Brief Reports

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Time delay in tunneling: Sojourn-time approach versus mean-position approach

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Our previous results [Phys. Rev. A **37**, 2843 (1988)], based on the concept of sojourn time, for the transmission and reflection time delays in one-dimensional scattering by a potential barrier are compared to the corresponding results of E. H. Hauge, J. P. Falck, and T. A. Fjedly [Phys. Rev. B **36**, 4203 (1987)], based on the idea of following the time evolution of the mean position of wave packets. It is shown that the mean-position approach agrees with the more general sojourn-time approach in the limit of well-defined momentum, and that, in general, the two approaches appear to be related in a classical-like manner.

In a recent paper[1] we showed how one can define the transmission and reflection times (time delays) for one-dimensional scattering by a potential barrier using the concept of sojourn time of the particle in a spatial region. Other approaches to the problem exist,[2–4] the oldest and conceptually simplest being based on the idea of extracting the interaction times from the study of the time evolution of quantum-mechanical wave packets treated as a sort of a classical-like "quasiparticle", the position of which is identified with the peak of the wave packet or its mean position (center of gravity). Recently, the mean-position approach received a thorough treatment in the paper of Hauge *et al.*[2] In the present paper we compare this approach with our own in order to clarify and contrast features of each approach, thus providing some insight into the nature of the time delays.

We deal with a one-dimensional quantum-mechanical system with the Hamiltonian $H=-\frac{1}{2}d^2/dx^2+V(x)$, where the potential (potential barrier) $V(x)$ vanishes for sufficiently large $|x|$, say, $V(x)=0$ for $|x|\geq R_0>0$. Using the notation of Ref. 1 we consider time-dependent wave packets $\Psi_t(x)$.

$$\Psi_t(x)=\int_0^\infty dE\,\exp(-itE)\Phi_1(E)\tilde{\varepsilon}_{1E}(x)\ , \tag{1}$$

approaching the barrier from the left and satisfying the Schrödinger equation $i\,\partial\Psi_t/\partial t=H\Psi_t$ (we put $\hbar=1$). $\tilde{\varepsilon}_{1E}$ denotes the stationary scattering state defined by the requirements

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+V(x)\right]\tilde{\varepsilon}_{1E}(x)=E\tilde{\varepsilon}_{1E}(x)\ , \tag{2}$$

$$\tilde{\varepsilon}_{1E}(x)=\varepsilon_{1E}(x)+S_{21}(E)\varepsilon_{2E}(x)\quad\text{for } x\leq -R_0\ , \tag{3}$$

$$\tilde{\varepsilon}_{1E}(x)=S_{11}(E)\varepsilon_{1E}(x)\quad\text{for } x\geq R_0\ . \tag{4}$$

Here

$$\varepsilon_{1E}(x)=(2\pi)^{-1/2}(2E)^{-1/4}\exp(ix\sqrt{2E}\,)$$

and

$$\varepsilon_{2E}(x)=(2\pi)^{-1/2}(2E)^{-1/4}\exp(-ix\sqrt{2E})$$

are the stationary states of the free Hamiltonian $H_0=-\frac{1}{2}d^2/dx^2$. Their normalization assures that

$$N=\int_{-\infty}^\infty dx\ |\Psi_t(x)|^2=\int_0^\infty dE\ |\Phi_1(E)|^2\ , \tag{5}$$

and we assume that $N=1$. $S_{11}(E)$ and $S_{21}(E)$ are elements of the full scattering matrix $S(E)=[S_{ij}(E)]$ of the system. Asymptotically, for $t\to\mp\infty$ the wave packet (5) behaves as free. For $t\to-\infty$ we have

$$\Psi_t(x)\approx\int_0^\infty dE\,\exp(-itE)\Phi_1(E)\varepsilon_{1E}(x)\ , \tag{6}$$

while for $t\to\infty$ we have

$$\Psi_t(x)\approx\int_0^\infty dE\,\exp(-itE)S_{11}(E)\Phi_1(E)\varepsilon_{1E}(x)+\int_0^\infty dE\,\exp(-itE)S_{21}(E)\Phi_1(E)\varepsilon_{2E}(x)\ . \tag{7}$$

In Eq. (7) the first component represents transmitted particles moving to the right and the second represents reflected particles moving to the left. The probabilities of transmission and reflection are

$$P_{\rm tr}=\int_0^\infty dE\ |S_{11}(E)|^2|\Phi_1(E)|^2\ ,\qquad P_r=\int_0^\infty dE\ |S_{21}(E)|^2|\Phi_1(E)|^2\ , \tag{8}$$

respectively.

