Compound nucleus in Livšic open-system theory: Factorization of the S matrix

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The compound-nucleus system fits into a mathematical theory of open systems in physics developed by the mathematician M. Livšic [*Translations of Mathematical Monographs* (American Mathematical Society, Providence, Rhode Island, 1973), Vol. 34]. In this article we review some basic concepts of the above theory and apply it to study the structure of the compound-nucleus S matrix. One of the results is a factorization of the S matrix in the form $S(\omega)=\prod_k[I+iA_k/(\tau_k-\omega)]$, where A_k are known matrices and τ_k are the complex resonance energies.

I. INTRODUCTION

Physicists are much accustomed to closed systems whose states $\mathbf{f}$ satisfy the equation of motion

$$-i\frac{d\mathbf{f}}{dt}=A\mathbf{f}\ ,$$

where A is a self-adjoint operator. The first example that comes to mind is of course the Schrödinger equation where A is the Hamiltonian. Another example is a free oscillator $m\ddot{x}+kx=0$. Defining $f_1=\sqrt{m}\,\dot{x}$, $f_2=\sqrt{k}\,x$, $\mathbf{f}=(f_1,f_2)$, and

$$A=\begin{pmatrix}0 & i\omega\\ -i\omega & 0\end{pmatrix}\ ,$$

where $\omega=(k/m)^{1/2}$, we get the above equation of motion for the state vector $\mathbf{f}$.

The scalar product in the space of states $\mathbf{f}$ is simply defined as $(\mathbf{f},\mathbf{g})=f_1^*g_1+f_2^*g_2$ so that $(\mathbf{f},\mathbf{f})$ is twice the energy of the oscillator. The self-adjointness of the operator A ensures the constancy of $(\mathbf{f},\mathbf{f})$ with time. Another example is Maxwell's equations with $\mathbf{f}=(\mathbf{E},\mathbf{H})$ and

$$A=\begin{pmatrix}0 & -i\epsilon^{-1}\mathrm{curl}\\ i\mu^{-1}\mathrm{curl} & 0\end{pmatrix}\ ,$$

where ϵ is the dielectric constant and μ is the magnetic susceptibility. The scalar product is defined by $\int(\epsilon\mathbf{E}_2^*\mathbf{E}_1+\mu\mathbf{H}_2^*\mathbf{H}_1)dv$ so that $(\mathbf{f},\mathbf{f})$ is twice the field energy.

Yet, in many cases it is required to deal with an open system which transfers energy or probability to its environment. (Although the open system together with its environment is a closed system). Loosely speaking, any physical system in which we can speak of input, output, and interior states can be regarded as an open system. Thus a quantum-mechanical scattering problem can be regarded as an open system in the sense that the interior states are linked with the asymptotic channels. If the asymptotic channels contain only output channels, we have a decaying system, the norm of which decreases with time, in contrast with the constant norm encountered in closed systems. The theory of electric circuits in which input and output currents are linked with circuit elements is another example of an open system.

The study of open systems is intimately related to the mathematical theory of non-self-adjoint operators. This theory has been thoroughly investigated by the mathematician M. Livšic. Its relation to open systems is summarized in Ref. 1.

Investigation of the structure of the compound-nucleus S matrix reveals a strong similarity with the characteristic S function defined by Livšic which is ubiquitous in every open system. Thus it is reasonable to follow the properties of the S function of Livšic and see what it implies on the compound-nucleus S matrix. Among other properties, Livšic has shown that the S operator of an open system can be written as a product of S operators belonging to simpler systems.

It is indeed sometimes preferable to have at hand a product form for the compound-nucleus S matrix. For example, if one resonance is to be singled out it is useful to start from a product expression and approximate the contribution from all other resonances as a multiplicative background matrix multiplying the explicit contribution of the singled-out resonance.

Although we introduce the basic concepts through the S matrix of the compound nucleus, it is easy to perceive their generalities. The concept of an S matrix is common to many physical systems, classical and quantum mechanical. The idea behind the factorization expression is nothing but writing the S matrix of a large system as a product of the S matrices of its elementary components.

In Sec. II we combine the compound-nucleus system with the theory of open systems and determine a product expression for the S matrix of the compound nucleus. In Sec. II we also introduce the basic concepts and definitions of open systems detailed in Ref. 1. In Sec. III we go further and discuss the resolution of open systems. This process leads finally to the factorization of the compound nucleus S matrix.

II. COMPOUND NUCLEUS AND OPEN SYSTEMS

In problems involving compound nucleus formation, the S matrix can be written as

$$S=I+i\Gamma^{\dagger}(T-\omega I)^{-1}\Gamma \ . \tag{1}$$

We shall now explain the right-hand side of Eq. (1) in a form suitable for our further discussion. The operator S is defined in the Hilbert space E of channels which is assumed to be finite-dimensional ($\dim E=m$, $m<\infty$). On the other hand, the operator T is defined in other space, namely, the Hilbert space H of compound-nucleus bound states whose dimension N is large but also finite ($\dim H=N$, $N\gg 1$). The operator T can be written as

$$T=h-\frac{i}{2}\Gamma\Gamma^{\dagger} , \tag{2}$$

where h (it operates in H) is the Hamiltonian of the compound nucleus ($h=h^{\dagger}$, and in its matrix representation h can be chosen to be a real symmetric matrix if time reversal invariance is respected). The operator Γ ($\Gamma^{\dagger}$) appearing in both Eqs. (1) and (2) transforms E into H (H into E) and represents, physically, the coupling between the channel space and the interior (that is, compound-nucleus) space. Apart from trivial phase factors, Eq. (1) is an operator form of Eq. (4.2.30) in Ref. 2 with the identifications $\Gamma\leftrightarrow\sqrt{2\pi}V$, $(T-\omega I)\leftrightarrow -d$, and $\omega\leftrightarrow E$ (the total energy).

