Propagators for driven coupled harmonic oscillators

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Propagators for coupled and driven coupled harmonic oscillators are evaluated exactly by the path-integral method. The propagators for coupled harmonic oscillators are used to obtain explicitly the energy expectation values.

I. INTRODUCTION

Although the Feynman path-integral formulation[1] offers a general approach for treating quantum-mechanical systems, only several problems can be solved exactly. Two of these are the driven harmonic oscillator with a quadratic Hamiltonian[2] and the time-dependent damped driven harmonic oscillator.[3] A number of situations such as superconducting quantum-interference devices,[4] quantum-nondemolition measurements,[5] magnetohydrodynamics,[6] etc., can be described by driven coupled harmonic oscillators. Introducing the Caldirola-Kanai Hamiltonian,[7] one can obtain the time-dependent Schrödinger equation for the damped harmonic oscillator. However, it has been a matter of debate as to whether or not this Schrödinger equation represents the quantum-mechanical dissipative system.[8] Some workers[9] claim affirmation while others[10] object to it. This problem has been critically reviewed by Greenberger[11] and Cervero and Villaroel.[12]

The purpose of this paper is to derive the propagator for a driven coupled harmonic oscillators (DCHO) system from our previous work[13] for both coupled and coupled driven harmonic oscillators by means of the path-integral method. We introduce two harmonic oscillators that are coupled together with another spring. We review the classical case and construct the form of the propagator for DCHO, respectively, in Secs. II and III. Section IV gives the exact derivation of the propagator for the coupled harmonic oscillators (CHO), and in Sec. V we evaluate the exact propagator for DCHO by using the results obtained in Sec. IV. The energy expectation values of CHO are evaluated in Sec. VI, and finally we give results and discussion in Sec. VII.

II. CLASSICAL CASE

In this section we consider a system of two harmonic oscillators which are coupled together by means of another spring. We assume that the masses of the oscillators and three spring constants are all the same. Let the forces $f_1(t)$ and $f_2(t)$ exerted on the two oscillators and their displacements be x_1 and x_2. Then the Hamiltonian for DCHO can be written as

$$H=\frac{1}{2m}(p_1^2+p_2^2)+m\omega^2(x_1^2-x_1x_2+x_2^2)-f_1(t)x_1 -f_2(t)x_2 , \tag{2.1}$$

where $\omega^2=k/m$. Hamilton's equations of motion for Eq. (2.1) are

$$\dot{x}_1=p_1/m , \tag{2.2}$$

$$\dot{x}_2=p_2/m , \tag{2.3}$$

$$\dot{p}_1=m\omega^2(x_2-2x_1)+f_1(t) , \tag{2.4}$$

$$\dot{p}_2=m\omega^2(x_1-2x_2)+f_2(t) . \tag{2.5}$$

Equations (2.1)–(2.5) yield the Lagrangian

$$\begin{aligned}L&=(p_1\dot{x}_1+p_2\dot{x}_2)-H\\ &=\frac{m}{2}(\dot{x}_1^2+\dot{x}_2^2)-m\omega^2(x_1^2-x_1x_2+x_2^2)+f_1(t)x_1 +f_2(t)x_2 ,\end{aligned} \tag{2.6}$$

with the corresponding equations of motion

$$x_1+\omega^2(2x_1-x_2)=f_1(t)/m , \tag{2.7}$$

$$x_2+\omega^2(2x_2-x_1)=f_2(t)/m . \tag{2.8}$$

The classical solutions of Eqs. (2.7) and (2.8) are given by

$$\begin{aligned}x_1(t)=&A\sin(\omega t)+B\cos(\omega t)+C\sin(\sqrt{3}\omega t)\\ &+D\cos(\sqrt{3}\omega t)\\ &+\int^t d\tau\int^\tau d\nu\, e^{i\omega(2\tau-\nu-t)}[f_1(\nu)+f_2(\nu)]\end{aligned} \tag{2.9}$$

and

$$x_2(t)=A\sin(\omega t)+B\cos(\omega t)-C\sin(\sqrt{3}\omega t)-D\cos(\sqrt{3}\omega t)+\int^t d\tau\int^\tau d\nu e^{i\omega(2\tau-\nu-t)}[f_1(\nu)-f_2(\nu)] \;. \quad (2.10)$$

