fact only under the condition (compare with Sec. IV B) $\eta\sqrt{-z_0} \gg 1$, i.e.,

$$\epsilon \ll (\mu^2/\beta)\exp(-2\pi U_b/\hbar\omega_b) .$$

That is to say, for exponentially small values of the friction, such that

$$\lambda/\omega_b \ll (\beta\hbar\omega_b/2\pi)(\hbar/2\pi I_b)\exp(-2\pi U_b/\hbar\omega_b) ,$$

the metastable system as described by (3.1) does not maintain the proper local equilibrium distribution. Actually, this could have been expected as in that case the local equilibration rate becomes much smaller than the global decay rate Γ. Such exceptionally small friction values will be disregarded throughout the sequel.

B. The decay rate

The decay rate is basically defined by

$$\Gamma = \sum_{n=0}^{\infty} \Gamma(E_n)\rho(E_n) , \tag{3.9}$$

corresponding to the average value of the escape term in (2.1). Within the semiclassical approximation the sum in (3.9) is replaced by an integral in the usual manner. Noticing that $dn/dE = (2\pi\hbar)^{-1} dI/dE$ on grounds of the standard WKB analysis[54,55,65] (see also Sec. III A) and that $dI/dE = \tau(E)$ represents the (semi)classical period of the system's local dynamics, one obtains

$$\Gamma = \int_0^\infty \tau(E)\Gamma(E)\rho(E)dE/2\pi\hbar . \tag{3.10}$$

Introducing (2.3) for $\Gamma(E)$ and realizing that always $\nu(E)\tau(E) = 1$—which as noted before is the crucial property that makes the result for Γ practically insensitive to the (semi)classical approximations—one has

$$\Gamma = \int_0^\infty \kappa P(E)\rho(E)dE/2\pi\hbar , \tag{3.11}$$

which is evidently equal to the integral over the right-hand side of (2.18), divided only by $2\pi\hbar$. But then it is slightly simpler to do the equivalent integral on the left-hand side of (2.18), which readily yields

$$\Gamma = -\frac{\lambda I_b}{\pi\hbar} \left[\rho + k_B T \frac{d\rho}{dE} \right]_{E=0} . \tag{3.12}$$

Finally, invoking the explicit solution (3.6) for $\rho(E)$ and using the standard formula for the derivative of a hypergeometric function,[78–80] the general result may be written as

$$\Gamma = \omega_0 \frac{\epsilon}{\hbar\omega_b} \left[\frac{ab}{1-\mu} e^{y_0} F(1-b, 1-a; 2-\mu; z_0) + \frac{\Gamma(a)\Gamma(1+a)\Gamma(-\mu)}{\Gamma(\mu)\Gamma(1-b)\Gamma(-b)} e^{\mu y_0} \left[\frac{ab}{1+\mu} e^{y_0} F(1+a, 1+b; 2+\mu; z_0) - \mu F(a,b;1+\mu;z_0) \right] \right] \Big/ F(-b,-a;1-\mu;z_0) . \tag{3.13}$$

The parameters a and b have been defined in (3.2), while $z_0=-\exp(y_0)$ with $y_0=-2\pi U_b/\hbar\kappa\omega_b$ so that $\mu y_0=-\beta U_b$.

IV. DISCUSSION

It turns out that all previously known pertinent formulas for the escape rate in separate parameter regimes are contained even in a further simplified version of (3.13). Considering the underlying distribution $\rho(E)$ from (3.6), it appears that—in view of (3.12)—all cases of practical interest are still included even if the argument $z\approx z_0$ is set equal to zero in the already relatively small nonequilibrium terms in the numerator—so that $F(a,b;1+\mu;z)=1$—and also setting z_0 equal to zero in the denominator—so that $F(-b,-a;1-\mu;z_0)=1$. That is, the subsequent discussion will be based upon

$$\rho(E)=\beta\hbar\omega_0[e^{-\beta E}F(-b,-a;1-\mu;z)+e^{-\beta U_b}\Gamma(a)\Gamma(1+a)\Gamma(-\mu)/\Gamma(\mu)\Gamma(1-b)\Gamma(-b)]\ , \tag{4.1}$$

which implies the reduction of (3.13) to the most useful result

$$\Gamma=\omega_0\frac{\epsilon}{\hbar\omega_b}\left[\frac{ab}{1-\mu}e^{y_0}F(1-b,1-a;2-\mu;z_0)+e^{\mu y_0}\Gamma(a)\Gamma(1+a)\Gamma(1-\mu)/\Gamma(\mu)\Gamma(1-b)\Gamma(-b)\right]\ . \tag{4.2}$$

Notice the general structure (see also Ref. 16 cited in Ref. 26) of this result in the sense that $\Gamma=\Gamma$(quantum tunneling with thermal corrections)$+\Gamma$(classical hopping with quantum corrections).

