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Symmetry transforms of spherically symmetric Hamiltonians for different values of pertinent space dimensions

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Symmetry transforms in N_2 space dimensions of N_1-dimensional spherically symmetric Schrödinger Hamiltonians have been treated for $N_2 \neq N_1$. Accordingly, the quantum number of the angular momentum and the number of space dimensions become subject to related mappings. One proceeds using suitable transformations of the radial coordinate and of the radial state function so as to exhibit the form invariance of the Laplace operator. The symmetries established in this way concern potentials which can be represented by power-series expansions. Such symmetry transforms are generated by rational values of underlying power exponents. Symmetry properties of $1/N$ energy estimates are also discussed.

Mutual relationships between Hamiltonians with power-law potentials have received much attention.[1] In particular, the equivalence between the three-dimensional ($N_1=3$) Coulomb problem and the four-dimensional ($N_2=4$) harmonic oscillator has also been analyzed with the help of the Kustaanheimo-Stiefel transformation.[2] This latter transformation can be extended towards $N_1=5$ and $N_2=8$.[3] Further, using maps between the corresponding radial equations, the equivalence mentioned above has been generalized for an arbitrary number of space dimensions, such that $N_2=2(N_1-1)$.[4] Here we shall prove that this latter equation is subject itself to further generalizations concerning, this time, arbitrary power potentials, or potentials containing at least one power term. We would like to recall that quasiclassical equivalence relationships referring to power potentials or to superpositions between them have also been discussed before.[5] However, such equivalences only work by removing corresponding Langer terms.[6] We obtain such terms by keeping invariant the quantum number of the angular momentum, now for $N_1=N_2$. Thus the Langer terms characterizing, e.g., Eqs. (4.9) and (4.10) of Ref. 6 are given by $-3/4x^2$ and $3/16x^2$, respectively.[7] This means that it is of interest to look for a refined theoretical formulation enabling us to derive exact symmetry transforms of various Hamiltonians. For this purpose, we have to consider that the number of space dimensions, as well as the quantum number of the angular momentum, are not invariant under symmetry transformations. Then the Langer terms characterizing previous quasiclassical relationships are able to be incorporated self-consistently into the centrifugal barrier of the transformed Hamiltonian. Proceeding in this way enables us to establish the exact symmetry transforms of Hamiltonians with power terms anticipated before. In general, the present symmetry transforms can also be viewed as a manifestation of a genuine "shape-invariant" behavior of power potentials.[8] Infinite series expansions, such as those for exponential, Yukawa, or other potentials, can also be considered, at least as limiting cases. It should be mentioned that the equivalence between the present Hamiltonian-partners works in terms of conversions of energies into couplings and vice versa.[9] This represents an equivalent formulation of the same physics.

Let us start from the N_1-dimensional radial Schrödinger equation

$$\left[-\Delta_x+\frac{l_1}{x^2}(l_1+N_1-2)+V(x)\right]\psi(x)=\mathcal{E}\psi(x)\ , \tag{1}$$

where x denotes the radial coordinate and $\mathcal{E}$ is the eigenvalue, whereas $\psi(x)$ denotes the radial state function. The radial constituent of the N_1-dimensional Laplace operator is

$$\Delta_x=\frac{\partial^2}{\partial x^2}+\frac{N_1-1}{x}\frac{\partial}{\partial x}\ , \tag{2}$$

as usual. Generalizing our earlier treatment,[6] let us choose the new radial coordinate y and the new radial state function $\varphi(y)$ as $x=g(y)$ and $\psi(x)=f(y)\varphi(y)$.[10] Then Eq. (1) reads

$$-\left[\frac{\partial^2\varphi}{\partial y^2}+\frac{N_1-1}{y}\frac{\partial\varphi}{\partial y}F_1(y)\right]+F_2(y)\varphi+V[g(y)](g')^2\varphi + \frac{l_1}{y^2}(l_1+N_1-2)\left[y\frac{g'}{g}\right]^2\varphi=\mathcal{E}(g')^2\rho\ , \tag{3}$$

where the primes denote differentiations with respect to the radial coordinate. Above

$$F_1(y)=y\left[\frac{g'}{g}+\frac{2}{N_1-1}\frac{f'}{f}-\frac{1}{N_1-1}\frac{g''}{g'}\right]\ , \tag{4}$$

and

$$F_2(y)=\frac{g''}{g'}\frac{f'}{f}-\frac{f''}{f}-(N_1-1)\frac{g'f'}{gf}\ . \tag{5}$$