III. PATH INTEGRAL OF DRIVEN COUPLED HARMONIC OSCILLATORS

In the path-integral formulation, the solution of the Schrödinger equation is given as the path-dependent integral equations with propagator K,

$$\psi(x_1,x_2,t)=\int dx_1'dx_2'K(x_1,x_2,t;x_1',x_2',0)\psi(x_1',x_2',0) \;, \quad (3.1)$$

which gives the wave function $\psi(x_1,x_2,t)$ at time t in terms of the wave function $\psi(x_1',x_2')$ at time $t=0$. The propagator in Eq. (3.1) can be written by means of the Feynman path integral

$$K(x_1,x_2,t;x_1',x_2',0)=\int_{(x_1',x_2',0)}^{(x_1,x_2,t)}Dx(t)\exp[(i/\hbar)S(x_1,x_2,x_1',x_2';t)] \;, \quad (3.2)$$

where

$$Dx(t)=\lim_{N\to\infty}\frac{1}{A}\prod_{j=1}^{N-1}(dx_{1j}dx_{2j}/A^2) \quad (3.3)$$

and $S(x_1,x_2,x_1',x_2';t)$ is the action defined as the time integral over the Lagrangian $L(\dot{x}_1,\dot{x}_2,x_1,x_2;t)$ between $t=t$ and $t=0$,[1]

$$S(x_1,x_2,x_1',x_2';t)=\int_0^t dt\, L(\dot{x}_1,\dot{x}_2,x_1,x_2;t) \;. \quad (3.4)$$

In Eq. (3.3) A is the normalization factor given by

$$A=[2\pi i\hbar\epsilon/m]^{1/2}, \quad \epsilon=\lim_{N\to\infty}(t/N) \;. \quad (3.5)$$

Substituting Eq. (2.6) into (3.4), the action becomes

$$S(x_1,x_2,x_1',x_2';t)=S_c(x_1,x_2,x_1',x_2';t)+\int_0^t d\tau\frac{m}{2}\{\dot{y}_1^2(\tau)+\dot{y}_2^2(\tau)-2\omega^2[y_1^2(\tau)-y_1(\tau)y_2(\tau)+y_2^2(\tau)]\} \;, \quad (3.6)$$

where S_c is the classical action and y_i is the deviation of $x_i(t)$ from its classical path x_{ci} given as

$$y_i=x_i-x_{ci} \quad (i=1,2) \;. \quad (3.7)$$

Then the propagator [Eq. (3.2)] can be expressed as

$$K(x_1,x_2,t;x_1',x_2',0)=F(t)e^{iS_c/\hbar} \;. \quad (3.8)$$

Here, $F(t)$ is the multiplicative function given in the form

$$F(t)=\int_0^0 Dx(t)\left[\exp\left[(im/2\hbar)\int_0^t dt[\dot{y}_1^2+\dot{y}_2^2-2\omega^2(y_1^2-y_1y_2+y_2^2)]\right]\right] \;. \quad (3.9)$$

It is easy to show that $F(t)$ has the same form for CHO and DCHO. Therefore the propagator depends only on the classical action in both cases. In Eq. (3.9), by changing the variables $x_1\pm x_2$ into

$$z_1=\frac{1}{\sqrt{2}}(x_1-x_2) \;, \quad (3.10)$$

$$z_2=\frac{1}{\sqrt{2}}(x_1+x_2) \;, \quad (3.11)$$

we can reduce the condition $(y_1,y_2)=(0,0)$ to $(z_1,z_2)=(0,0)$. Applying Eqs. (3.10) and (3.11) to Eq. (3.9), the multiplicative function becomes

$$F(t)=J\int_0^0 Dz(t)\left[\exp\left[(im/2\hbar)\int_0^t Dz(t)[(\dot{z}_1^2-\omega^2z_1^2)+(\dot{z}_2^2-3\omega^2z_2^2)\right]\right] \;, \quad (3.12)$$

$$\begin{pmatrix} y_{11}\\ y_{12}\\ y_{21}\\ \vdots\\ y_{N-1,2}\\ y_{N1}\\ y_{N2}\end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 & 0 & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & \frac{1}{\sqrt{2}} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{2}} & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix} z_{11}\\ z_{12}\\ z_{21}\\ \vdots\\ z_{N-1,2}\\ z_{N1}\\ z_{N2}\end{pmatrix} \;. \quad (3.13)$$

In Eq. (3.12), J becomes unity.