Of crucial importance for the whole theory is the resonance structure of the S operator. Most naturally, one is led to a Breit-Wigner-type decomposition

$$S_{ab}=\delta_{ab}+i\sum_{k=1}^{N}\frac{\gamma_{ka}^{1/2}\gamma_{kb}^{1/2}}{\omega-\tau_k} , \tag{3}$$

where a,b are channel indices and k runs over compound-nucleus levels, $\gamma_{ka}^{1/2}$ are partial-width amplitudes, and $\tau_k=\zeta_k-\frac{1}{2}i\eta_k$ are the complex eigenvalues of the operator T.

In the present work we suggest a *product* expression for the S matrix

$$S=\prod_{k=1}^{N}\left[I+i\frac{A_k}{\tau_k-\omega}\right] , \tag{4}$$

where A_k are matrices (operators) in the channel space E whose relation to the operators Γ and T in Eqs. (1) and (2) will be determined below. If Γ and T are independent of ω so are the operators A_k.

At first glance, a derivation of an expression like (4) from Eqs. (1) and (2) might look like a simple exercise in operator algebra. However, this is probably not the case here. The compound-nucleus system is described by the non-self-adjoint operator T. Due to the coupling with the channel space (through the operator Γ) the compound nucleus system is an *open system*. The norm of its wave function decays with time. Notice of course that the system of compound nucleus *and* asymptotic channels is a *closed system* and is described by a self-adjoint operator (its total Hamiltonian).

Intuitively, any physical S-matrix theory deals with input states ϕ^{-}, interior states ψ, and output states ϕ^{+}. Input and output states are vectors in a Hilbert space E ($\dim E=m$, $m<\infty$), whereas interior states belong to another Hilbert space H (the so-called interior space, with $\dim H=N$). The dynamics are such that the input should uniquely determine both the internal and output states. The pair of Hilbert spaces E and H for which relations of the type

$$\psi=R\phi^{-}, \quad \phi^{+}=S\phi^{-} \quad (\phi^{-},\phi^{+}\in E,\ \psi\in H) \tag{5}$$

are defined which transform E into H and E into itself is termed as an open system, symbolized as

$$F\begin{bmatrix}\phi^{-}\rightarrow\phi^{+}\\ \phi^{-}\rightarrow\psi\end{bmatrix} \text{ or } \begin{bmatrix}E\rightarrow E\\ E\rightarrow H\end{bmatrix} , \tag{6}$$

henceforth it will be assumed that the transformations $R(E\rightarrow H)$ and $S(E\rightarrow E)$ are linear.

These concepts are not limited to quantum-mechanical systems. In fact, as a first example Livšic considers a uniform, infinite string attached at the point $x=0$ to a transverse spring. Assuming that the density and the tension in the string are both equal to unity, we have the following equations for small oscillations:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0 \quad (-\infty<x<0,\ 0<x<\infty) , \tag{7}$$

while at $x=0$ the condition

$$\left[\frac{\partial u}{\partial x}\right]_{0+}-\left[\frac{\partial u}{\partial x}\right]_{0-}=ku(0,t) \tag{8}$$

is satisfied. For oscillations of frequency ω, the solution $u(x,t)$ reads

$$u(x,t)=\begin{cases}e^{i\omega t}(A^{-}e^{-i\omega x}+B^{-}e^{i\omega x}), & x\le 0\\ e^{i\omega t}(A^{+}e^{-i\omega x}+B^{+}e^{i\omega x}), & x\ge 0 .\end{cases} \tag{9}$$

Now let $v=\partial u/\partial t$ and $w=-\partial u/\partial x$ be the transverse velocity of a point and the transverse force at that point, respectively. Then we have

$$\begin{aligned} &v(0-,t)=v(0+,t)=i\omega u(0,t) ,\\ &w(0-,t)-w(0+,t)=-i(k/\omega)v(0-,t) . \end{aligned} \tag{10}$$

Thus the complex amplitudes $v^{+}=v(0+,t)e^{-i\omega t}$, $w^{+}=w(0+,t)e^{-i\omega t}$, and $u_0=u(0,t)e^{-i\omega t}$ are expressed in terms of $v^{-}=v(0-,t)e^{-i\omega t}$, $w^{-}=w(0-,t)e^{-i\omega t}$ in the form

$$\begin{aligned} &\begin{bmatrix}v^{+}\\ w^{+}\end{bmatrix}=\begin{bmatrix}1 & 0\\ ik/\omega & 1\end{bmatrix}\begin{bmatrix}v^{-}\\ w^{-}\end{bmatrix} ,\\ &u_0=(-i/\omega,0)\begin{bmatrix}v^{-}\\ w^{-}\end{bmatrix} . \end{aligned} \tag{11}$$

Thus the space E in this example is two dimensional while the interior space H is one dimensional. The first of the above equations defines the S operator and the second one defines the R operator.