Next we are faced with the question of how to implement the N_2-dimensional Laplace operator Δ_y and the related centrifugal barrier within the φ description. This proceeds via

$$F_1(y)=\frac{N_2-1}{N_1-1}\ , \tag{6}$$

and

$$y\frac{g'}{g}=\text{const}=-\tilde{\rho}\ , \tag{7}$$

where the constant has been quoted by $-\tilde{\rho}$. Clearly, Eqs. (6) and (7) are not the most general solution to the symmetry problem, as they serve to establish just relationships between radial Schrödinger equations. Other cases, such as mappings between radial and one-dimensional Schrödinger equations, deserve a special treatment.[11] We then get the solutions

$$g(y)=c_1y^{-\tilde{\rho}}\ , \tag{8}$$

and

$$f(y)=c_2y^{1/2[N_2-2+\tilde{\rho}(N_1-2)]}\ , \tag{9}$$

in which $c_1=c_2=1$. This is a boundary condition which gives $\tilde{\rho}=-1$ for the identity transformation ($N_1=N_2$). Using Eq. (5) leads to

$$F_2(y)=\frac{\Lambda}{y^2}=-\frac{1}{4y^2}[(N_2-2)^2-\tilde{\rho}^2(N_1-2)^2]\ , \tag{10}$$

which reproduces precisely Eq. (3.7) of Ref. 6 if $N_1=N_2=3$. So far the symmetry transform of the ordinary Hamiltonian $H=\mathbf{p}^2+V(x)=\mathcal{E}\neq 0$ exhibits the intermediary form

$$\mathbf{q}^2+(\rho^2-1)\frac{l_2}{y^2}(l_2+N_2-2)+\frac{\Lambda}{y^2}+\tilde{\rho}^2[V(y^{-\tilde{\rho}})-\mathcal{E}]y^{-(2+2\tilde{\rho})}=0\ , \tag{11}$$

where $\mathbf{p}$ and $\mathbf{q}$ denote the momenta canonically conjugated to $\mathbf{x}$ and $\mathbf{y}$. In addition, we have accounted for a new parameter ρ such that

$$\tilde{\rho}^2l_1(l_1+N_1-2)=\rho^2l_2(l_2+N_2-2)\ , \tag{12}$$

which allows to describe the transformations of the centrifugal barriers in a quite direct manner. Next we have to recognize that it is theoretically simplest to consider that $\rho^2=1$, thereby eliminating the second superfluous term from Eq. (11). One would then have $\tilde{\rho}^2\neq\rho^2=1$, excepting, of course, the identity transformation, for which $\tilde{\rho}^2=\rho^2=1$. This differs from the previous quasiclassical transforms, which have been established via $\tilde{\rho}^2\equiv\rho^2$.[6] So Eq. (12) leads to the condition

$$\tilde{\rho}_0=\left[\frac{l_2(l_2+N_2-2)}{l_1(l_1+N_1-2)}\right]^{1/2}\ , \tag{13}$$

where $\tilde{\rho}_0=|\tilde{\rho}|$ and where we have assumed that $N_1>2$ and $N_2>2$.

As in the quasiclassical case, we have now to determine selected $\tilde{\rho}$ values, say, $\tilde{\rho}\in\{\tilde{\rho}_j\}$, enabling us to rewrite Eq. (11) under the transformed form

$$\tilde{H}=\mathbf{q}^2+\tilde{V}(y)=\tilde{\mathcal{E}}\neq 0\ , \tag{14}$$

where both $\tilde{\rho}$ and $\tilde{\mathcal{E}}$ are independent of y. This means that $\tilde{\mathcal{E}}$ can be interpreted as the transformed eigenvalue, whereas $\tilde{V}(y)$ stays for the transformed potential. Obviously, only the potentials for which such $\tilde{\rho}$ values are definable become subject to Eq. (11). We realize that the above procedure yields the equivalence class $\{\tilde{H}_j\}$ of transformed Hamiltonians, in which $\tilde{\mathcal{E}}=\tilde{\mathcal{E}}_j$ if $\tilde{\rho}=\tilde{\rho}_j$. Specifically, one has $j=1,2,\ldots,M$, if $V(x)$ contains a number of M different monomials like power potentials $V_n(x)=\gamma(n)/x^n$. Note, however, that the $n=2$ potential remains invariant under the present symmetry transformations, so that it should not be counted. It can also be easily verified that the set $\{\tilde{\rho}_1,\tilde{\rho}_2,\ldots,\tilde{\rho}_M\}$ exhibits the structure of a multiplicative cyclic group. At this point we are in a position to eliminate the Langer term Λ/y^2 putting simply $\Lambda=0$. This is an exact procedure, which is different from the quasiclassical description of Langer-type Λ/y^2 corrections. Combining Eq. (13) with $\Lambda=0$ then gives the matching condition