If the action is separated into the functionals with only same variables in the path integral, then this integral can be represented by the multiplication of path integrals with each variable. Therefore Eq. (3.12) becomes

$$F(t)=F_1(t)F_2(t)=\left[\int_0^0 Dz_1(t)\exp\left[(im/2\hbar)\int_0^t dt(\dot z_1^2-\omega^2 z_1^2)\right]\right]\left[\int_0^0 Dz_2(t)\exp\left[(im/2\hbar)\int_0^t dt(\dot z_2^2-3\omega^2 z_2^2)\right]\right] . \quad (3.14)$$

Since $F_1(t)$ and $F_2(t)$ are the path integrals of the harmonic oscillator, the evaluation of Eq. (3.14) gives

$$F(t)=\frac{m\omega}{2\pi i\hbar}\left[\frac{\sqrt{3}}{\sin(\omega t)\sin(\sqrt{3}\omega t)}\right]^{1/2} . \quad (3.15)$$

Hence, the propagator of DCHO can be written as

$$K(x_1,x_2,t;x_1',x_2',0)=\frac{m\omega}{2\pi i\hbar}\left[\frac{\sqrt{3}}{\sin(\omega t)\sin(\sqrt{3}\omega t)}\right]^{1/2} e^{iS_c/\hbar} . \quad (3.16)$$

IV. PROPAGATOR FOR THE COUPLED HARMONIC OSCILLATORS

To evaluate the exact propagator expressed by Eq. (3.16), we should first obtain the propagator for CHO. The classical action of CHO is

$$S_c=\int_0^t d\tau\left[\frac{m}{2}(\dot x_{c1}^2+\dot x_{c2}^2)-m\omega^2(x_{c1}^2-x_{c1}x_{c2}+x_{c2}^2)\right] , \quad (4.1)$$

where x_{c1} and $\dot x_{c1}$ are the classical path and velocity, respectively. Integrating Eq. (4.1) over the time, we get

$$\begin{aligned}S_c&=\frac{m}{2}(x_{c1}\dot x_{c1}+x_{c2}\dot x_{c2})|_0^t-\int_0^t d\tau\frac{m}{2}x_{c1}[\ddot x_{c1}+\omega^2(2x_{c1}-x_{c2})]-\int_0^t d\tau\frac{m}{2}x_{c2}[\ddot x_{c2}+\omega^2(2x_{c2}-x_{c1})]\\&=\frac{m}{2}[x_{c1}(t)\dot x_{c1}(t)+x_{c2}(t)\dot x_{c2}(t)-x_{c1}(0)\dot x_{c1}(0)-x_{c2}(0)\dot x_{c2}(0)] . \end{aligned}\quad (4.2)$$

Here the second and third terms become zero because of the equations of motion [see Eqs. (2.7) and (2.8)], given as

$$\ddot x_1+\omega^2(2x_1-x_2)=0 , \quad (4.3)$$

$$\ddot x_2+\omega^2(2x_2-x_1)=0 . \quad (4.4)$$

To obtain the exact expression of Eq. (4.2), we solve Eqs. (4.3) and (4.4) to obtain

$$x_1=x_1(t)=A\sin(\omega t)+B\cos(\omega t)+C\sin(\sqrt{3}\omega t)+D\cos(\sqrt{3}\omega t) , \quad (4.5)$$

$$x_2=x_2(t)=A\sin(\omega t)+B\cos(\omega t)-C\sin(\sqrt{3}\omega t)-D\cos(\sqrt{3}\omega t) , \quad (4.6)$$

and $\dot x_1$ and $\dot x_2$ are given, respectively, by

$$\dot x_1=\dot x_1(t)=\omega[A\cos(\omega t)-B\sin(\omega t)+\sqrt{3C}\cos(\sqrt{3}\omega t)-\sqrt{3}D\sin(\sqrt{3}\omega t)] , \quad (4.7)$$

$$\dot x_2=\dot x_2(t)=\omega[A\cos(\omega t)-B\sin(\omega t)-\sqrt{3}C\cos(\sqrt{3}\omega t)+\sqrt{3}D\sin(\sqrt{3}\omega t)] . \quad (4.8)$$

