Series analysis of multivalued functions

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A recurrent problem in mathematical physics, for example, in the theory of critical phenomena, is the need to study the structure of physically interesting functions at a branch point of a complex structure usually called a "confluent singularity." In such a neighborhood the function is necessarily multivalued. In addition, the value of such a function is sometimes required on a branch cut or even off the first Riemann sheet. Our approach to this problem is inspired by the Riemann (global) monodromy theorem and consists of using series expansions to form integral approximants (special case of Hermite-Padé approximants) to represent multivalued functions on multiple Riemann sheets. We prove an analogous local-monodromy-theorem, functional-representation results. We further identify the important "separation property" and use it to prove a convergence theorem for horizontal sequences of integral approximants. We make an extensive numerical investigation, using horizontal, diagonal, and constrained diagonal sequences, and find that these methods give excellent results on a wide variety of test functions of rather complex structure.

I. INTRODUCTION AND SUMMARY

The problem of obtaining information about the analytic structure of a function from a finite number of coefficients of a power-series expansion is one which occurs in many areas of physics, notably in the theory of critical phenomena. Here one is normally interested in estimating the location and nature of one or more singularities of the function.

Much effort has been denoted to this problem over the past 30 years. Provided that the behavior of the function is sufficiently simple, for example, a single branch point or a small number of isolated poles, the standard ratio and Padé approximant methods yield essentially exact information. Unfortunately the function of interest appears often to be more complex than this. In particular, it appears that confluent singularities are often present, and this feature seems to have led to erroneous estimates of critical exponents in the past. The possibility of essential singularities of the Kosterlitz-Thouless type also provides a much sterner challenge for series-analysis methods. These problems have led to the development of more sophisticated methods of analysis of power series, e.g., integral approximant methods, special transformations, etc. It is not our intention here to review these methods in any detail, rather we refer the reader to the forthcoming review article by Guttmann.[1]

In this paper we describe a new method of analysis of power series which, in principle, ought to be well suited to functions with confluent singularities. The method, which is based on the construction of inhomogeneous linear differential equations for the function, consistent with the known series coefficients, is a generalization of the familiar Padé-approximant methods. This is, of course, the basic idea behind the "integral-approximant" methods[2] which have been used for many years. Our contribution is to combine this approach with some ideas from the classical analysis of functions of a complex variable,[3] in particular, the ideas of monodromy and the theorem of Riemann. We are able, in this way, to provide a systematic method of testing for confluent singularities and of estimating their exponents. Furthermore, some theorems are now available[4] regarding the convergence of the sequences of approximants ("Hermite-Padé" approximants) to the function of interest, which are obtained in this way.

To make these comments explicit, suppose we are given the first $\mathcal{N}$ coefficients of a power series

$$f(z)=\sum_{j=0}^{\infty} f_j z^j \, . \tag{1.1}$$

We then construct polynomials $P(z), Q(z), \ldots, S(z)$ of degrees $p, q, \ldots, s, t$, such that

$$P(z)\frac{d^m f}{dz^m}+Q(z)\frac{d^{m-1}f}{dz^{m-1}}+\cdots+S(z)f+T(z)=O(z^{\mathcal{N}+1}) \, , \tag{1.2}$$

where $m+p+q+\cdots+s+t=\mathcal{N}+1$ and we choose the normalization $P(0)=1$. Such a set of polynomials can always be found, by solution of a set of linear equations. We then define an approximant, denoted

$$y(z)=[t/s;\cdots;q;p] \, , \tag{1.3}$$

as the solution, or integral function $y(z)$, of the differential equation,

$$P(z)\frac{d^m y}{dz^m}+Q(z)\frac{d^{m-1}y}{dz^{m-1}}+\cdots+S(z)y+T(z)=0 \, , \tag{1.4}$$

subject to the initial conditions $y(0)=f(0);\ldots;$ $y^{(m-1)}(0)=f^{(m-1)}(0)$. The singularity structure of the functions $y(z)$ will then provide estimates of the singularity structure of the unknown function $f(z)$.

The outline of the paper is as follows. In Sec. II we give a brief review of the classical theory of multivalued functions of a complex variable, introduce the idea of the monodromic dimension as a useful way of classifying such functions, and relate these ideas to the problem of series analysis. In Sec. III we define "horizontal" and "diagonal" sequences of approximants and discuss some of the convergence properties of these sequences. Section IV is devoted to an analysis of a large set of test functions using our method. It is shown that a number of quite complicated test functions can be successfully analyzed in this way, and that the accuracy is generally superior to that achieved by other methods on a series of equal length.

