Fast, accurate algorithm for numerical simulation of exponentially correlated colored noise

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The normal procedure for numerical simulation of exponentially correlated colored noise is superseded by the novel algorithm presented here. A differential algorithm is replaced by an integral algorithm which is faster, more accurate, and permits the use of longer step sizes.

I. INTRODUCTION

Traditionally, stochastic differential equations used in the physical sciences have involved Gaussian white noise.[1] In recent times, however, white noise has been replaced by colored noise in a variety of contexts. Laser noise problems[2] and first passage time problems[3] have been shown to necessitate the use of colored noise instead of white noise. Even the mathematical foundations for the theory of stochastic differential equations call for colored noise if the Stratonovich perspective is adopted, as it is when physical arguments are invoked.[1,4]

In each of these contexts, many specific problems require numerical simulation as a component of a complete analysis. This is usually a consequence of nonlinearity and the resulting intractability in purely analytic terms. Consequently, numerical-simulation algorithms have been developed, originally for white noise, and recently for colored noise as well.[5] The simplest type of colored noise to generate is exponentially correlated colored noise. Such noise introduces only one more parameter, the correlation time for the exponential correlation, and it is easily generated by a linear damping equation driven by white noise. Our new algorithm is for this kind of colored noise.

In Sec. II we review the white-noise algorithm and the differential version of the exponentially correlated, colored-noise algorithm. In Sec. III we present the integral version of the colored-noise algorithm and demonstrate its superior properties.

II. DIFFERENTIAL ALGORITHM FOR COLORED NOISE

In order to be concrete, we consider the one variable problem

$$\dot{x}=f(x)+g_w \tag{1}$$

in which x is the variable of interest, $f(x)$ is a nonlinear function, and g_w is Gaussian white noise. The noise has the properties

$$\langle g_w(t)\rangle=0 , \tag{2}$$

$$\langle g_w(t)g_w(s)\rangle=2D\delta(t-s) , \tag{3}$$

which completely determines all of its statistical features because of the Gaussian quality. The white-noise quality of g_w is contained in the Dirac δ-function correlation (3).

The Box-Mueller algorithm[6] is used to generate Gaussian noise from two random numbers which are uniformly distributed on the unit interval. The Euler version of the integration of (1) is given by (Δt is the step size)

$$a=\text{random number} , \tag{4}$$

$$b=\text{random number} , \tag{5}$$

$$g_w=[-4D\Delta t\ln(a)]^{1/2}\cos(2\pi b) , \tag{6}$$

$$x|_{t+\Delta t}=x+f(x)\Delta t+g_w . \tag{7}$$

After Eq. (7) the algorithm loops back to Eq. (4) and continues as many times as is desired. For sufficiently small step size, Δt, this algorithm is known to work extremely well.

The straightforward way to obtain exponentially correlated colored noise to drive Eq. (1) instead of using g_w is to replace (1) with the pair of equations

$$\dot{x}=f(x)+\epsilon , \tag{8}$$

$$\dot{\epsilon}=-\lambda\epsilon+\lambda g_w , \tag{9}$$

in which g_w is still Gaussian, white noise in accord with (2) and (3). The driven noise ϵ is now exponentially correlated colored noise (Ornstein-Uhlenbeck process) with the properties

$$\langle\epsilon(t)\rangle=0 , \tag{10}$$

$$\{\langle\epsilon(t)\epsilon(s)\rangle\}=D\lambda\exp(-\lambda|t-s|) , \tag{11}$$

in which $\{\cdots\}$ denotes averaging over the distribution of initial ϵ_0 values which is given by

$$P(\epsilon_0)=\frac{1}{(2\pi D\lambda)^{1/2}}\exp\left[-\frac{\epsilon_0^2}{2D\lambda}\right] . \tag{12}$$