In order to obtain information regarding R and S one must impose restrictions on the open system F and its mode of connection with its channels. This restriction is termed by Livšic as an "extension hypothesis," namely, that an open system together with its coupling channels forms a closed system. A state of a closed system may be represented by vectors of the form

$$\mathbf{f}=\begin{bmatrix}\boldsymbol{\phi}\\ \boldsymbol{\psi}\end{bmatrix},$$

where $\boldsymbol{\phi}$ represents the state of the coupling channels and $\boldsymbol{\psi}$ the interior state of the open system F. The state $\mathbf{f}$ of a closed system obeys the equation of motion $-i(d\mathbf{f}/dt)=Q\mathbf{f}$, where Q is a self-adjoint operator. For oscillations with frequency ω we set $\mathbf{f}_t=\mathbf{f}e^{i\omega t}$ so that

$$Q\mathbf{f}=\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{\phi}\\ \boldsymbol{\psi}\end{bmatrix}=\omega\begin{bmatrix}\boldsymbol{\phi}\\ \boldsymbol{\psi}\end{bmatrix}. \tag{12}$$

The open system F is said to be stationary if Q_{12}, Q_{21}, and Q_{22} are independent of ω.

We will now relate the operators Q_{ij} appearing in the equations of motion to the operators T,Γ appearing in Eq. (1) and to the operators R and S appearing in the definition of the open system F in Eqs. (5) and (6). Instead of simple channel states appearing in Eq. (9) we may consider more general states whose x dependence is given by

$$\boldsymbol{\phi}(x)=e^{-i\omega x v^{-1}}\boldsymbol{\phi}_0, \tag{13}$$

where $\boldsymbol{\phi}_0$ on the rhs (right-hand side) of Eq. (13) is a constant vector in E and v is a given operator in E possessing an inverse (which physically plays the role of a constant velocity). The dependence of $\boldsymbol{\phi}$ on x will be dropped since in analogy with Eqs. (10) and (11), transition occurs at $x=\mp 0$.

We will now give a more explicit representation of the operators Q_{ij} $(i,j=1,2)$ in Eq. (12). The operator Q_{11} can be shown to act on $\boldsymbol{\phi}$ as (see Ref. 1, pp. 16–19)

$$Q_{11}\boldsymbol{\phi}=\omega\boldsymbol{\phi}+iv(\boldsymbol{\phi}^{+}-\boldsymbol{\phi}^{-})\delta(x)\equiv\omega\boldsymbol{\phi}+\mathbf{c}(\psi)\delta(x), \tag{14}$$

where $\boldsymbol{\phi}^{\pm}=\boldsymbol{\phi}(x=0\pm)$. From Eqs. (12) and (14) it is evident that the second term on the right-hand side of Eq. (14) is equal to $-Q_{12}\boldsymbol{\psi}$. We choose in E an arbitrary normalized basis $\mathbf{a}_\alpha$ $(\alpha=1,\ldots,m)$ and write

$$Q_{12}\boldsymbol{\psi}=\sum_{\alpha=1}^{m}c_\alpha(\psi)\mathbf{a}_\alpha\delta(x), \tag{15}$$

where $c_\alpha(\psi)$ are the components of $\mathbf{c}(\psi)$ in the basis $\mathbf{a}_\alpha$. Since each $c_\alpha(\psi)$ is a linear functional on H, we write it as $c_\alpha(\psi)=(\boldsymbol{\psi},\mathbf{g}_\alpha)$, where $g_\alpha\in H$, $\alpha=1,\ldots,m$. Thus

$$Q_{12}\boldsymbol{\psi}=-iv(\boldsymbol{\phi}^{+}-\boldsymbol{\phi}^{-})\delta(x)=\sum_{\alpha=1}^{m}(\boldsymbol{\psi},\mathbf{g}_\alpha)\boldsymbol{\delta}_\alpha(x), \tag{16}$$

where $\boldsymbol{\delta}_\alpha(x)\equiv\mathbf{a}_\alpha\delta(x)$. From the self-adjointness of Q it can be shown that

$$Q_{21}\boldsymbol{\phi}=\sum_{\alpha=1}^{m}(\boldsymbol{\phi},\boldsymbol{\delta}_\alpha)\mathbf{g}_\alpha \tag{17}$$

and that Q_{22} is self-adjoint in H, namely,

$$Q_{22}=Q_{22}^{\dagger}. \tag{18}$$

Equations (14)–(18) may be somewhat simplified if we take as a basis $\mathbf{a}_\alpha$ $(\alpha=1,\ldots,m)$ in E the orthonormalized system of eigenvectors of the operator $B=(vv^{\dagger})^{1/2}$, so that

$$B\mathbf{a}_\alpha=b_\alpha\mathbf{a}_\alpha\quad(b_\alpha>0). \tag{19}$$

To shorten our notation we further define

$$\begin{aligned}&\mathbf{e}_\alpha=-b_\alpha^{-1}\mathbf{g}_\alpha,\\ &\boldsymbol{\phi}_\alpha^{\pm}=(\boldsymbol{\phi}^{\pm},\mathbf{a}_\alpha).\end{aligned} \tag{20}$$