II. MONODROMY GROUP AND CONFLUENT SINGULARITIES

We begin this section by briefly reviewing some aspects of the classical theory[3] of functions of a complex variable. Suppose we have a Taylor series

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n , \tag{2.1}$$

which is convergent in some neighborhood of $z=z_0$ and hence defines a unique, single-valued regular function of the complex variable z in this neighborhood. By standard methods of analytic continuation one can construct from this "functional element" a finite, or at most denumerably infinite number p of "coverings" $y_1(z),y_2(z),\ldots,y_p(z)$ of the complex plane, which are the Riemann sheets of the function. Not all of those coverings are necessarily Linearly independent. We call the number of independent coverings the "monodromic dimension" of the function. Some examples may make this clearer:

Example (a),

$$f(z)=z^{1/p},\quad y_n(z)=e^{2\pi i(n-1)/p}f(z) ;$$

example (b),

$$f(z)=\ln z,\quad y_n(z)=f(z)+2\pi i(n-1) .$$

Both functions are singular at $z=0$. Analytic continuation around a path which encircles the origin n times yields the function $y_n(z)$. The function of example (a) has exactly p sheets, but only one linearly independent covering, and hence its monodromic dimension is $m=1$. Example (b) has an infinite number of sheets but has $m=2$ since there are only two linearly independent coverings.

We now consider a monogenic analytic function of monodromic dimension m and with exactly n singular points in the extended complex plane. At some regular point z_0 we can use it to define, by analytic continuation, the m linearly independent coverings of the complex plane in the neighborhood of z_0. We will call them $y_1,\ldots,y_m$. Each one has a Taylor series about z_0 and so defines a complete monogenic analytic function as did the function $f(z)$. If we encircle the ith singularity a_i of the n singular points, then $y_j(z)$ changes into

$$\mathcal{Y}_j(z)\rightarrow\tilde{y}_j(z)=\sum_{k=1}^{m} M_{jk}^{(i)}y_k(z) , \tag{2.2}$$

since it represents an analytic continuation of the original $f(z)$ and any such is a linear combination of our basis set $y_1,\ldots,y_m$. This continuation defines a number of $m\times m$ matrices $M^{(i)}$, known as the "monodromy matrices," one for each singularity of the function. One can prove, by an application of Cauchy's theorem, that

$$M^{(1)}M^{(2)}\cdots M^{(n)}=I \tag{2.3}$$

It is a consequence of (2.3) that each of these matrices $M^{(i)}$ has an inverse, and hence they generate a group called the monodromy group of the **y** system.

A simple example may be useful at this point. Let us choose

$$f(z)=(1-z)^{-1.5}+(1+z)^{-1.25} ,$$

which has three singular points, at $z=\pm1,\infty$. If we encircle the singular point $z=1$ we generate two linearly independent coverings of the complex plane,

$$y_1(z)=(1-z)^{-1.5}+(1+z)^{-1.25} ,$$
$$y_2(z)=-(1-z)^{-1.5}+(1+z)^{-1.25} .$$

The corresponding monodromy matrices are found to be

$$M^{(1)}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$
$$M^{(-1)}=\begin{bmatrix}\frac{1-i}{2} & \frac{-1-i}{2}\\ \frac{-1-i}{2} & \frac{1-i}{2}\end{bmatrix},$$
$$M^{(\infty)}=\begin{bmatrix}\frac{-1+i}{2} & \frac{1+i}{2}\\ \frac{1+i}{2} & \frac{-1+i}{2}\end{bmatrix},$$

and it is easy to verify that $M^{(1)}M^{(-1)}M^{(\infty)}=I$. The eigenvalues are ±1 for $M^{(1)}$, 1 and $-i$ for $M^{(-1)}$, and -1 and i for $M^{(\infty)}$. These reflect the phase factors obtained for each of the two parts of $f(z)$ on encircling each of the three singular points.