This secondary averaging is essential for the stationary correlation given in (11). Clearly, λ^{-1} is the correlation time for the colored noise. Once again, the Box-Mueller algorithm can be used to realize both (12) and the g_w in (9). This yields the Euler version of the integration of (8) and (9) given by

$$m=\text{random number} , \tag{13}$$

$$n = \text{random number} , \tag{14}$$

$$\epsilon = [-2D\lambda \ln(m)]^{1/2} \cos(2\pi n) , \tag{15}$$

$$a = \text{random number} , \tag{16}$$

$$b = \text{random number} , \tag{17}$$

$$p = f(x) + \epsilon , \tag{18}$$

$$x|_{t+\Delta t} = x + p\,\Delta t , \tag{19}$$

$$g_w = [-4D\,\Delta t \ln(a)]^{1/2} \cos(2\pi b) , \tag{20}$$

$$\epsilon|_{t+\Delta t} = \epsilon - \Delta t \lambda \epsilon + \lambda g_w . \tag{21}$$

After Eq. (21) the algorithm loops back to Eq. (16) and continues for as long as one likes. Repeating this process starting from Eq. (13), and subsequently averaging over many such realizations of the algorithm realizes the average over the distribution in (12). We have emphasized the importance of this secondary average and have utilized this algorithm to high accuracy in a very recent study of first passage times.[7]

It should be noted that the integration of (8) without the noise term may be accurately accomplished using a bigger step size than is permitted by the integration of (9). In such a case, the limiting step in the integration of the coupled equations is created by the noise. It is desirable to overcome this limitation.

III. INTEGRAL ALGORITHM FOR COLORED NOISE

In addition to the motivation expressed in the last sentence of Sec. II, three other motivations lead us to the integral algorithm presented below. A laser-noise study[8] was done using an algorithm for colored noise which putatively produced the colored noise directly and efficiently without integrating an equation such as (9). Study of this algorithm showed that the stated algorithm did not generate a Gaussian process, but it did suggest how to do so, and resulted in the algorithm given below. In addition, discussion of noise algorithms resulted in the suggestion that an integrated version of the algorithm would be more efficient. Finally, Horsthemke[9] also suggested that an integrated form of the algorithm should exist and would work both faster and more accurately. The algorithms were implemented on the computer to test their accuracy and speed, and resulted in the numerical results presented here.

The new algorithm is obtained as follows. Integrate (9) and obtain

$$\epsilon(t) = e^{-\lambda t}\epsilon(0) + \lambda \int_0^t ds\, e^{-\lambda(t-s)} g_w(s) , \tag{22}$$

and

$$\epsilon(t+\Delta t) = e^{-\lambda(t+\Delta t)}\epsilon(0) + \lambda \int_0^{t+\Delta t} ds\, e^{-\lambda(t+\Delta t-s)} g_w(s) . \tag{23}$$

Consequently,

$$\begin{aligned}\epsilon(t+\Delta t) &= e^{-\lambda \Delta t}\epsilon(t) + \lambda \int_t^{t+\Delta t} ds\, e^{-\lambda(t+\Delta t-s)} g_w(s) \\ &= e^{-\lambda \Delta t}\epsilon(t) + h(t,\Delta t) ,\end{aligned} \tag{24}$$

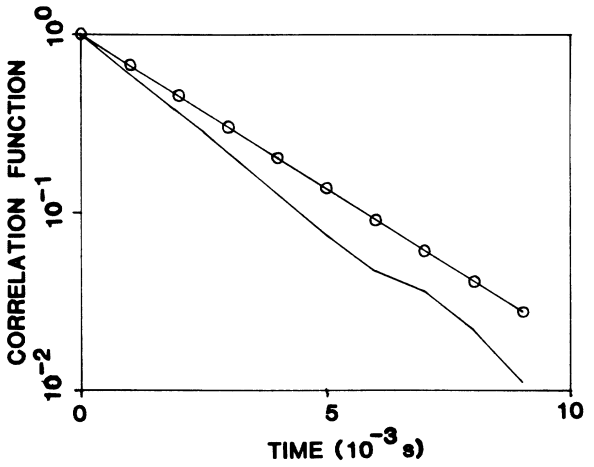

FIG. 1. Correlation function for the differential algorithm. Step size $\Delta t = 10^{-3}$ s and correlation time $\lambda^{-1} = 2.5 \times 10^{-3}$ s. The open circles are on the exact curve and the lower curve is the average over 100 realizations of the algorithm.

in which the last equation defines $h(t,\Delta t)$. Now, $h(t,\Delta t)$ is Gaussian (because g_w is) and has zero mean (because g_w does). Therefore all of its properties are determined by its second moment

$$\langle h^2(t,\Delta t)\rangle = D\lambda(1 - e^{-2\lambda \Delta t}) . \tag{25}$$