Then Eqs. (14)–(18) may be rewritten as

$$\begin{aligned}&Q_{11}\boldsymbol{\phi}=\omega\boldsymbol{\phi}-Q_{12}\boldsymbol{\psi},\\ &Q_{12}\boldsymbol{\psi}=-v(\boldsymbol{\phi}^{+}-\boldsymbol{\phi}^{-})\delta(x)=v\sum_{\alpha=1}^{m}(\boldsymbol{\psi},\mathbf{e}_\alpha)\mathbf{a}_\alpha\delta(x),\\ &Q_{21}\boldsymbol{\phi}=-\sum_{\alpha=1}^{m}\boldsymbol{\phi}_\alpha^{-}\mathbf{e}_\alpha+\tfrac{1}{2}i\sum_{\alpha=1}^{m}(\boldsymbol{\psi},\mathbf{e}_\alpha)\mathbf{e}_\alpha,\\ &Q_{22}\boldsymbol{\psi}=\omega\boldsymbol{\psi}+\sum_{\alpha=1}^{m}\boldsymbol{\phi}_\alpha^{-}e_\alpha-\tfrac{1}{2}i\sum_{\alpha=1}^{m}(\boldsymbol{\psi},\mathbf{e}_\alpha)\mathbf{e}_\alpha.\end{aligned} \tag{21}$$

It now remains to find expressions for the operators R and S defined in Eq. (5). In order to do so we collect some operators appearing on the rhs of Eqs. (21), that is,

$$\begin{aligned}&T\boldsymbol{\psi}=\omega\boldsymbol{\psi}+\sum_{\alpha=1}^{m}\boldsymbol{\phi}_\alpha^{-}\mathbf{e}_\alpha,\\ &\Gamma\boldsymbol{\phi}=\sum_{\alpha=1}^{m}(\boldsymbol{\phi},\mathbf{a}_\alpha)\mathbf{e}_\alpha\quad(\boldsymbol{\phi}\in E,\ \mathbf{e}_\alpha\in H),\\ &\Gamma^{\dagger}\boldsymbol{\psi}=\sum_{\alpha=1}^{m}(\boldsymbol{\psi},\mathbf{e}_\alpha)\mathbf{a}_\alpha\quad(\boldsymbol{\psi}\in H,\ \mathbf{a}_\alpha\in E).\end{aligned} \tag{22}$$

The operator T transforms H into itself. The operator Γ transforms E into H whereas $\Gamma^{\dagger}$ transforms H into E. Moreover, they are adjoint in the sense that

$$(\Gamma\boldsymbol{\phi},\boldsymbol{\psi})=(\boldsymbol{\phi},\Gamma^{\dagger}\boldsymbol{\psi}). \tag{23}$$

From Eqs. (21) it follows that

$$\begin{aligned}&(T-\omega I)\boldsymbol{\psi}=\Gamma\boldsymbol{\phi}^{-},\\ &\boldsymbol{\phi}^{+}-\boldsymbol{\phi}^{-}=i\Gamma^{\dagger}\boldsymbol{\psi},\\ &T-T^{\dagger}=-i\Gamma\Gamma^{\dagger}.\end{aligned} \tag{24}$$

A set of two spaces H and E, together with operators $T,\Gamma,\Gamma^{\dagger}$ satisfying Eqs. (24), is called by Livšic an *operator cluster*. In fact, the operator-cluster concept is more convenient in formulating a general theory than by the use of equations of motion with the operator Q.

Finally, if the resolvent $(T-\omega I)^{-1}$ exists then $\boldsymbol{\psi}=(T-\omega I)^{-1}\Gamma\boldsymbol{\phi}^{-}$. Using this relation together with its substitution into the second equation (24) yield expressions for R and S in the operator-cluster formulation,

$$\begin{aligned}&R=(T-\omega I)^{-1}\Gamma,\\ &S=I+i\Gamma^{\dagger}R=I+i\Gamma^{\dagger}(T-\omega I)^{-1}\Gamma.\end{aligned} \tag{25}$$

The identity of the second equation with Eq. (1) is of course not just semantic. We regard the nucleon-nucleus channel space E as a finite-dimensional Hilbert space in which $\boldsymbol{\phi}^{+}$ and $\boldsymbol{\phi}^{-}$ correspond to output and input states, respectively. The compound-nucleus states $\boldsymbol{\psi}$ belong to the interior space H, which is coupled to E through the operators $\Gamma,\Gamma^{\dagger}$. The interaction between the compound-

nucleus states is given by the non-self-adjoint operator T of Eq. (2). Thus Eqs. (24) are valid, and so the physical system of compound nucleus fits nicely into the mathematical concept of operator clusters, so that Eqs. (1) and (25) are conceptually identical. In Sec. III we shall follow the mathematical ideas which lead to the factorization expression (4).