Returning now to the general theory, we will say that the **y** system of a monodromy group $\mathcal{M}$ is of finite order if there exist A and r both finite such that

$$\left|\frac{y_k'}{y_k}\right|\leq\frac{A}{|z-a_i|^r} \tag{2.4}$$

holds as z approaches a_i for all i and k. Let us next define a function class (given A and r), denoted by

$$Q\begin{bmatrix}a_1,\ldots,a_n\\ M^{(1)},\ldots,M^{(n)};z\end{bmatrix} , \tag{2.5}$$

to consist of all the $\mathbf{y}$ systems of the above specified finite order with the n singular points at a_i with the corresponding monodromy matrices $M^{(i)}$. We can now state Riemann's[5] monodromy theorem.

Theorem (Riemann). For any $m+1$ systems of functions $(\mathbf{y}_j; j=1,\ldots,m+1)$ belonging to the same class Q there exists a linear homogeneous relation of the form

$$\sum_{j=1}^{m+1} A_j(z)\mathbf{y}_j(z)=0\ , \tag{2.6}$$

with the $A_j(z)$ being polynomials in z.

Now, since if $\mathbf{y}$ belongs to class Q, then so do the derivatives $\mathbf{y}', \mathbf{y}'', \ldots, \mathbf{y}^{(m)}$, as can be seen by differentiating Eq. (2.2), we have an immediate corollary.

Corollary. If $y(z)$ is a monogenic analytic function which generates exactly m linearly independent coverings of the extended complex plane and these coverings form a $\mathbf{y}$ system belonging to class Q as defined above, then there exist polynomials $A_j(z)$, $j=0,\ldots,m$, such that

$$\sum_{j=0}^{m} A_j(z)y^{(j)}(z)=0\ . \tag{2.7}$$

We conclude therefore that, for functions generating $\mathbf{y}$ systems in class Q, the integral approximants (1.3), for high enough order, will be exact. Baker[4] has suggested that it is reasonable to consider the idea of monodromic dimension in connection with the use of integral approximants. With this aim in mind and since often we lack global information about a function of interest, we define the idea of *local monodromic dimension* and state a theorem, analogous to Riemann's monodromy theorem.

Definition. Given a convergent Taylor series $f(z)$ about $z=0$ and a disk $\mathcal{D}=\{z \mid |z| \le R\}$, we say that $f(z)$ has local monodromic dimension m if analytic continuation along all paths in $\mathcal{D}$ generates exactly m linearly independent coverings of $\mathcal{D}$.

An illustrative example might be, on the disk $|z| \le 1$,

$$f(z)=(1+2z)^{1/2}+(1+z/3)^{1/3}+e^z\ .$$

Inside the disk only the first term displays a branch point at $z=\frac{1}{2}$ and there are exactly two linearly independent coverings so the local monodromic dimension is 2. Globally there are three singular points and six Riemann sheets; however, only three linearly independent coverings so the monodromic dimension is 3.

We next prove a theorem, using only a minor variant of Riemann's original proof of his monodromy theorem. The purpose of our theorem is to establish a representation analogous to (2.6), valid in $\mathcal{D}$ when we only have information about the function in $\mathcal{D}$ and not necessarily in the extended complex plane.

Theorem (Disk Monodromy). Let $f(z)$ be a convergent Taylor series about $z=0$ and of local monodromic dimension m in a disk $\mathcal{D}\{z \mid |z| \le R\}$. Furthermore, let there be $n<\infty$ singular points a_k of finite order in $\mathcal{D}$ and $|a_k|<R$, $k=1,\ldots,n$. Then

$$\sum_{j=0}^{m} \rho_j(z)f^{(j)}(z)=0\ , \tag{2.8}$$

where $\rho_m(z)$ is a polynomial of finite degree and $\rho_j(z)$ are analytic in $\mathcal{D}$.

Proof. First we observe from (2.2) that at a specific singular point, a_k for example, we may introduce a change of basis

$$u_l(z)=\sum_{j=1}^{m} U_{k;lj}y_j(z)\ , \tag{2.9}$$

where U_k is a constant matrix and y_j are the m linearly independent coverings generated by $f(z)$ in $\mathcal{D}$, such that, provided the eigenvalues of $M^{(k)}$ are not degenerate,

$$\Lambda_k=U_kM^{(k)}U_k^{-1} \tag{2.10}$$

is a diagonal matrix. This result can be shown from the existence of a basis which gives $M^{(k)}$ the structure of a companion matrix; see Eq. (2.26) below. Then