Equations (4.5)-(4.8) give

$$x_1'=x_1(0)=B+D , \quad (4.9)$$

$$x_2'=x_2(0)=B-D , \quad (4.10)$$

$$\dot x_1'=\dot x_1(0)=\omega(A+\sqrt{3}C) , \quad (4.11)$$

$$\dot x_2'=\dot x_2(0)=\omega(A-\sqrt{3}C) . \quad (4.12)$$

The time-dependent constants A, B, C, and D obtained from Eqs. (4.5) and (4.6), and Eqs. (4.9) and (4.10) can be expressed as

$$A=[\tfrac{1}{2}\sin(\omega t)][x_1+x_2-(x_1'+x_2')\cos(\omega t)] , \quad (4.13)$$

$$B=\tfrac{1}{2}(x_1'+x_2') , \quad (4.14)$$

$$C=[\tfrac{1}{2}\sin(\sqrt{3}\omega t)][x_1-x_2+(x_1'-x_2')\cos(\sqrt{3}\omega t)] , \quad (4.15)$$

$$D=\tfrac{1}{2}(x_1'-x_2') . \quad (4.16)$$

Substitution of Eqs. (4.5)–(4.16) into (4.2) gives the classical action,

$$S_c=\frac{m\omega}{4}\{(x_1^2+x_2^2+x_1'^2+x_2'^2)[\cot(\omega t)+\sqrt{3}\cot(\sqrt{3}\omega t)]\}+2(x_1x_2+x_1'x_2')[\cot(\omega t)-\sqrt{3}\cot(\sqrt{3}\omega t)]$$
$$-2(x_1x_1'+x_2x_2')\{[1/\sin(\omega t)+\sqrt{3}/\sin(\sqrt{3}\omega t)]\}+2(x_1x_2'+x_2x_1')[-1/\sin(\omega t)+\sqrt{3}/\sin(\sqrt{3}\omega t)]\ . \tag{4.17}$$

Combining Eqs. (4.17) and (3.16), we obtain the propagator for CHO,

$$K(x_1,x_2,t;x_1',x_2',0)=\frac{m\omega}{2\pi i\hbar}[\sqrt{3}/\sin(\omega t)\sin(\sqrt{3}\omega t)]^{1/2}$$
$$\times\exp\{(im\omega/4\hbar)[(x_1^2+x_2^2+x_1'^2+x_2'^2)[\cot(\omega t)+\sqrt{3}\cot(\sqrt{3}\omega t)]$$
$$+2(x_1x_2+x_1'x_2')[\cot(\omega t)-\sqrt{3}\cot(\sqrt{3}\omega t)]$$
$$-2(x_1x_1'+x_2x_2')[1/\sin(\omega t)+\sqrt{3}/\sin(\sqrt{3}\omega t)]$$
$$+2(x_1x_2'+x_2x_1')[-1/\sin(\omega t)+\sqrt{3}/\sin(\sqrt{3}\omega t)]]\}\ . \tag{4.18}$$

V. PROPAGATOR FOR DRIVEN COUPLED HARMONIC OSCILLATORS

When we set $f_1(t)=f_2(t)=0$, DCHO reduces to CHO, whereby we can write the propagator for DCHO as

$$K(x_1,x_2,t;x_1',x_2',0)=\exp[a(t)x_1^2+b(t)x_1x_2+c(t)x_2^2+d(t)x_1+g(t)x_2+h(t)]\ . \tag{5.1}$$