A. Strong damping

The regime of strong damping is bounded from below by the requirement $\beta\epsilon\gg 4$, which in the original parameters amounts to $\lambda/\omega_b\gg 2\kappa k_B T/I_b\omega_b$. For instance, for the typical case of a quartic barrier[46] $I_b=\frac{16}{3}U_b/\omega_b$ so that one should demand that $\lambda/\omega_b\gg(3\kappa/8)k_BT/U_b$. Notice that since always $k_BT/U_b\ll 1$ this case can very well accommodate the intermediate (or moderate) damping range where $\lambda/\omega_b\approx 1$. If $\beta\epsilon$ is sufficiently large, then following (3.2) $a\approx\mu$. And if $\beta\epsilon\gg\mu$ [which for the quartic potential requires $\lambda/\omega_b\gg(3\kappa^2/64\pi)\hbar\omega_b/U_b$, and which is obviously implied in the previous condition if $\mu\le 4$], one may also neglect b relative to unity. Using some elementary properties of gamma functions [78–80] [e.g., $\Gamma(\mu)\Gamma(1-\mu)=\pi/\sin(\pi\mu)$] and noticing that always $ab=-\mu^2/\beta\epsilon$, (4.2) then becomes

$$\Gamma=\kappa\frac{\omega_0}{2\pi}\left[\frac{\pi\mu}{\sin(\pi\mu)}e^{\mu y_0}+\frac{\mu}{\mu-1}e^{y_0}F(1,1-\mu;2-\mu;-e^{y_0})\right]\ . \tag{4.3}$$

If, moreover, one only considers the moderate damping regime $\lambda/\omega_b\approx 1$ such that $\kappa\approx 1$, then (4.3) reduces to Bell's result for the parabolic barrier.[3,63] It is now evident that Bell's original, damping-independent expression is strictly speaking valid only in the range

$$\max[2k_BT/I_b\omega_b;\hbar/4\pi I_b]\ll\lambda/\omega_b\ll 1\ .$$

As long as $\mu\lesssim 2$, (4.3) further simplifies to

$$\Gamma=\kappa\frac{\omega_0}{2\pi}\left[\frac{\pi\mu}{\sin(\pi\mu)}e^{\mu y_0}+\frac{\mu}{\mu-1}e^{y_0}\right]\ , \tag{4.4}$$

which—like (4.3), and with $|y_0|\gg 1$—implies

$$\Gamma=-\kappa y_0\frac{\omega_0}{2\pi}e^{y_0} \tag{4.5}$$

at $\mu=1$. This particular value of μ defines the crossover temperature T_0 between thermal activation and quantum tunneling. Noticing κ from (2.7) one obtains

$$T_0=\frac{\hbar\omega_b}{2\pi k_B}[(1+\lambda^2/\omega_b^2)^{1/2}-\lambda/\omega_b]\ . \tag{4.6}$$

Clearly, T_0 significantly depends on λ. The stronger the dissipation, the more the classical hopping regime extends down to lower temperatures. The result (4.6) is in full agreement with recent findings by means of a dissipative version of Langer's "imaginary part of the free energy" method (see the Introduction and Refs. 13, 32, 59, and 81). Below T_0 [i.e., if $\mu>1+\mathcal{O}(\exp y_0)$] the formula (4.3) yields a finite result even at the integer values $\mu=2,3,\ldots$. A simple limiting procedure shows that the expression

$$\Gamma=\kappa\frac{\omega_0}{2\pi}\frac{\mu}{\mu-1}e^{y_0} \tag{4.7}$$

also contains the precise integer $\mu\ge 2$ values as given by (4.3). Fortunately, if $\mu>1+\mathcal{O}(\exp y_0)$ this is just the exponentially leading contribution in the simplified formula (4.4). The result (4.7) will be compared with some recent experimental data from metastable Josephson junction devices in Sec. IV D.