$$\Lambda_k u_l=\lambda_{k;l}u_l\ , \tag{2.11}$$

which, again provided the λ_l are nondegenerate, implies

$$u_l=(z-a_k)^{\nu_{k,l}}h_{k,l}(z)\ , \tag{2.12}$$

where $h_{k,l}(z)$ is uniform (single valued) in the neighborhood of a_k and $\nu_{k,l}-(2\pi i)^{-1}\ln\lambda_{k;l}$ is a whole number. If $r=1$ [Eq. (2.4)] holds exactly, then $h_{k,l}(a_k)\neq 0,\infty$ and $h_{k,l}(z)$ is analytic at $z=a_k$. We will not discuss either the special cases $r\neq 1$ or degenerate λ's as they add to the complexity of the proof and obscure the essential features. These extensions follow without modification from the classical results.

We illustrate the case for $r=\frac{3}{2}$ with the example $f(z)=\exp(z^{-1/2})$. This function can be written as

$$f(z)=[\cosh(z^{-1/2})]+[z^{1/2}\sinh(z^{-1/2})]/z^{1/2}\ ,$$

where each [ ] is a uniform function. Clearly $m=2$ for this case. It takes only a little work to verify that

$$z^3f''+1.5z^2f'-0.25f=0$$

and so, to anticipate, (2.8) holds here, but not with $\rho_2(z)=z$ as one might have guessed. When degenerate eigenvalues or eigenvalues differing by integers occur logarithmic terms are expected to appear.

As remarked above, $\mathbf{y}^{(j)}(z)$ satisfies the same monodromy equations as $\mathbf{y}(z)$. The same is true of the $\mathbf{u}(z)$ and its derivatives by differentiating (2.9). It is elementary to derive from (2.12)

$$u_l^{(j)}(z)=(z-z_k)^{\nu_{k,l}-j}h_{j;k,l}(z)\ , \tag{2.13}$$

where $h_{j;k,l}$ is uniform in the neighborhood of a_k.

With these preliminaries, we now consider the set of equations

$$\sum_{j=0}^{m} c_jy_i^{(j)}(z)=0,\quad i=1,\ldots,m\ . \tag{2.14}$$

Cramer's rule gives the solution for c_j as

$$c_{j_0}=\Delta_{j_0}(z)=\det[y_i^{(j)}(z)]\ ;$$
$$i=1,\ldots,m;\quad j=0,\ldots,m,\quad j\neq j_0\ . \tag{2.15}$$

The $\Delta_{j_0}(z)\neq 0$ because by hypothesis the y_i are all linearly independent. Next, following (2.9), we introduce a change of basis $y\to u$ so that

$$\Delta_{j_0}(z)=\det U_k \det[u_l^{(j)}(z)]\ ;\quad l=1,\ldots,m;\ \ j=0,\ldots,m;\ \ j\neq j_0\ . \tag{2.16}$$

The $\det U_k$ is a constant and by factoring the $u_l^{(i)}(z)$ as in (2.13) we find that

$$\prod_{l=1}^{m}(z-a_k)^{-\nu_{k,l}+m}\Delta_{j_0}(z) \tag{2.17}$$

is analytic in the neighborhood of $z=a_k$. By repeating this argument at each singular point a_k we conclude that

$$P_{j_0}(z)=\prod_{k=1}^{n}\prod_{l=1}^{m}(z-a_k)^{-\nu_{k,l}+m}\Delta_{j_0}(z) \tag{2.18}$$

is analytic at $z=a_k$, $k=1,\ldots,n$ and, of course, by construction is analytic elsewhere in $\mathcal{D}$. Thus by multiplying the solution (2.15) by the factor found in (2.18) we conclude from (2.14) that

$$\sum_{j=0}^{m}P_j(z)y_i^{(j)}(z)=0\ , \tag{2.19}$$

where $P_j(z)$ exist, are not identically zero and are analytic in $\mathcal{D}$. By standard theorems $P_m(z)$ has only a finite number of zeros in $\mathcal{D}$. We anticipate that there will be zeros of $P_m(z)$ at $z=a_k$, $k=1,\ldots,n$. There may, however, be other zeros. For example, if

$$f(z)=(1-z)^{-1.5}+e^{-z}$$

Hunter and Baker[2] find

$$(1-z)(\tfrac{5}{2}-z)f''-(\tfrac{11}{4}+2z-z^2)f'+(\tfrac{3}{2}z-\tfrac{21}{4})f=0$$