Here $a(t)$, $b(t)$, $c(t)$, $d(t)$, $g(t)$, and $h(t)$ are time-dependent functions including x_1' and x_2', which need to be determined. Equation (5.1) must satisfy the Schrödinger equation

$$i\hbar(\partial K/\partial t)=HK\ . \tag{5.2}$$

Substitution of Eq. (5.1) into (5.2) gives the time-dependent coefficients

$$a(t)=\frac{i\hbar}{2m}[4a^2(t)+c^2(t)]+m\omega^2/i\hbar\ , \tag{5.3}$$

$$b(t)=\frac{i\hbar}{2m}[4b^2(t)+c^2(t)]+m\omega^2/i\hbar\ , \tag{5.4}$$

$$c(t)=\frac{2i\hbar}{m}[a(t)c(t)+b(t)c(t)]-m\omega^2/i\hbar\ , \tag{5.5}$$

$$d(t)=\frac{i\hbar}{m}[2a(t)d(t)+c(t)g(t)]+\left[\frac{i}{\hbar}\right]f_1(t)\ , \tag{5.6}$$

$$g(t)=\frac{i\hbar}{m}[2b(t)g(t)+c(t)d(t)]+\left[\frac{i}{\hbar}\right]f_2(t)\ , \tag{5.7}$$

$$h(t)=\frac{i\hbar}{2m}[d^2(t)+g^2(t)+2a(t)+2b(t)]\ . \tag{5.8}$$

Since Eqs. (5.3) and (5.4) have the same form, we get

$$a(t)=b(t)\ . \tag{5.9}$$

Substituting Eq. (5.9) into (5.5) and changing the variables a and c into

$$\eta=a+c/2\ , \tag{5.10}$$

$$\zeta=a-c/2\ , \tag{5.11}$$

we obtain two ordinary differential equations,

$$\dot{\eta}=\frac{2i\hbar}{m}\eta^2+\frac{m\omega^2}{2i\hbar}\ , \tag{5.12}$$

$$\dot{\zeta}=\frac{2i\hbar}{m}\zeta^2+\frac{3m\omega^2}{2i\hbar}\ . \tag{5.13}$$

The solutions of Eqs. (5.12) and (5.13) are given by

$$\eta=\frac{i\omega m}{2\hbar}\cot(\omega t+\theta_1)\ , \tag{5.14}$$

$$\zeta=\frac{\sqrt{3}i\omega m}{2\hbar}\cot(\sqrt{3}\omega t+\theta_2)\ , \tag{5.15}$$

where θ_1 and θ_2 are the constants to be determined. The time-dependent coefficients $a(t)$, $b(t)$, and $c(t)$ obtained in comparison with Eqs. (5.10), (5.11), (5.14), and (5.15) are given as

$$a(t)=b(t)=\frac{i\omega m}{4\hbar}[\cot(\omega t+\theta_1)+\sqrt{3}\cot(\sqrt{3}\omega t+\theta_2)]\ , \tag{5.16}$$

$$c(t)=\frac{i\omega m}{2\hbar}[\cot(\omega t+\theta_1)-\sqrt{3}\cot(\sqrt{3}\omega t+\theta_2)]\ . \tag{5.17}$$

Equations (5.16) and (5.17) do not include the driven forces $f_1(t)$ and $f_2(t)$. Therefore, through setting $f_1(t)=f_2(t)=0$, Eqs. (5.16) and (5.17) do not change at all and should be equal to the coefficients of x_1^2 and x_2^2 in Eq. (4.18). Comparison of these two equations shows θ_1 and θ_2 to be zero. Substituting Eq. (5.9) into Eqs. (5.6) and (5.7) and changing variables d and g into

$$\rho=d+g\ , \tag{5.18}$$

$$\sigma=d-g\ , \tag{5.19}$$

we obtain the two differential equations

$$\dot{\rho}=\frac{i\hbar}{m}[2a(t)+c(t)]\rho+\frac{i}{\hbar}[f_1(t)+f_2(t)]\ , \tag{5.20}$$

$$\dot{\sigma}=\frac{i\hbar}{m}[2a(t)+c(t)]\sigma+\frac{i}{\hbar}[f_1(t)-f_2(t)]\ . \tag{5.21}$$