In the classical high-temperature regime (where $\mu\ll 1$) it is of course the first term in (4.4) which is exponentially leading contribution. At sufficiently high temperature (and/or friction) it reduces to Kramers's result[12]

$$\Gamma=\kappa\frac{\omega_0}{2\pi}e^{\mu y_0}\ , \tag{4.8}$$

which connects the transition state value (where $\kappa=1$) with the correct value in the Smoluchovsky limit (where $\kappa=\omega_b/2\lambda$). The validity of this latter result is now seen to be restricted to values

$$\lambda/\omega_b\gg\max[1;\beta\hbar\omega_b/2\pi]\ .$$

B. Weak damping

The weak damping regime is defined by $\beta\epsilon \ll 4$. Under that condition, following (3.2) one has[82] $a \approx \frac{1}{2}\mu + \eta$ and $b \approx \frac{1}{2}\mu - \eta$, with $\eta = \mu/\sqrt{\beta\epsilon} \gg 1$ if $\beta\epsilon \ll \mu^2$. In contrast with the considerations of Sec. III A 2, however, exponentially small—local equilibrium violating—values of the friction will presently be excluded by requiring $\eta\sqrt{-z_0} \ll 1$. In that case the hypergeometric function in (4.2) remains essentially equal to unity, and using Stirling's formula[79–80] to give

$$\Gamma(\tfrac{1}{2}\mu+\eta)/\Gamma(1-\tfrac{1}{2}\mu+\eta) \approx \eta^{\mu-1} ,$$

it is straightforward to reduce (4.2) to

$$\Gamma = \kappa \frac{\omega_0}{2\pi} \left[\frac{\mu}{\mu-1} e^{y_0} + (\beta\epsilon)^{1-\mu} \frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \mu^{2\mu} e^{\mu y_0} \right] . \quad (4.9)$$

The first term in (4.9) again represents the quantum ($\mu \gg 1$) limit value (4.7). The second contribution in (4.9) now precisely equals the quantum mechanically corrected version of Kramers's classical ($\mu \ll 1$) weak damping result[12]

$$\Gamma = \frac{\omega_0}{2\pi} \beta\epsilon\kappa e^{\mu y_0} , \quad (4.10)$$

as found previously by Melnikov[22,42] from an integral equation which, however, connected neither with the pure quantum limit nor with the heavy damping regime.

The result (4.9) is easily seen to be valid in terms of the original variables only if

$$(\beta\hbar^2\omega_b/8\pi^2 I_b)\exp y_0 \ll \lambda/\omega_b \ll \min[2/\beta I_b\omega_b ; \beta\hbar^2\omega_b/2I_b] ,$$

recalling that $y_0 = -2\pi U_b/\hbar\kappa\omega_b$. At the crossover temperature T_0 [see (4.6)] defined by $\mu=1$, (4.9) again yields the finite value (4.5). Notice finally that at low temperatures the friction range of validity of the Melnikov term in (4.9) shrinks to zero like $1/\mu$, but that for $\mu \gg 1$ this part of Γ is always exponentially smaller than the quantum contribution (4.7).

C. Classical limit

There is another interesting special case contained in (4.2), namely, the classical limit $\hbar \to 0$. Letting Planck's constant go to zero is obviously a simple theoretical device. Experimentally, however, one should be cautious concerning the validity of the ensuing formulas. For instance, although at first glance it would appear from the expression (2.19) for the tunnel probability $P(E)$ that the classical limit can be reached by letting $\omega_b \to 0$, a consistent application of that procedure would always make $\lambda/\omega_b \to \infty$, necessarily implying the strong damping Smoluchovsky result. On the other hand, the frequent suggestion that the classical limit is strictly identical to the high-temperature limit finds a counterexample in the present theory because letting $\beta\epsilon \to 0$ would always imply the extremely weak damping Kramers result.

Let us then consider the limit $\hbar \to 0$, keeping λ/ω_b and $\beta\epsilon$ fixed. Let $\mu \ll 1$ so that following (3.2) both $a \ll 1$ $b \ll 1$. With $\Gamma(\mu) \approx 1/\mu$, and similarly for a and b, the second contribution in (4.2) then immediately reduces to

$$\Gamma = \frac{\omega_0}{2\pi} \beta\epsilon\kappa \frac{\sqrt{1+4/\beta\epsilon}-1}{\sqrt{1+4/\beta\epsilon}+1} e^{\mu y_0} , \quad (4.11)$$

which for $\kappa=1$ becomes equal to the original result of Büttiker, Harris, and Landauer.[19] In the weak damping limit it reduces to Kramers's pertinent formula (4.10), whereas in the strong damping limit (4.11) improves upon the original formula (see also Refs. 46 and 50) as it now