and the zero at $z=\frac{5}{2}$ corresponds to a solution

$$f(z)=(1-z)^{-1.5}-(-1.5/e)^{-1.5}e^{1-z}\ ,$$

which has the behavior $f(z)\sim(z-2.5)^2+\cdots$ for z near 2.5. In any case we may factor

$$P_m(z)=\rho_m(z)Q(z)\ , \tag{2.20}$$

where $\rho_m\neq 0$ for $|z|>R$ and $Q\neq 0$ for $|z|\leq R$. We now define

$$\rho_j(z)=P_j(z)/Q(z),\quad j=0,\ldots,m-1 \tag{2.21}$$

and so by (2.19) we conclude (2.8), the conclusion of the theorem.

We remark that once $\rho_m(z)$ is fixed the linear independence of $y_1,\ldots,y_m$ fix the other ρ_j uniquely. If fewer than m y's were independent then an equation of lower order could be found and multiplied by an arbitrary factor and added to (2.10), destroying the uniqueness.

In the theory of critical phenomena one of the major complications to the analysis of series is the occurrence, or possible occurrence, of confluent singularities[6] of the form

$$f(z)\sim\sum_j h_j(z)(z-a)^{\gamma_j}\ , \tag{2.22}$$

where $h_j(z)$ are analytic functions of z in the neighborhood of $z=a$. As we have seen above, a homogeneous differential equation of mth order can represent such a confluent singularity exactly when there are m terms in the sum (2.22). According to the standard theory of differential equations[7] we expect that $A_m(z)\propto(z-a)^m$ for such a case [or $\rho_m(z)\propto(z-a)^m$ in Eq. (2.8)]. Singularities of lower degrees of confluence will have $A_m(z)$ proportional to a lower power of $(z-a)$ even in the context of monodromic dimension m. From the point of view of approximation theory, the main problem is that $A_m(z)$ will not generally have a single, high-order zero but rather a cluster of zeros. This problem has been observed and discussed previously[2] but not from the monodromy point of view. A further, closely related, problem is the difficulty of distinguishing between a true confluent singularity and two or more nearby separate singularities.

We illustrate our approach to the problem of confluent singularities using second-order inhomogeneous differential equations with polynomial coefficients, i.e.,

$$P(z)\frac{d^2y}{dz^2}+Q(z)\frac{dy}{dz}+R(z)y+T(z)=0\ . \tag{2.23}$$

The solutions of such equations are a subclass of the functions of monodromic dimension three. To investigate a confluent singularity we look, as did Rehr *et al.*,[2] for two close zeros of $P(z)$. We expect that, on a distance scale of the order of the separation of the two zeros, the change in the solution will be substantial as the approximation improves and tends towards a single double zero. However, at a distance large compared to the separation of the zeros, as the zeros merge into one, the monodromy structure of the equation should go smoothly from that of two isolated singularities to that of one confluent singularity.

We therefore propose to compute the monodromy matrix $M=M^{(1)}M^{(2)}$, the product of the two monodromy matrices for the two singularities, by integrating around a contour which encloses both singularities (and no others). We first integrate (2.23) from the origin, with the given boundary conditions, to a point z_0 convenient to the singularities z_1 and z_2, and then integrate counterclockwise around both singularities five times ($2m+1$ times in general). This will produce the quantities $y_0(z_0),y_1(z_0),\ldots,y_5(z_0)$. By the monodromy theory discussed previously we then have

$$\begin{bmatrix}y_{1+j}\\ y_{2+j}\\ y_{3+j}\end{bmatrix}=M^{j+1}\begin{bmatrix}y_0\\ y_1\\ y_2\end{bmatrix},\quad j=0,1,2\ . \tag{2.24}$$

From (2.24) we deduce the linear equations

$$y_{i+j}=\sum_{k=1}^{3}M_{ik}y_{k+j-1},\quad i=1,2,3,\quad j=0,1,2 \tag{2.25}$$

from which we find at once that

$$M=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ M_{31} & M_{32} & M_{33} \end{bmatrix}, \tag{2.26}$$

which has the structure of a companion matrix. The elements M_{31}, M_{32}, and M_{33} are then obtained from the solution of the set of equations (2.11). Since one of the eigenvalues of M is known to be unity, corresponding to the analytic background term, it follows that

$$M_{31}+M_{32}+M_{33}=1$$

and this provides a check on the numerical integration. It then follows that the nontrivial eigenvalues of M are

$$\lambda_\pm=\tfrac{1}{2}\{M_{33}-1\pm[(M_{33}-1)^2-4M_{31}]^{1/2}\} . \tag{2.27}$$