Combining Eqs. (5.20) and (5.21) with Eqs. (5.16) and (5.17), we obtain the solutions

$$\rho=[1/\sin(\omega t)]\left[\int_0^t d\tau \frac{i}{\hbar}[f_1(\tau)+f_2(\tau)]\times\sin(\omega t)+\alpha\right], \tag{5.22}$$

$$\sigma=[1/\sin(\sqrt{3}\omega t)]\left[\int_0^t d\tau \frac{i}{\hbar}[f_1(\tau)-f_2(\tau)]\times\sin(\sqrt{3}\omega t)+\beta\right], \tag{5.23}$$

where α and β are constants to be determined. We can obtain the time-dependent coefficients $d(t)$ and $g(t)$ by substituting Eqs. (5.22) and (5.23) into Eqs. (5.18) and (5.19),

$$\begin{aligned}d(t)=&[i/2\hbar\sin(\omega t)]\int_0^t d\tau[f_1(\tau)+f_2(\tau)]\sin(\omega\tau)\\&+[i/2\hbar\sin(\sqrt{3}\omega t)]\int_0^t d\tau[f_1(\tau)-f_2(\tau)]\times\sin(\sqrt{3}\omega\tau)\\&+[\alpha/2\sin(\omega t)]+[\beta/2\sin(\sqrt{3}\omega t)]\ ,\end{aligned} \tag{5.24}$$

$$\begin{aligned}g(t)=&[i/2\hbar\sin(\omega t)]\int_0^t d\tau[f_1(\tau)+f_2(\tau)]\sin(\omega\tau)\\&-[i/2\hbar\sin(\sqrt{3}\omega t)]\int_0^t d\tau[f_1(\tau)-f_2(\tau)]\times\sin(\sqrt{3}\omega\tau)\\&+[\alpha/2\sin(\omega t)]-[\beta/2\sin(\sqrt{3}\omega t)]\ .\end{aligned} \tag{5.25}$$

Substitution of Eqs (5.16), (5.17), (5.24), and (5.25) into (5.8) yields

$$\begin{aligned}h(t)=&-\frac{i\hbar}{4m\omega}[\alpha^2\cot(\omega t)+(\beta^2/\sqrt{3})\cot(\sqrt{3}\omega t)]-[\alpha/m\omega\sin(\omega t)]\int_0^t d\tau[f_1(\tau)+f_2(\tau)]\sin[\omega(t-\tau)]\\&-[\beta/\sqrt{3}m\omega\sin(\sqrt{3}\omega t)]\int_0^t d\tau[f_1(\tau)-f_2(\tau)]\sin[\sqrt{3}\omega(t-\tau)]\\&+[1/4i\hbar m\omega\sin(\omega t)]\int_0^t d\tau\int_0^t d\nu[f_1(\tau)+f_2(\tau)][f_1(\nu)+f_2(\nu)]\sin[\omega(t-\tau)]\sin(\omega\nu)\\&+[1/4\sqrt{3}i\hbar m\omega\sin(\sqrt{3}\omega t)]\int_0^t d\tau\int_0^t d\nu[f_1(\tau)-f_2(\tau)][f_1(\nu)-f_2(\nu)]\sin[\sqrt{3}\omega(t-\tau)]\\&\times\sin(\sqrt{3}\omega\nu)-\ln[\sin(\omega t)\sin(\sqrt{3}\omega t)]+\delta\ .\end{aligned} \tag{5.26}$$

Here, δ is also a constant to be determined. When setting $f_1(t)=f_2(t)=0$, Eqs. (5.24) and (5.25) should be reduced to the coefficients of x_1 and x_2, and Eq. (5.26) should also be reduced to the terms in the exponent in Eq. (4.18). Comparison between them gives the constants α, β, and δ,

$$\alpha=\frac{m\omega}{i\hbar}(x_1'+x_2')\ , \tag{5.27}$$

$$\beta=\frac{m\omega}{i\hbar}(x_1'-x_2')\ , \tag{5.28}$$

$$\delta=\ln\left[\frac{3^{1/4}m\omega}{2\pi i\hbar}\right]. \tag{5.29}$$

Substitution of the preceding results into Eq. (5.1) gives the propagator for DCHO,