In Ref. 1 we described a simple *Gedankenexperiment* in which mean transmission and reflection time delays are measured directly. The experiment is a straightforward quantum-mechanical analogue of the classical experiment in which transit times from a source s of particles, located far away on the left of the barrier, to detectors a and b are measured. Detector a picks up the reflected particles, while detector b picks up the transmitted ones ($s,a \ll -R_0 < R_0 \ll b$). The transit times are then compared to the free transit time (no barrier) from the source to detector b; the latter is subtracted from the former. This defines transmission and reflection time delays.

To theoretically describe the quantum-mechanical analogue of the preceding measurements we employed the concept of the mean sojourn time of the particle in a spatial region. The generally accepted expression for the mean sojourn time in a spatial interval (α,β) $(-\infty \le \alpha < \beta \le \infty)$ during a time interval (t_1,t_2) reads

$$\tau((\alpha,\beta),t_1,t_2;\Psi)=\int_{t_1}^{t_2}dt\int_{\alpha}^{\beta}dx\ |\Psi_t(x)|^2 . \tag{9}$$

To calculate the mean transmission time delay we considered the difference

$$\tau_0((b,\infty),t_1,t_2;\Psi)-P_{\mathrm{tr}}^{-1}\tau((b,\infty),t_1,t_2;\Psi) ,$$

where the second term is the sojourn time (9) for the wave packet Ψ_t given by (1), divided by the transmission probability (8), and the first term is the sojourn time for the wave packet given by the right-hand side of Eq. (6). Taking $b > R_0$ and passing to the limits $t_1 \to -\infty$, $t_2 \to \infty$ yields the mean transmission time delay

$$\begin{aligned}\Delta\tau_{\mathrm{tr},b}=&\int_0^\infty dE\frac{|S_{11}(E)\Phi_1(E)|^2}{P_{\mathrm{tr}}}\\ &\times\left[\frac{b}{\sqrt{2E}}+\frac{\partial\varphi_1(E)}{\partial E}\right.\\ &\left.+\mathrm{Re}\left[-iS_{11}^{-1}(E)\frac{\partial S_{11}(E)}{\partial E}\right]\right]\\ &-\int_0^\infty dE\ |\Phi_1(E)|^2\left[\frac{b}{\sqrt{2E}}+\frac{\partial\varphi_1(E)}{\partial E}\right] ,\end{aligned} \tag{10}$$

where $\varphi_1(E)$ is the phase of $\Phi_1(E)$, i.e.,

$$\Phi_1(E)=|\Phi_1(E)|\ \exp[i\varphi_1(E)] .$$

The corresponding expression for the reflection-time-delay results, essentially, from comparison of the sojourn time of the reflected particle in the region $(-\infty,a)$ with the sojourn time of the free particle in the region (b,∞) (use of the same reference time for both transmission and reflection is motivated in Ref. 1). The idea of the sojourn time (and the time delay) for the reflected particle is considerably more subtle than the corresponding idea of the sojourn time for the transmitted one [in the region (b,∞) the latter is simply $P_{\mathrm{tr}}^{-1}\tau((b,\infty),t_1,t_2;\Psi)$]. This is because in the region $(-\infty,a)$ both the incoming and the reflected parts of the wave packet are present. We showed how to overcome this difficulty using the concept of the total time delay as an auxiliary device. The result is (for $a\to-\infty$, $b>R_0$)

$$\begin{aligned}\Delta\tau_{r,a,b}=&\int_0^\infty dE\frac{|S_{21}(E)\Phi_1(E)|^2}{P_r}\\ &\times\left[\frac{-a}{\sqrt{2E}}+\frac{\partial\varphi_1(E)}{\partial E}\right.\\ &\left.+\mathrm{Re}\left[-iS_{21}^{-1}(E)\frac{\partial S_{21}(E)}{\partial E}\right]\right]\\ &-\int_0^\infty dE\ |\Phi_1(E)|^2\left[\frac{b}{\sqrt{2E}}+\frac{\partial\varphi_1(E)}{\partial E}\right] .\end{aligned} \tag{11}$$