Before doing so, we remark on the appearance of the resolvent operator $(T-\omega I)^{-1}$. It may be tempting to draw some parallelism with system in which the resolvent operator is $(H-\omega I)^{-1}$, where H is the Hamiltonian of the system. However, H is self-adjoint while T is not. Hence, investigation of the resolvent $(T-\omega I)^{-1}$ requires elements of the theory of non-self-adjoint operators, which is much less familiar, and has been investigated by Livšic.

III. RESOLUTION OF OPEN SYSTEMS AND DERIVATION OF THE PRODUCT FORMULA

One of the most important processes in the analysis of open systems is the resolution of an open system into a chain of elementary systems. This procedure will lead us to the factorization expression (4). The mathematical basis for the above procedure requires acquaintance with the concept of coupling and resolution of operator clusters. We recall that operator cluster is a collection of spaces H,E and operators T,Γ, satisfying Eq. (24). Let us start with coupling of two clusters $M_i=(H_i,E,T_i,\Gamma_i)$ $(i=1,2)$ with identical channel space E. We form the direct sum $H=H_1\oplus H_2$ and in the space H we define the linear operator T by means of

$$T=T_1P_1+T_2P_2-i\Gamma_1\Gamma_2^{\dagger}P_2\equiv T_1*T_2\ , \tag{26}$$

where P_i is the projection operator of H onto H_i. We call T the coupling of T_1 and T_2. The third term on the rhs of Eq. (26) is called the coupling coefficient. It maps H_2 into H_1 and annihilates H_1. The reason for its appearance will be clear immediately. (Notice that the subspace H_1 is invariant under T.)

First we define the imaginary part of T as

$$\mathrm{Im}(T)=\frac{1}{i}(T-T^{\dagger})$$

[note that the operator $(T-T^{\dagger})$, which is the skew Hermitian part of T, is anti-Hermitic, while the prefactor $1/i$ implies that $\mathrm{Im}(T)$ is Hermitian].

The special form (26) is chosen so that $\mathrm{Im}(T)=(1/i)(T-T^{\dagger})=-\Gamma\Gamma^{\dagger}$, where $\Gamma=\Gamma_1+\Gamma_2=P_1\Gamma+P_2\Gamma$. To prove it we compute

$$\begin{aligned}\mathrm{Im}(T)&=\mathrm{Im}(T_1)P_1+\mathrm{Im}(T_2)P_2-\Gamma_1\Gamma_2^{\dagger}P_2-\Gamma_2\Gamma_1^{\dagger}P_1\\&=-(\Gamma_1+\Gamma_2)(\Gamma_1^{\dagger}P_1+\Gamma_2^{\dagger}P_2)=-\Gamma\Gamma^{\dagger}\ .\end{aligned}$$

The above construction shows also that if the rank of the imaginary parts of T_1 and T_2 are identical (say, m) then the rank of $\mathrm{Im}(T)$ is also equal to m. We now ask the reverse question in the following way: Let there be given an operator cluster $M=(H,E,T,\Gamma)$ and let H_1 be a nontrivial subspace of H invariant under T, and $H_2=H\ominus H_1$. Is T a coupling of its projections $T_i=P_iTP_i$ in the sense of Eq. (26)? We show by construction that the answer is positive. Taking into account that $P_2TP_1=0$ we multiply by $(P_1+P_2)=I$ on the left and right of T to get

$$\begin{aligned}T&=(P_1+P_2)T(P_1+P_2)\\&=P_1TP_1+P_2TP_2+P_1TP_2\\&=P_1TP_1P_1+P_2TP_2P_2+P_1TP_2-P_1T^{\dagger}P_2\\&=T_1P_1+T_2P_2+P_1(T-T^{\dagger})P_2\end{aligned}$$

(we have subtracted $P_1T^{\dagger}P_2=0$). The desired form (26) follows immediately if we use $\mathrm{Im}(T)=(1/i)(T-T^{\dagger})=-\Gamma\Gamma^{\dagger}$ and identify $\Gamma_i=P_i\Gamma$. Furthermore, since $T_i=P_iTP_i$ we trivially get

$$\mathrm{Im}(T_i)=-P_i(\Gamma\Gamma^{\dagger})P_i=-\Gamma_i\Gamma_i^{\dagger}$$

so that the "components" T_i of T can be embedded into operator clusters. It should be pointed out that the requirements $T_i=P_iTP_i$ and H_1 being an invariant subspace of T [without the requirement $\mathrm{Im}(T)=(1/i)(T-T^{\dagger})=-\Gamma\Gamma^{\dagger}$] do not imply a unique decomposition. For example, the operator $T'=T_1P_1+T_2P_2$ satisfies these last two requirements but does not have the form (26). The operator T' does not fit into our procedure since $\mathrm{Im}(T')=-(\Gamma_1\Gamma_1^{\dagger}+\Gamma_2\Gamma_2^{\dagger})$. Thus if the rank of $\mathrm{Im}(T_i)=m$ the rank of $\mathrm{Im}(T')=2m$. In our decomposition of the S matrix all the components must act in the same space E so that the equality $\mathrm{rank}[\mathrm{Im}(T)]=\mathrm{rank}[\mathrm{Im}(T_1)]=\mathrm{rank}[\mathrm{Im}(T_2)]$ must hold. Thus the additional requirement $\mathrm{Im}(T)=-\Gamma\Gamma^{\dagger}$ is essential in order to couple two operator clusters with identical E spaces into an operator cluster with the same E space.