The behavior of $y(z)$ near the confluent singularity is then given by

$$Y(z)\approx A_+(z-z_c)^{\gamma_+}+A_-(z-z_c)^{\gamma_-}+A_0 , \tag{2.28}$$

where

$$2\pi i\gamma_+=\ln\lambda_+,\quad 2\pi i\gamma_-=\ln\lambda_- . \tag{2.29}$$

A difficulty appears at this point because we do not immediately know which sheet of the logarithm to choose in (2.29), since the monodromy matrix is a topological object and is insensitive to shifts of the power of the singularity by an exact integer. To resolve this problem we introduce the following approximate method to compute $\gamma_\pm$. A better method, in principle, but harder to compute, is given at the end of this section. The approximate method is based on a quadratic approximation to the coefficient $P(z)$ of y'', a linear approximation to the coefficient $Q(z)$ of y' and a constant approximation to the coefficient $R(z)$ of y in (2.9), near z_1 and z_2. We first compute the local singularity indices

$$\beta_1=1-\frac{Q(z_1)}{P'(z_1)},\quad \beta_2=1-\frac{Q(z_2)}{P'(z_2)} , \tag{2.30}$$

and the auxilliary quantities

$$\begin{aligned} r_1&=\frac{R(z_1)}{\displaystyle\lim_{z\to z_1}\frac{P(z)}{(z-z_1)(z-z_2)}} ,\\ r_2&=\frac{R(z_2)}{\displaystyle\lim_{z\to z_2}\frac{P(z)}{(z-z_1)(z-z_2)}} , \end{aligned} \tag{2.31}$$

and, from these, the quantities

$$\beta=1-\beta_1-\beta_2,\quad r=\tfrac{1}{2}(r_1+r_2) . \tag{2.32}$$

The approximate critical indices are then given by

$$\gamma_\pm=-\tfrac{1}{2}[\beta\pm(\beta^2-4r)^{1/2}] , \tag{2.33}$$

which should suffice to determine which Riemann sheet of the logarithm to use in (2.29).

To determine fully the representation (2.28) of $y(z)$ near the confluent singularity, we need to find the value of z_c and of the amplitudes A_+, A_-, and A_0. Rehr *et al.*[2] suggest using z_1 or z_2 as an estimate of z_c with the difference z_1-z_2 as a measure of the error. In the numerical results on test functions, to be described later, we find that the z_i which has the most divergent local singularity associated with it is the closest estimate and so we adopt this criterion. In order to compute the amplitudes we separate $y(z)$ into

$$y(z)=y_+(z)+y_-(z)+\hat{y}_0(z) . \tag{2.34}$$

This decomposition can be computed in terms of the functions $y_0(z)$, $y_1(z)$, and $y_2(z)$ defined in (2.24), giving

$$\begin{aligned} y_+(z)&=\frac{\lambda_-(y-y_1)-(y_1-y_2)}{(\lambda_--\lambda_+)(1-\lambda_+)} ,\\ y_-(z)&=\frac{\lambda_+(y-y_1)-(y_1-y_2)}{(\lambda_+-\lambda_-)(1-\lambda_-)} ,\\ \hat{y}_0(z)&=y_0(z)-y_+(z)-y_-(z) . \end{aligned} \tag{2.35}$$