Thus, to start the simulation, an initial value for ϵ is needed and is obtained in accord with lines (13)–(15), and set $E = \exp(-\lambda\,\Delta t)$. After that, the exponentially correlated, colored noise is obtained by the lines

$$a = \text{random number} , \tag{26}$$

$$b = \text{random number} , \tag{27}$$

$$h = [-2D\lambda(1 - E^2)\ln(a)]^{1/2} \cos(2\pi b) , \tag{28}$$

$$\epsilon|_{t+\Delta t} = \epsilon E + h . \tag{29}$$

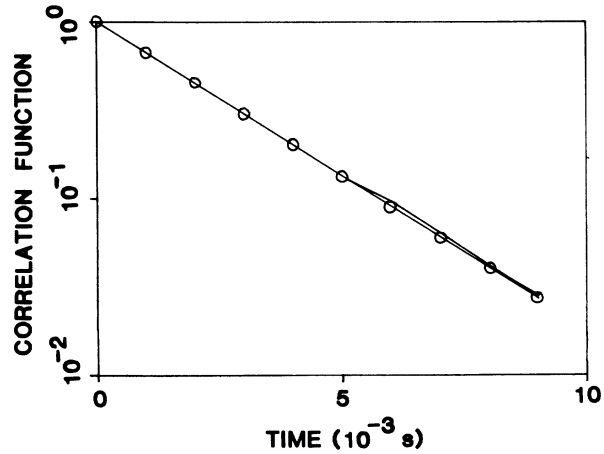

FIG. 2. Correlation function for the integral algorithm. Step size $\Delta t = 10^{-3}$ s and correlation time $\lambda^{-1} = 2.5 \times 10^{-3}$ s. The second curve is nearly indistinguishable from the exact results, and is the average over 100 realizations of the algorithm.

After Eq. (29), the algorithm loops back to Eq. (26) and continues as long as one would like. Equations (26)–(29) replace Eqs. (16), (17), (20), and (21). For very small Δt, Eqs. (28) and (29) clearly reduce to Eqs. (20) and (21). However, Eqs. (28) and (29) continue to give excellent results for larger Δt, which are too large for Eqs. (20) and (21) to accurately reproduce the intended correlation time. These results are exhibited in Figs. 1 and 2. In both figures, the curve on which the open circles appear is the exact, exponential correlation curve computed for the input correlation time λ^{-1}. In Fig. 1 the lower curve shows the result of averaging 100 realizations of the loop given by Eqs. (16), (17), (20), and (21) using step size $\Delta t = 0.001$ and correlation time $\lambda^{-1} = 0.0025$. The 100 realizations utilize 100 distinct sequences of random numbers. In Fig. 2 the second curve (nearly indistinguishable from the exact result) is obtained from Eqs. (26)–(29) for the same step size and correlation time as were used in Fig. 1. Moreover, it is the result of averaging over precisely the same set of 100 distinct sequences of random numbers. Thus the much improved quality of the results is not an artifact of the random numbers but is a result of the superior nature of the algorithm. By reducing the step size to 10^{-4}, both approaches give comparable results because now the Euler integration, i.e., Eqs. (20) and (21), is very accurate. By increasing the step size to 10^{-1}, we find that Eqs. (20) and (21) lead to numerical overflow, while Eqs. (28) and (29) continue to yield qualitatively useful results.

The efficacy of the integral method depends on two features: (a) $f(x)$ is relatively simple to compute as compared to the Box-Mueller computation, and (b) the x integration does not require such small values of Δt as does the ϵ integration. However, feature (b) may easily be relaxed by changing the x integration to a higher-order routine such as Runge-Kutta fourth-order, Runge-Kutta-Fehlberg, or Runge-Kutta-Verner routine. In extreme cases a routine which keeps Δt constant but has an adaptive order (e.g., Bulirsch-Stoer or Gear methods) may be invoked. Unfortunately, library routines cannot be used directly since the x integration and ϵ variation must be intertwined. In summary, our new algorithm is faster, more accurate, and more useful for larger step sizes than the old algorithm.

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