It may be helpful to illustrate the above discussion with a matrix example. Let us take H to be a two-dimensional complex Hilbert space and let the operator T be represented by the matrix

$$\begin{bmatrix}-ia^2/2 & -iab\\ 0 & -ib^2/2\end{bmatrix}\quad (a,b\ \text{real})\ .$$

Then the requirement $\mathrm{Im}(T)=(1/i)(T-T^{\dagger})=-\Gamma\Gamma^{\dagger}$ is satisfied for

$$\Gamma=\begin{bmatrix}a\\ b\end{bmatrix}\ ,$$

hence

$$\Gamma\Gamma^{\dagger}=\begin{bmatrix}a\\ b\end{bmatrix}\quad (a,b)$$

so that it is an operator of rank one. The components

$$T_1=P_1TP_1=\begin{bmatrix}-ia^2/2 & 0\\ 0 & 0\end{bmatrix}$$

and

$$T_2=P_2TP_2=\begin{bmatrix}0 & 0\\ 0 & -ib^2/2\end{bmatrix}$$

have $\mathrm{Im}(T_1)=-\Gamma_1\Gamma_1^{\dagger}$, $\mathrm{Im}(T_2)=-\Gamma_2\Gamma_2^{\dagger}$, where

$$\Gamma_1=\begin{bmatrix}a\\0\end{bmatrix} \text{ and } \Gamma_2=\begin{bmatrix}0\\b\end{bmatrix} .$$

It is trivial to show that if we couple T_1 and T_2 according to Eq. (26) the result will be the operator T. Thus we have $\mathrm{rank}[\mathrm{Im}(T)]=\mathrm{rank}[\mathrm{Im}(T_1)]=\mathrm{rank}[\mathrm{Im}(T_2)]=1$, whereas for

$$T'=T_1P_1+T_2P_2=\begin{bmatrix}-ia^2/2 & 0\\ 0 & -ib^2/2\end{bmatrix}$$

we have $\mathrm{rank}[\mathrm{Im}(T')]=2$.

We are now in a position to define the coupling of two operator clusters $M_i=(H_i,E,T_i,\Gamma_i)$ $(i=1,2)$. We will say that the operator cluster $M(H,E,T,\Gamma)$ is the coupling of the clusters M_1 and M_2 (and write it as $M=M_1*M_2$) if the following conditions hold: (1)

$$T=T_1*T_2 , \tag{27a}$$

(2)

$$\Gamma\equiv\Gamma_1+\Gamma_2=P_1\Gamma+P_2\Gamma . \tag{27b}$$

A little algebra will show that M is indeed a cluster, satisfying Eqs. (24). The third of these equations is satisfied due to the peculiar definition (26), hence, the role of the coupling coefficient $-i\Gamma_i\Gamma_2^{\dagger}P_2$. It can easily be shown that the coupling operation is associative:

$$(M_1*M_2)*M_3=M_1*(M_2*M_3) . \tag{28}$$

Having defined the coupling of clusters, we now turn to the inverse procedure, namely, a resolution of a cluster. If we are given any cluster $M=(H,E,T,\Gamma)$ and an invariant subspace H_1 of the interior operator T, then, with $H_2=H\ominus H_1$, we define $M_i=(H_i,E,P_iTP_i,P_i\Gamma)$ $(i=1,2)$, and call M_i the projection of the cluster M onto the subspaces H_i $(i=1,2)$. It is trivially shown that the cluster M is the coupling of its two projections, namely,

$$M=M_1*M_2 . \tag{29}$$

Equation (29) then expresses the cluster M as the coupling of its projections and will be termed as the resolution of M into its components M_1 and M_2.

The generalization to n components can be formulated as follows. Let $M=(H,E,T,\Gamma)$ be an operator cluster and $H=H_0\supset H_1\supset H_2\cdots\supset H_{n-1}\supset H_n=0$ be a decreasing sequence of invariant subspaces of the interior operator T. Performing the resolution (29) a sufficient number of times we obtain the resolution of M into its components,

$$M=\prod_{k=1}^{n}(*M_k) , \tag{30}$$

where M_k is the projection of M onto the subspace $\overline{H}_k\equiv H_{k-1}\ominus H_k$ $(k=1,2,\ldots,n)$.