Now asymptotically as $z\to z_c$

$$\begin{aligned} y_+(z)&\sim\phi_+(z)(z-z_c)^{\gamma_+} ,\\ y_-(z)&\sim\phi_-(z)(z-z_c)^{\gamma_-} ,\\ \hat{y}_0(z)&\sim\phi_0(z) , \end{aligned} \tag{2.36}$$

where the ϕ's are analytic near $z=z_c$. To obtain the amplitudes A_+, A_-, and A_0 of (2.28) we compute, by integration of (2.9), the quantities

$$\begin{aligned} \phi_+(z)&=(z-z_c)^{-\gamma_+}y_+(z) ,\\ \phi_-(z)&=(z-z_c)^{-\gamma_-}y_-(z) ,\\ \phi_0(z)&=\hat{y}_0(z) , \end{aligned} \tag{2.37}$$

and extrapolate them to the singular point z_c by the linear extrapolation

$$\phi_j(z_c)\approx\phi_j(z)+\phi_j'(z)(z_c-z) . \tag{2.38}$$

Since the ϕ_j' are directly available from the numerical solution of (2.23), this extrapolation is directly computable. Some care is required in selecting the point z to extrapolate from since one wants to be close enough for the linear extrapolation to be reliable but not close that the z_1,z_2 difference will seriously perturb the estimate. We also remark that study of the decomposition (2.36) provides an alternative way to fix the correct Riemann sheet of the logarithm in (2.29).

In the case of a single isolated singularity $[P(z_0)=0, P'(z_0)\neq 0]$ we follow the method Rehr *et al.*[2] The singular point is, of course, z_0 and the exponent β in

$$y(z)\sim\phi_1(z)(z-z_0)^{\beta}+\phi_2(z) \tag{2.39}$$

is given by

$$\beta=1-\frac{Q(z_0)}{P'(z_0)} \tag{2.40}$$

as in (2.30). In order to compute $\phi_1(z_0)$ and $\phi_2(z_0)$ we first note that it is numerically unstable to integrate (2.23) into or out from a regular singular point. Instead we compute the power-series expansions

$$y_\beta(z)=(z-z_0)^\beta\left[1+\sum_{n=1}^{\infty}a_n(z-z_0)^n\right],$$

$$y_0(z)=1+\sum_{n=1}^{\infty}b_n(z-z_0)^n, \tag{2.41}$$

from the homogeneous version of (2.23) and

$$y_{00}(z)=\sum_{n=1}^{\infty}c_n(z-z_0)^n \tag{2.42}$$

from the full inhomogeneous equation (2.23). The full inhomogeneous equation (2.23), subject to the prescribed initial conditions at $z=0$, is then integrated to some convenient intermediate point[5] where the series (2.41) and (2.42) converge rapidly, and the equations

$$\begin{aligned} y(\xi)&=A_\beta y_\beta(\xi)+A_0y_0(\xi)+y_{00}(\xi),\\ y'(\xi)&=A_\beta y'_\beta(\xi)+A_0y'_0(\xi)+y'_{00}(\xi), \end{aligned} \tag{2.43}$$

are then solved for A_β and A_0. It then follows by construction that

$$\phi_1(z_0)=A_\beta,\quad \phi_2(z_0)=A_0\ . \tag{2.44}$$

III. DIAGONAL AND HORIZONTAL SEQUENCES OF APPROXIMANTS

As stated in the Introduction, we seek to analyze power series since by means of approximants denoted by $[t/s;\ldots;q;p]$ which are solutions of the differential equation (1.4) with polynomial coefficients, subject to appropriate initial conditions. A sequence of approximants corresponds to a sequence of choices of the integers $(t,s,\ldots)$, the degrees of the polynomial coefficients. In order to make reliable estimates of the singular quantities it is essential to study a sequence of approximants, and it is therefore not only desirable but necessary to investigate the invariance and convergence properties of various types of sequences.

For the case of ordinary Padé approximants, sequences $[L/M]$ for which $\lim_{M\to\infty}L/M=1$ appear to converge more rapidly and over a wider domain of z than other types of sequences. We term these "diagonal sequences." It is well known[7] that the strictly diagonal approximants $[L/L]$ are invariant with respect to Euler transformations, and it appears that the convergence property is not unrelated to this invariance. That is to say, the value of the $[L/L](z)$ to $f(z)$ is the same as the value of $[L/L](u)$ to $g(u)$, where $z=au/(1+bu)$ and $g(u)=f(au/(1+bu))$. Another type of sequences for Padé approximants, which we term a "horizontal sequence" has the form $[L/m]$ with $L\to\infty$ and m fixed. A theorem of de Montessus de Ballore (see Ref. 8 for details) states that, for an appropriate m and $L\to\infty$, the sequence of $[L/m]$ Padé approximants to a meromorphic function $f(z)$ in $|z|<R$ converges to $f(z)$ in compact subsets of the largest punctured open disk $|z|<R/\{p_j\}$ containing exactly m poles of $f(z)$. We take the point of view that the usefulness of the horizontal sequences of Padé approximants is their convergence properties in a disk to functions whose properties outside the disk are unknown, and perhaps horrible, in contradistinction to the diagonal sequences whose natural domain is the extended complex plane.