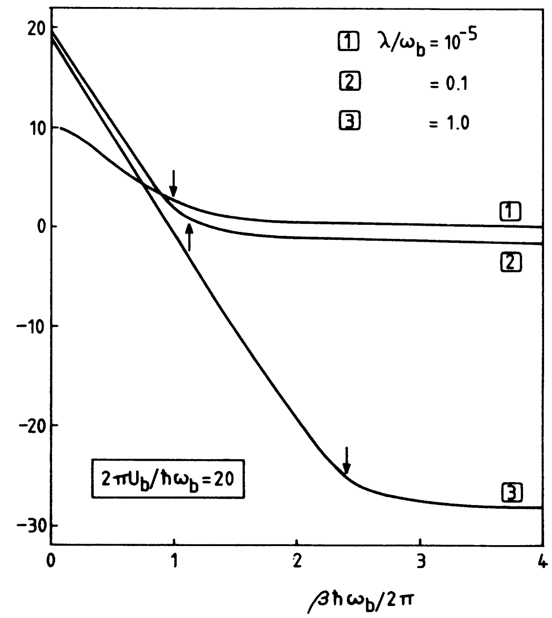

FIG. 2. Logarithm of the decay rate $\Gamma(\beta,\lambda)$ as a function of inverse temperature for three dissipative cases. $\Gamma_0 = \Gamma(\infty, +0)$, while arrows indicate T_0 following (4.6).

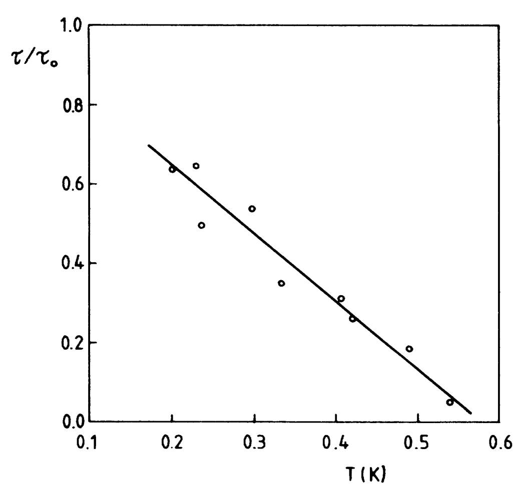

FIG. 3. Lifetime $\tau = 1/\Gamma$ of a metastable flux state as a function of temperature. The experimental dots are according to Refs. 83 and 84; the straight line is following (4.7).

neatly connects with the Kramers-Smoluchovsky result (4.8). The validity of (4.11) is clearly restricted to not too low temperatures $T \gg T_0$, i.e., well above the crossover temperature (4.6).

In Fig. 2 $\ln[\Gamma/\Gamma(\beta=\infty,\lambda=+0)]$ has been sketched as a function of $\mu/\kappa=\beta\hbar\omega_b/2\pi$ (i.e., proportional to $1/T$) for three cases of the dissipation. Following (4.9): *case 1*, $\lambda/\omega_b=10^{-5}$ so that $\kappa\approx 1.000$, with the assumption of a quartic potential,[46] i.e., $I_b=\frac{16}{3}U_b/\omega_b$, so that for $-\kappa y_0=2\pi U_b/\hbar\omega_b=20$ one has $2\pi\epsilon/\hbar\omega_b\approx 0.002$; and, with the same value for κy_0, following (4.4): *case 2*, $\lambda/\omega_b=0.1$ so that $\kappa\approx 0.905$ and *case 3*, $\lambda/\omega_b=1.0$ so that $\kappa\approx 0.414$. Arrows indicate T_0 (i.e., $\mu=1$). Notice the influence of friction on the classical rate at both weak and strong damping via the prefactor and the enormous effect on the quantum rate at strong damping via the exponential (which agrees with recent findings from Langer's theory.[13,20,25,32] Also note the rather slow approach to the final zero-temperature value (see further Sec. IV D and Fig. 3).

D. An experimental case

If one applies (4.7) to the most recent experimental data of de Bruyn Ouboter and co-workers[83,84] (see also Refs. 6 and 26) on metastable flux states—in superconducting rings interrupted by a Josephson weak link—one arrives at Fig. 3, where $\tau=1/\Gamma$. The theoretical straight line has been found—with only ω_b as a variational parameter—for $\omega_b=0.98\omega_0$, whereas the device calculated ratio (not measured *in situ*) would have been $\omega_b/\omega_0=0.80$. The quantity τ_0 used in Fig. 3 in order to normalize the lifetime τ is just the zero-temperature value of (4.7). The present theory fits the data at least as well as other theories.[8,20,32,83–85]

V. CONCLUSIONS

In this paper an exactly solvable stochastic quantum model is discussed for the escape process of a particle in a metastable local potential hole coupled with arbitrary strength to an Ohmic thermal environment. Both the system's quasiequilibrium distribution $\rho(E)$ and the ensuing decay rate Γ are obtained. With the exception of exponentially small values of the rate of dissipation (i.e., $\lambda\lesssim\Gamma_0$) the distribution remains locally exponentially close to the Boltzmann distribution. The decay rate is shown to include earlier results of, e.g., Bell, Büttiker *et al.*, Kramers, and Melnikov as special cases.

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