We now recall that an operator cluster is related to the transformations R and S of an open system according to Eqs. (25). The coupling of operator clusters defined above will be very helpful when the concept of coupling of open systems is introduced, which must have a physical meaning beside its mathematical structure. In fact, guided by physical reasoning let us consider the set

$$F_k\begin{bmatrix}E\to E\\ E\to H_k\end{bmatrix} \quad (k=1,2,\ldots,n)$$

of open systems with identical E spaces. We construct a new open system

$$F\begin{bmatrix}E\to E\\ E\to H\end{bmatrix}$$

with interior space $H=\oplus_{k=1}^{n}H_k$ in such a way that the operators R and S of F are given by

$$\begin{aligned} R&=R_1+R_2S_1+R_3S_2S_1+\cdots+R_nS_{n-1}S_{n-2}\cdots S_1 ,\\ S&=S_nS_{n-1}\cdots S_1 , \end{aligned} \tag{31}$$

where R_k and S_k are the corresponding operators for the system F_k. Equations (31) indicate that in coupling the systems F_k, the output of each system is delivered to the input of the succeeding system, the input of F coincides with the input of F_1 and the output of F coincides with the output of F_n. The system F determined by Eqs. (31) is termed by Livšic as the kymological coupling of the systems $F_1,F_2,\ldots,F_n$:

$$F=\prod_{k=1}^{n}(*F_k) . \tag{32}$$

It has the associative property but the components F_k commute only in very special cases.

The question now arises what should be the coupling of the operator clusters M_k corresponding to the open systems F_k in such a way that Eqs. (31) will be guaranteed. The answer is that the cluster $M=\prod_{k=1}^{n}(*M_k)$ defined by Eq. (30) is the cluster belonging to the kymological chain $F=\prod_{k=1}^{n}(*F_k)$. The proof is by induction. For two systems F_1,F_2 with clusters (H_k,E,T_k,Γ_k) $(k=1,2)$ we have according to Eq. (25)

$$\begin{aligned} R=(T-\omega I)^{-1}\Gamma&=(T_1-\omega I)^{-1}\Gamma_1+(T_2-\omega I)^{-1}\Gamma\\ &\quad+i(T_2-I)^{-1}\Gamma_2\Gamma_1^{\dagger}(T_1-\omega I)^{-1}\Gamma_2\\ &=R_1+R_2S_1 , \end{aligned}$$

since by definition

$$R_k=(T_k-\omega I)^{-1}\Gamma_k ,$$

$$S_k=I+i\Gamma_k^{\dagger}R_k .$$

Thus the first of Eqs. (31) is satisfied.

For the S operator we have

$$\begin{aligned} S=I+i\Gamma^{\dagger}R&=I+i\Gamma_1^{\dagger}R_1+i\Gamma_2^{\dagger}R_2S_1\\ &=S_1+i\Gamma_2^{\dagger}R_2S_1=S_2S_1 , \end{aligned}$$

which is the second of Eqs. (31).

We are now close to our initial goal of factorizing the S operator. What is needed is of course the "inverse" of a

kymological coupling, namely, a kymological resolution. Since we know already how to resolve an operator cluster [see discussion leading to Eq. (30)], then a resolution of an open system F into its components F_k is as follows: Let $M=(H,E,T,\Gamma)$ be the operator cluster belonging to F and let $H=H_0\supset H_1\supset H_2\cdots\supset H_n=0$ be a decreasing sequence of invariant subspaces of T. Then F can be resolved as $F=\prod_{k=1}^{n}(*F_k)$, where F_k are open systems with clusters M_k (the projection of M onto $\overline{H}_k=H_{k-1}\ominus H_k$).

If the system F has a finite number of degrees of freedom ($\dim H=N<\infty$) then the sequence of invariant subspaces can always be chosen so that

$$\dim(H_{k-1}\ominus H_k)=1 \quad (k=1,2,\ldots,N) .$$

In this case, the interior spaces of all the links in the kymological chain will be one dimensional, and the links themselves will not admit any further resolution. The transformations R_k and S_k for the link F_k are then

$$\begin{aligned} R_k&=(\tau_k-\omega)^{-1}\Gamma_k , \\ S_k&=I+i(\tau_k-\omega)^{-1}\Gamma_k^{\dagger}\Gamma_k , \end{aligned} \tag{33}$$

where τ_k is a number and Γ_k an operator mapping the space E into the one-dimensional space $\overline{H}_k=H_{k-1}\ominus H_k$. Thus knowing Γ_k and τ_k we can construct the S operator according to Eq. (31), and it will have the product form, Eq. (4). We note that if the open system F is stationary [see the definition after Eq. (12)] then τ_k and Γ_k are independent of ω.

The above discussion leads to the following result: In order to obtain the resolution of the open system into a chain of elementary systems we need to reduce the operator T to its triangular form by means of a *unitary* transformation.