Our time delays depend on the parameters a,b which we interpreted as positions of detectors in our *Gedankenexperiment.*

In the mean-position approach of Hauge *et al.*[2] no consideration is given to any *Gedankenexperiments.* The authors study asymptotic time evolution of the mean position of the wave packet before colliding with the barrier $(t\to-\infty)$, and of the mean positions of the transmitted and reflected wave packets after the collision $(t\to\infty)$. It can be shown[5] that for the wave packet (1)

$$\bar{x}_t=\int_{-\infty}^{\infty}dx\ |\Psi_t(x)|^2x\approx t\int_0^\infty dE\sqrt{2E}\ |\Phi_1(E)|^2+\int_0^\infty dE\sqrt{2E}\ |\Phi_1(E)|^2\left[-\frac{\partial\varphi_1(E)}{\partial E}\right] ,\quad t\to-\infty \tag{12}$$

$$\begin{aligned}P_{\mathrm{tr}}\bar{x}_{\mathrm{tr},t}=&\int_\beta^\infty dx\ |\Psi_t(x)|^2x\\ \approx& t\int_0^\infty dE\sqrt{2E}\ |S_{11}(E)\Phi_1(E)|^2\\ &+\int_0^\infty dE\sqrt{2E}\ |S_{11}(E)\Phi_1(E)|^2\left[-\frac{\partial\varphi_1(E)}{\partial E}+\mathrm{Re}\left[iS_{11}^{-1}(E)\frac{\partial S_{11}(E)}{\partial E}\right]\right] ,\quad t\to\infty\end{aligned} \tag{13}$$

$$\begin{aligned}P_r\bar{x}_{\mathrm{tr},t}=&\int_{-\infty}^{\alpha}dx\ |\Psi_t(x)|^2x\\ \approx& -t\int_0^\infty dE\sqrt{2E}\ |S_{21}(E)\Phi_1(E)|^2\\ &-\int_0^\infty dE\sqrt{2E}\ |S_{21}(E)\Phi_1(E)|^2\left[-\frac{\partial\varphi_1(E)}{\partial E}+\mathrm{Re}\left[iS_{21}^{-1}(E)\frac{\partial S_{21}(E)}{\partial E}\right]\right] ,\quad t\to\infty\end{aligned} \tag{14}$$

(here β,α are arbitrary finite parameters; they do not affect the asymptotics for $t\to\infty$). Comparing the time of arrival $t_{\mathrm{tr},b}$ of the mean position $\bar{x}_{\mathrm{tr},t}$ at a point b ($b \gg R_0$) with the time of arrival t_b of the mean position $\bar{x}_t$ at the same point, one finds

$$\Delta t_{\mathrm{tr},b}=t_{\mathrm{tr},b}-t_b=\int_0^\infty dE\frac{|S_{11}(E)\Phi_1(E)|^2}{P_{\mathrm{tr}}}\left[\frac{b}{\sqrt{2E}}+\frac{\partial\varphi_1(E)}{\partial E}+\mathrm{Re}\left[-iS_{11}^{-1}(E)\frac{\partial S_{11}(E)}{\partial E}\right]\right]\frac{\sqrt{2E}}{\bar{v}_{\mathrm{tr}}}$$
$$-\int_0^\infty dE\;|\Phi_1(E)|^2\left[\frac{b}{\sqrt{2E}}+\frac{\partial\varphi_1(E)}{\partial E}\right]\frac{\sqrt{2E}}{\bar{v}}\;. \tag{15}$$

Here

$$\bar{v}_{\mathrm{tr}}=P_{\mathrm{tr}}^{-1}\int_0^\infty dE\;|S_{11}(E)\Phi_1(E)|^2\sqrt{2E}\;,$$
$$\bar{v}=\int_0^\infty dE\;|\Phi_1(E)|^2\sqrt{2E}$$

are the mean velocities of the transmitted and the incoming wave packets, respectively.