$$\tilde{\rho}_0=\frac{N_2-2}{N_1-2}=\frac{l_2}{l_1}=\tilde{\rho}_j^0\equiv|\tilde{\rho}_j|\ , \tag{15}$$

where $\tilde{\rho}\in\{\tilde{\rho}_j\}$. Admissible values of $\tilde{\rho}_j$, N_1, and N_2 should then be determined in accord with the quantization of l_1 and l_2. Summarizing, we can then say that the symmetry transforms of H come from

$$[V(y^{-\tilde{\rho}})-\mathcal{E}]\tilde{\rho}^2y^{-(2+2\tilde{\rho})}=-\tilde{\mathcal{E}}+\tilde{V}(y)\ , \tag{16}$$

via Eqs. (14) and (15), where $\tilde{\rho}\in\{\tilde{\rho}_j\}$. In practice, we have to look for selected $\tilde{\rho}$ values for which the left-hand side of Eq. (16) can be rewritten as a sum between a nonzero constant and a remaining y-dependent contribution. Now it is clear that the conditions $\mathcal{E}\neq 0$ and $\tilde{\mathcal{E}}\neq 0$ are necessary in order to convert energies into couplings and vice versa.

The simplest case is again the Hamiltonian $H=\mathbf{p}^2+V_n$ with a power-law potential. This yields the well-known symmetry transform

$$\tilde{H}=\mathbf{q}^2+\frac{\tilde{\gamma}(\tilde{n})}{y^{\tilde{n}}}=\tilde{\mathcal{E}}=-\gamma(n)\tilde{\rho}^2\ , \tag{17}$$

in which $n\gamma(n)<0$, $n<2$, and $\tilde{n}=2n/(n-2)$, whereas $\tilde{\rho}=\tilde{\rho}_1=2/(n-2)$ and $\tilde{\gamma}(\tilde{n})=-\mathcal{E}\tilde{\rho}_1^2$. This time $N_1\neq N_2$ and $l_1\neq l_2$, in contradistinction to the earlier descriptions for which $N_1=N_2$ and $l_1=l_2$.[1,6] Next we get

$$\frac{N_2-2}{N_1-2}=\frac{l_2}{l_1}=\frac{2}{2-n}\ , \tag{18}$$

by virtue of Eq. (15), which agrees with Eq. (21) of Ref. 12. We remark that nonrational n values are not subject to the symmetry description, as it follows from Eq. (18). In general, the present equivalence proceeds in terms of the l_1 (l_2) subsequence for which the mapping $l_1\rightarrow l_2$ ($l_2\rightarrow l_1$) preserves integral values needed. Under such circumstances Eq. (18) produces the admissible N_2 values. In particular, Eqs. (17) and (18) show that the N_1-dimensional Coulomb problem ($n=1$) has been converted into the N_2-dimensional eigenvalue problem of the related harmonic oscillator ($\tilde{n}=-2$), such that

$$N_2=2(N_1-1)\ , \tag{19}$$

in accord with Eq. (2.11) of Ref. 4, where $l_2=2l_1$. Then l_2 exhibits the subsequence $l_2=0,2,4,\ldots$ of even values if $l_1=0,1,2,\ldots$. Furthermore, we realize that a potential like $V(x)=V_n(x)+R(x)$ gives, via Eq. (18), at least one symmetry transform. Accordingly,

$$\tilde{V}(y)=\frac{\tilde{\gamma}(\tilde{n})}{y^{\tilde{n}}}+\frac{1}{y^{\tilde{n}}}\tilde{\rho}_1^2 R(y^{-\tilde{\rho}_1})\ , \tag{20}$$

where $\tilde{\mathcal{E}}=-\tilde{\rho}_1^2\gamma(n)$ and $|\tilde{\rho}_1|=2/|n-2|$.

Quasiclassically motivated representations for $\mathcal{E}$ and $\tilde{\mathcal{E}}$ in terms of minimizations such as[5]

$$\mathcal{E}=\min\left[\frac{d_0^2}{x^2}+V(x)\right]=\frac{d_0^2}{x_0^2}+V(x_0)\ , \tag{21}$$

and

$$\tilde{\mathcal{E}}=\min\left[\frac{\tilde{d}_0^2}{y^2}+\tilde{V}(y)\right]=\frac{\tilde{d}_0^2}{y_0^2}+\tilde{V}(y_0) \tag{22}$$

can also be analyzed in connection with the above results. The locations of these minima satisfy the condition $x_0=y_0^{-\tilde{\rho}}$. This leads to the exact matching condition $\tilde{d}_0=\tilde{\rho}_0 d_0$, where Eqs. (16), (21), and (22) have been used so that $x=x_0$ and $y=y_0$. In other words, Eq. (21) can be rewritten equivalently in terms of Eq. (22) and vice versa. Other details concern the $1/N$ description of the phase-space quanta d_0 and $\tilde{d}_0$. Restricting ourselves to the first $1/N$ order[13] gives[14]