We come now to the final and practical stage of determining the factors S_k of Eq. (33) from the original S matrix of Eq. (1), so that Eq. (4) can be constructed. We assume that all the operators in Eq. (1) are given in their matrix representation,

$$\begin{aligned} &S=S_{ab} \quad (a,b=1,2,\ldots,m) , \\ &T=T_{ij} \quad (i,j=1,2,\ldots,N) , \\ &\Gamma=\Gamma_{ia} \quad (i=1,2,\ldots,N,\ a=1,2,\ldots,m) , \end{aligned} \tag{34}$$

and denote by U the $N\times N$ unitary matrix which triangularizes the matrix T,

$$U^{\dagger}TU=\Delta=\begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1N} \\ 0 & \Delta_{22} & \cdots & \Delta_{2N} \\ 0 & & & \\ \vdots & & & \\ 0 & 0 & \cdots 0 & \Delta_{NN} \end{pmatrix} . \tag{35}$$

The diagonal elements of Δ,

$$\Delta_{ii}=\tau_i , \tag{36}$$

are of course the complex eigenvalues of the non-self-adjoint operator T. It can now be shown that the $1\times m$ matrix Γ_k appearing in Eq. (33) is given by the following expression (summation convention on $j=1,2,\ldots,N$ is assumed henceforth):

$$\Gamma_k=(\Gamma_{j1}U_{jk}^{*},\Gamma_{j2}U_{jk}^{*},\ldots,\Gamma_{jm}U_{jk}^{*}) \quad (k=1,2,\ldots,N) , \tag{37}$$

where Γ_{ja} ($j=1,2,\ldots,N$, $a=1,2,\ldots,m$) are the elements of the original operator in Eqs. (1) and (34). Thus the $m\times m$ matrices A_k ($k=1,2,\ldots,N$) in Eq. (4) are given by $\Gamma_k^{\dagger}\Gamma_k$. For the sake of completeness we now rewrite Eq. (4) as

$$S=\prod_{k=1}^{N}\left[I+i\frac{\Gamma_k^{\dagger}\Gamma_k}{(\tau_k-\omega)}\right] , \tag{38}$$

where Γ_k are given in Eq. (37) in terms of the original Γ [Eq. (1)] and the matrix U [Eq. (35)], and τ_k are the eigenvalues of T given by Eq. (36).

Example. Consider the case $N=2$, $m=1$, namely, two internal states ($\dim H=2$) and one external state ($\dim E=1$). The internal Hamiltonian h [see Eq. (2)] is written explicitly

$$h=\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \quad (h_{11},h_{22}\ \text{real},\ h_{12}=h_{21}^{*}) . \tag{39}$$

The coupling terms Γ_{ja} [$j=1,2$, $a=1$; see Eqs. (1) and (34)] are chosen as $\Gamma_{11}=\gamma_1$, $\Gamma_{21}=\gamma_2$, where γ_j are complex numbers. The non-self-adjoint operator T has then the form

$$T=\begin{pmatrix} h_{11}-\frac{i}{2}|\gamma_1|^2 & h_{12}-\frac{i}{2}\gamma_1\gamma_2^{*} \\ h_{21}-\frac{i}{2}\gamma_1^{*}\gamma_2 & h_{22}-\frac{i}{2}|\gamma_2|^2 \end{pmatrix} . \tag{40}$$

As a special case we take $h_{21}=(i/2)\gamma_1^{*}\gamma_2$ so that T has already the triangular form

$$T=\begin{pmatrix} h_{11}-\frac{i}{2}|\gamma_1|^2 & -i\gamma_1\gamma_2^{*} \\ 0 & h_{22}-\frac{i}{2}|\gamma_2|^2 \end{pmatrix} ; \tag{40a}$$

in this case U is the 2×2 unit matrix. The resonance energies τ_k [Eq. (36)] and the matrices (in this case c numbers) Γ_k [Eq. (37)] are given by

$$\begin{aligned} \tau_k&=h_{kk}-\frac{i}{2}|\gamma_k|^2 , \\ \Gamma_k&=\gamma_k . \end{aligned} \tag{41}$$

The S matrix then factorizes as

$$S=\prod_{k=1}^{n}\left[1+i\frac{|\gamma_k|^2}{\tau_k-\omega}\right] . \tag{42}$$

Simple algebra shows that in fact

$$S=\prod_{k=1}^{2}\frac{\tau_k^{*}-\omega}{\tau_k-\omega} , \tag{43}$$

which guarantees the unitarity of S and indicates that in the one-dimensional case, our factorization coincides with the one obtained from dispersion relations.

Before concluding, we would like to stress two points.

(1) The terms in the factorization expression do not commute. The order is determined uniquely by the decreasing sequence of invariant subspaces of the operator T, which thereby dictates a natural hierarchy.

(2) The transition from input to output in Livšic's theory is instantaneous, so that the factorization scheme does not imply any chronological order or sequential mechanism.

In conclusion, we hope to have drawn attention of physicists to the concept of open systems introduced by Livšic, which is very rich in physics. Our main concern here was the compound-nucleus S matrix, for which we have derived a product expression. So far, we have assumed that the interior space (that is, the compound-nucleus space of states) is finite dimensional ($\dim H = N < \infty$). If $N \to \infty$ the theory becomes even richer, but this goes beyond the scope of the present article.

Finally, it should be mentioned that the concept of open systems is very powerful in numerous branches of physics. In fact, most of the examples in Ref. 1 are taken from the theory of electric networks.

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[1] M. S. Livšic, *Translations of Mathematical Monographs* (American Mathematical Society, Providence, Rhode Island, 1973), Vol. 34.

[2] C. Mahaux and H. Weidenmüller, *Shell-Model Approach to Nuclear Reactions* (North-Holland, Amsterdam, 1969).