Similarly, comparing the time of arrival $t_{r,a}$ of the mean position $\bar{x}_{r,t}$ at a point a ($a \ll -R_0$) with the time t_b, one finds

$$\Delta t_{r,a,b}=t_{r,a}-t_b=\int_0^\infty dE\frac{|S_{21}(E)\Phi_1(E)|^2}{P_r}\left[\frac{-a}{\sqrt{2E}}+\frac{\partial\varphi_1(E)}{\partial E}+\mathrm{Re}\left[-iS_{21}^{-1}(E)\frac{\partial S_{21}(E)}{\partial E}\right]\right]\frac{\sqrt{2E}}{\bar{v}_r}$$
$$-\int_0^\infty dE\;|\Phi_1(E)|^2\left[\frac{b}{\sqrt{2E}}+\frac{\partial\varphi_1(E)}{\partial E}\right]\frac{\sqrt{2E}}{\bar{v}}\;, \tag{16}$$

with

$$\bar{v}_r=P_r^{-1}\int_0^\infty dE\;|S_{21}(E)\Phi_1(E)|^2\sqrt{2E}$$

being the mean velocity of the reflected wave packet.[6]

It is clear that, in general, $\Delta t_{\mathrm{tr},b}\neq\Delta\tau_{\mathrm{tr},b}$ and $\Delta t_{r,a,b}\neq\Delta\tau_{r,a,b}$. However, in the limit of well-defined momentum, i.e., when the energy distribution $|\Phi_1(E)|^2$ vanishes outside a narrow interval $(E_0, E_0+\delta)$ [narrow means that $S_{ij}(E)$ are slowly varying functions of E on $(E_0, E_0+\delta)$], one has approximately

$$\Delta t_{\mathrm{tr},b}\approx\Delta\tau_{\mathrm{tr},b}(\Psi)\approx\mathrm{Re}\left[-iS_{11}^{-1}(E_0)\frac{\partial S_{11}(E_0)}{\partial E}\right]\;, \tag{17}$$

$$\Delta t_{r,a,b}\approx\Delta\tau_{r,a,b}(\Psi)\approx-(a+b)(2E_0)^{-1/2}+\mathrm{Re}\left[-iS_{21}^{-1}(E_0)\frac{\partial S_{21}(E_0)}{\partial E}\right]\;. \tag{18}$$

In Ref. 2 the authors claim that "the scattering off barriers of wave packets with a wide (energy) distribution cannot, in general, be characterized by simple concepts like delay times." While we agree that the time delays $\Delta t_{\mathrm{tr},b}, \Delta t_{r,a,b}$ defined by the mean-value approach are certainly unreliable for wave packets with a wide energy distribution, we assert that the time delays $\Delta\tau_{\mathrm{tr},b}$ and $\Delta\tau_{r,a,b}$ remain meaningful for arbitrary wave packets.[7]

A look at Eqs. (10), (11), (15), and (16) reveals that the integrands of the corresponding integrals differ by the multiplicative factors $\sqrt{2E}/\bar{v}$, $\sqrt{2E}/\bar{v}_{\mathrm{tr}}$, or $\sqrt{2E}/\bar{v}_r$. This seems to be natural in view of the following classical analogy.