$$d_0=l_1+\frac{N_1-2}{2}+(n_r+\tfrac{1}{2})\left[3+x_0\frac{V''(x_0)}{V'(x_0)}\right]^{1/2}\ , \tag{23}$$

and, similarly for $\tilde{d}_0$,

$$\tilde{d}_0=l_2+\frac{N_2-2}{2}+(n_r+\tfrac{1}{2})\left[3+y_0\frac{\tilde{V}''(y_0)}{\tilde{V}'(y_0)}\right]^{1/2}\ . \tag{24}$$

The radial quantum number $n_r=0,1,2,\ldots$ remains invariant under the above symmetry transformations. Now we can verify that the condition $\tilde{d}_0=\tilde{\rho}_0 d_0$ is preserved to first $1/N$ order. Indeed, coming back to the power potentials one gets

$$d_0=l_1+\frac{N_1-2}{2}+(n_r+\tfrac{1}{2})(2-n)^{1/2}\ , \tag{25}$$

and

$$\tilde{d}_0=l_2+\frac{N_2-2}{2}+(n_r+\tfrac{1}{2})(2-\tilde{n})^{1/2}\ , \tag{26}$$

so that $\tilde{d}_0=2d_0/(2-n)$ by virtue of Eq. (18). However, this exact covariance criterion ceases to be fulfilled to higher $1/N$ orders. It is worthy of being mentioned that Eq. (25) produces the exact higher-dimensional results for $n=1$ and $n=-2$,[15] so that Eq. (26) does the same for $\tilde{n}=-2$ and $\tilde{n}=1$. At this point we are also able to realize that, within the quasiclassical approach ($N_1=N_2$ and $l_1=l_2$) discussed before,[6] matching conditions between d_0 and $\tilde{d}_0$ require an extra treatment[16] circumventing inherent difficulties with (nonzero) Langer terms. It should also be noted that the supersymmetric partners of $V(x)$ and $\tilde{V}(y)$ act within N_1+2 and N_2+2 space dimensions, respectively. This property can then be used to improve the convergence of related $1/N$ expansions.[17]

We conclude by remarking that the earlier quasiclassical approach[6] has been refined by defining the exact covariance criterion $\tilde{d}_0=\tilde{\rho}_0 d_0$ and a number of properties characterizing exact symmetry transforms for $N_1\neq N_2$ and $l_1\neq l_2$. Equations (15), (16), and (13) represent the main results of this paper. Equivalently, Eq. (3) can also be interpreted as a Schrödinger equation for the reduced radial state function by choosing $F_1(y)=0$. Then Eqs. (12) and (15), as well as the symmetry condition $\rho^2=1$, will be preserved, whereas

$$F_2(y)=-\frac{1}{4y^2}[1-\tilde{\rho}^2(N_1-2)^2]\ , \tag{27}$$

instead of Eq. (10). We can also apply, at least in principle, the above symmetry transformations to Yukawa potentials, or to other potentials, which are represented by infinite power-series expansions. One would then obtain an infinite sequence of Hamiltonian transforms, which work in terms of interrelated values of couplings. Starting, e.g., from the Yukawa potential[18] and selecting the Coulomb term $-\alpha/x$ ($\tilde{\rho}_1=-2$) of the series expansion, yields the transformed potential

$$\tilde{V}(y)=4y^2\left[-\mathcal{E}+\alpha\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}\mu^{n+1}y^{2n}\right]\ , \tag{28}$$

via Eq. (20), which can be resummed as

$$\tilde{V}(y)=-4\mathcal{E}_1y^2+4\alpha[1-\exp(-\mu y^2)]\ . \tag{29}$$

Above $\mathcal{E}=\mathcal{E}_1-\alpha\mu$, $\mathcal{E}_1$ is the eigenvalue of the Yukawa Hamiltonian, whereas the transformed energy reads $\tilde{\mathcal{E}}=4\alpha$. So, the Yukawa potential turns out to be equivalent to the superposition between the harmonic oscillator and the shifted Gaussian potential, as shown in Eq. (29). Other selections can be treated easily in a similar way.

Finally, we would like to say that the coupling-constant "metamorphosis"[9] exhibited by Eqs. (16), (17), (20), and (29) is in accord with similar opinions expressed before.[19] Moreover, our Eq. (17) corresponds to Eq. (12) of Ref. 20. However, in our case $c_1=c_2=1$, which results in separate transformations for l_1 and N_1, as given by Eq. (15).

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