$$\begin{aligned}K(x_1,x_2,t;x_1',x_2',0)=&\frac{m\omega}{2\pi i\hbar}\{\sqrt{3}/[\sin(\omega t)\sin(\sqrt{3}\omega t)]\}^{1/2}\\&\times\exp\left\{\frac{im\omega}{4\hbar}\left[(x_1^2+x_2^2+x_1'^2+x_2'^2)[\cot(\omega t)+\sqrt{3}\cot(\sqrt{3}\omega t)]\right.\right.\\&+2(x_1x_2+x_1'x_2')[\cot(\omega t)-\sqrt{3}\cot(\sqrt{3}\omega t)]\\&-2(x_1x_1'+x_2x_2')\{1/\sin(\omega t)+[\sqrt{3}/\sin(\sqrt{3}\omega t)]\}\\&+2(x_1x_2'+x_1'x_2)[-1/\sin(\omega t)+\sqrt{3}/\sin(\sqrt{3}\omega t)]\\&+\frac{2x_1}{m\omega}\left[[1/\sin(\omega t)]\int_0^t d\tau[f_1(\tau)+f_2(\tau)]\sin(\omega\tau)\right.\\&\left.+[1/\sin(\sqrt{3}\omega t)]\int_0^t d\tau[f_1(\tau)-f_2(\tau)]\sin(\sqrt{3}\omega\tau)\right]\\&+\frac{2x_2}{m\omega}\left[[1/\sin(\omega t)]\int_0^t d\tau[f_1(\tau)+f_2(\tau)]\sin(\omega\tau)\right.\\&\left.-[1/\sin(\sqrt{3}\omega t)]\int_0^t d\tau[f_1(\tau)-f_2(\tau)]\sin(\sqrt{3}\omega\tau)\right]\end{aligned}$$

$$+\frac{4x_1'}{m\omega}\left[[1/\sin(\omega t)]\int_0^t d\tau[f_1(\tau)+f_2(\tau)]\sin[\omega(t-\tau)] + [1/\sin(\sqrt{3}\omega t)]\int_0^t d\tau[f_1(\tau)-f_2(\tau)]\sin(\sqrt{3}\omega\tau)\right]$$

$$+\frac{4x_2'}{m\omega}\left[[1/\sin(\omega t)]\int_0^t d\tau[f_1(\tau)+f_2(\tau)]\sin[\omega(t-\tau)] - [1/\sin(\sqrt{3}\omega t)]\int_0^t d\tau[f_1(\tau)-f_2(\tau)]\sin(\sqrt{3}\omega\tau)\right]$$

$$-[1/m^2\omega^2\sin(\omega t)]\int_0^t d\tau\int_0^t d\nu[f_1(\tau)+f_2(\tau)][f_1(\nu)+f_2(\nu)]\times\sin[\omega(t-\tau)]\sin(\omega\nu)$$

$$-[1/\sqrt{3}m^2\omega^2\sin(\sqrt{3}\omega t)]\int_0^t d\tau\int_0^t d\nu[f_1(\tau)-f_2(\tau)][f_1(\nu)-f_2(\nu)]\times\sin[\sqrt{3}\omega(t-\tau)]\sin(\sqrt{3}\omega\nu)\Big]\Big\}. \quad (5.30)$$

VI. ENERGY EXPECTATION VALUES OF COUPLED HARMONIC OSCILLATORS

The Hamiltonian of CHO is

$$H=\frac{1}{2m}(p_1^2+p_2^2)+m\omega^2(x_1^2-x_1x_2+x_2^2)\ . \quad (6.1)$$

Using Eqs. (3.1) and (3.2) with Eq. (6.1), we obtain the Schrödinger equation

$$i\hbar(\partial/\partial t)\psi(x_1,x_2,t)=H_{\rm op}\psi(x_1,x_2,t)\ , \quad (6.2)$$

where $H_{\rm op}$ is the Hamiltonian operator in which the momentum p_i is changed into $p_i=(\hbar/i)(\partial/\partial x_i)$. Since Eq. (6.2) can be separated into time and coordinate parts, we may write

$$K(t)=e^{-iH_{\rm op}t/\hbar}, \quad (6.3)$$

$$H_{\rm op}|l,n\rangle=E_{ln}|l,n\rangle\quad (l,n=1,2,3,\dots)\ . \quad (6.4)$$