Consider a free classical particle of unit mass. Let E be its energy and q its initial position (at time $t=0$). Then the trajectory reads $q_t=\pm t\sqrt{2E}+q$ and the arrival time of the particle at a point c is $T_c=\pm(c-q)/\sqrt{2E}$. Suppose that we have an ensemble of such particles moving in one direction and described (at $t=0$) by the probability distribution $\rho(q,E)$ [with $\int_0^\infty dE\int_{-\infty}^{\infty}dq\,\rho(q,E)=1$]. Let $g(E)=\int_{-\infty}^{\infty}dq\,\rho(q,E)$ be the energy distribution and define

$$\xi(E)=(2E)^{-1/2}g^{-1}(E)\int_{-\infty}^{\infty}dq\,\rho(q,E)q\;.$$

Then the time evolution of the mean value of the position can be written as

$$\bar{q}_t=\pm t\int_0^\infty dE\,g(E)\sqrt{2E}+\int_0^\infty dE\int_{-\infty}^{\infty}dq\,\rho(q,E)q$$
$$=\pm t\bar{V}+\int_0^\infty dE\,g(E)\sqrt{2E}\,\xi(E)\;. \tag{19}$$

Note formal resemblance of this equation to Eqs. (12)–(14). The time of arrival of the mean value $\bar{q}_t$ at c is

$$\vartheta_c=\int_0^\infty dE\,g(E)\left[\frac{c}{\pm\sqrt{2E}}\mp\xi(E)\right]\frac{\sqrt{2E}}{\bar{V}}\;, \tag{20}$$

while for the mean arrival time at c we have

$$\bar{T}_c=\int_0^\infty dE\,g(E)\left[\frac{c}{\pm\sqrt{2E}}\mp\xi(E)\right]\;. \tag{21}$$

Formally one can represent the incoming, the transmitted, and the reflected wave packets by ensembles of classical free particles with appropriately chosen energy distributions $g(E)$ and the functions $\xi(E)$. The quantum-mechanical time delays (10), (11), (15), and (16) will be equal to the corresponding time delays for the classical ensembles. The appearance of the factors $\sqrt{2E}/\bar{v}$, $\sqrt{2E}/\bar{v}_{\rm tr}$, and $\sqrt{2E}/\bar{v}_r$ in Eqs. (15) and (16) can then be traced back to Eq. (20) and understood on classical grounds.

In our opinion the mean-position approach should only be viewed as an indirect, theoretical device aimed at justifying the time delay formulas (17) and (18) for the case of wave packets with well-defined momentum. In view of the uncertainty principle the mean value of position poorly describes such wave packets and therefore derivation of the time delays $\Delta\tau_{{\rm tr},b}$ and $\Delta\tau_{r,a,b}$ based on the idea of following the time evolution of the mean value (or the peak of the wave packet in another known approach) seems generally unreliable. It is therefore interesting and nontrivial that the mean-position approach and the more general approach of Ref. 1 not only agree in the limit of well-defined momentum, but are related in a classical-like manner.

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[1]W. Jaworski and D. M. Wardlaw, Phys. Rev. A **37**, 2843 (1988).

[2]E. H. Hauge, J. P. Falck, and T. A. Fjedly, Phys. Rev. B **36**, 4203 (1987).

[3]A recent assessment of various theoretical approaches to the tunneling-time problem is given by S. Collins, D. Lowe, and J. R. Baker, J. Phys. C **20**, 6213 (1987).

[4]See also references cited in Refs. 1 and 2.

[5]Equations (12)–(14) are justified in Ref. 2 using a different notation. A rigorous derivation can be based upon the properties of one-dimensional free motion, applying a technique similar to that of Appendixes A and B of Ref. 1.

[6]The right-hand sides of Eqs. (15) and (16) result from the use of the asymptotics (12)–(14). The conditions $b \gg R_0$, $a \ll -R_0$ are essential. They are imposed in order to ensure that the exact arrival times of the mean positions of the transmitted and reflected wave packets at b and a, respectively, are very large. Only then can these arrival times be assessed by the use of the asymptotics (13) and (14).

[7]Let us emphasize here that time delays $\Delta\tau_{{\rm tr},b}$ and $\Delta\tau_{r,a,b}$ are, according to our interpretation, statistical mean values and exactly as such do they remain meaningful for arbitrary wave packets. For wave packets with a wide energy distribution one can expect that the transmitted and reflected packets are highly distorted with respect to the initial packet. It may happen that a reasonably localized initial packet with a single peak gives rise to transmitted or reflected packets with more than one peak. In such cases the mean time delay poorly describes the actual distribution of time delays and the theory calls for an extension providing such a distribution or at least some adequate description of it.