Here the states $|l,n\rangle$ are the complete set with energy eigenvalues of $H_{\rm op}$. Since the function with states $|l,n\rangle$ can be expressed by

$$\phi_{ln}(x_1,x_2)=\langle x_1,x_2|l,n\rangle\ , \quad (6.5)$$

the propagator at $t>0$ becomes

$$K(x_1,x_2,t;x_1',x_2',0)=\langle x_1,x_2|e^{-iH_{\rm op}t/\hbar}|x_1',x_2'\rangle$$
$$=\sum_l\sum_n\sum_{l'}\sum_{n'}\langle x_1,x_2|ln\rangle\langle ln|e^{iH_{\rm op}t/\hbar}|l',n'\rangle\langle l',n'|x_1',x_2'\rangle$$
$$=\sum_l\sum_n\phi_{ln}(x_1,x_2)e^{-iE_{ln}t/\hbar}\phi_{ln}^*(x_1',x_2')\ . \quad (6.6)$$

Equation (6.6) should be the same as Eq. (4.18). Setting $x_1'=x_1$ and $x_2'=x_2$ in Eq. (4.18) and integrating over x_1 and x_2, we get

$$\sum_l\sum_n\int\int dx_1dx_2\phi_{ln}^*(x_1,x_2)e^{-iE_{ln}t/\hbar}\phi_{ln}(x_1,x_2)=e^{-iE_{ln}t/\hbar} \quad (6.7)$$

and

$$\int\int dx_1dx_2\frac{m\omega}{2\pi i\hbar}\{\sqrt{3}/[\sin(\omega t)\sin(\sqrt{3}\omega t)]\}^{1/2}\times\exp\left[\frac{im\omega}{2\hbar}[(x_1+x_2)^2-\sqrt{3}(x_1-x_2)^2][\cot(\omega t)-1/\sin(\omega t)]\right]=-\tfrac{1}{2}[\sin(\omega t/2)\sin(\sqrt{3}/2\omega t)]^{-1}\ . \quad (6.8)$$

Hence we have

$$\sum_l \sum_n e^{-iE_{ln}t/\hbar} = -\tfrac{1}{2}[\sin(\omega t/2)\sin(\sqrt{3}/2\omega t)]^{-1}$$
$$= [e^{-i\omega t/2}/(1-e^{-i\omega t})][e^{-i\sqrt{3}\omega t/2}/(1-e^{-i\sqrt{3}\omega t})]$$
$$= \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \exp\{-i\omega t[(l+\tfrac{1}{2})+\sqrt{3}(n+\tfrac{1}{2})]\} . \quad (6.9)$$

Therefore the expectation values of CHO becomes

$$E_{ln} = [(l+\tfrac{1}{2})+\sqrt{3}(n+\tfrac{1}{2})]\hbar\omega . \quad (6.10)$$

VII. RESULTS AND DISCUSSION

In the previous sections we have obtained the exact propagators [Eqs. (4.18) and (5.30] for CHO and DCHO by the path-integral method. The forms of the propagators are new. Setting $f(t)=0$, Eq. (5.30) is reduced to Eq. (4.18). Although DCHO is a nonconservative system, the quantum-mechanical problem for the momentum operator does not appear because the canonical momentum is equal to the kinetic momentum in our derivation.[13]

Making use of Eq. (4.81), we have obtained the energy expectation values [Eq. (6.10)] for CHO, given by the sum of two energy expectation values corresponding to the quantum states of two oscillators. Even though we have not evaluated the wave function of CHO, we may easily surmise that the wave function will be given by the multiplication of two wave functions for two oscillators. In the case of DCHO, one cannot easily apply Eq. (5.20) to obtain the energy expectation values, since this equation cannot be expressed in the form of Eq. (6.6), and one should recognize that the energy operator is not equal to the Hamiltonian operator in a nonconservative system.[9]

The evaluations for the wave functions, energy expectation values for CHO and DCHO, and propagator and other physical quantities for n coupled and n driven coupled harmonic oscillators (arbitrary n) are in progress and will be reported in the near future.

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