In an entirely similar way we define diagonal and horizontal sequences for the more general Hermite-Padé approximants being discussed here. A diagonal sequence consists of approximants of the form

$$[L/M_0;M_1;\ldots;M_m]\ \text{ with }\lim_{L\to\infty}L/M_i=1\ ,$$

whereas a horizontal sequence consists of approximants $[L/M_0;\ldots;M_m]$ with $L\to\infty$ and the M_i finite. A number of theorems are known for this class of approximants, and we mention these here for completeness. For details readers are referred elsewhere.[4,9]

In particular, Baker[9] has studied the invariance of Hermite-Padé approximants with respect to Euler transformations, and has obtained a general invariant form. For the case of inhomogeneous second-order differential equations, on which we concentrate here, the invariant form is

$$P_{N+4}(z)\frac{d^2f}{dz^2}+Q_{N+3}(z)\frac{df}{dz}+R_N(z)f+T_N(z)=O(z^{4N+q})\ , \tag{3.1}$$

where, if we write

$$P_{N+4}(z)=\sum_{j=0}^{N+4}p_jz^j,\quad Q_{N+3}(z)=\sum_{j=0}^{N+3}q_jz^j\ , \tag{3.2}$$

we have the linear restriction

$$q_{N+3}=2p_{N+4} \tag{3.3}$$

This restriction does not create any difficulties in the computation of the polynomials P, Q, R, and T since the system of equations remains linear. Riemann's P equation is an example of (3.1) subject to (3.3) where the restriction (3.3) corresponds exactly to the well-known requirement that the sum of the six exponents at the three regular singular points must sum up to unity. Riemann's study of the P equation is closely related to his monodromy theorem which we quoted in Sec. II.

The behavior of diagonal approximants in the limit $N\to\infty$ is not yet well understood in general. For ordinary Padé approximants Stahl[10] has recently shown that $[L/M]$, $L,M\to\infty$, $L/M\to1$ converges to the defining function $f(z)$ in logarithmic capacity in a domain $\mathcal{D}$. His set of conditions is that (1) the set $\mathcal{E}$ of singularities of $f(z)$ be of capacity zero and not include the origin and (2) the domain $\mathcal{D}$ is defined by the three properties: (a) $f(z)$ has a single-valued continuation in $\mathcal{D}$, (b) the logarithmic capacity of the boundary $\partial\mathcal{D}$ is minimal in the $1/z$ plane among all domains satisfying assertion (a), and (c) $\mathcal{D}$ contains all domains satisfying (a) and (b).

Nuttall[11] has obtained results for a special case of the diagonal integral approximants (and also for the same special case of more general Hermite-Padé approximants). Chudnovsky[12] has also obtained some results for

TABLE I. Test functions $A-W$ used to study the Hermite-Padé approximant method of series analysis.

Series	Test function
A	$(1-z)^{-1.5}+e^{-z}$
D	$(1-z)^{-1.5}+(1+\frac{1}{4}z^2)^{-1.25}+(1+\frac{15}{112}z-\frac{1}{4}z^2)^{-1.25}$
E	$(1-z)^{-1.5}(1+\frac{1}{2}z)^{1.5}+(1+\frac{1}{4}z^2)^{-1.25}+(1+\frac{15}{112}z-\frac{1}{4}z^2)^{-1.25}$
H^*	$(1-z)^{-1.5}+(1+\frac{1}{2}z)^{-1.5}+\left[\frac{2(1-z)(2-z)^6}{(2-z)^7-z^7}\right]^{1.25}$
J	$(1-z)^{-1.5}+(1+\frac{4}{5}z)^{-1.25}$
K	$(1-z)^{-1.5}+(1+\frac{4}{5}z)^{-1.25}+e^{-z}$
L	$(1-z)^{-7/4}+(1-z)^{-1}+(1-z)^{-1/4}+(1-z)^{1/2}+(1-z)^{5/4}+e^{-z}$
M	$\tan\sqrt{z}\,/\sqrt{z}$
O	$\{(1-z)^{1/2}[1+\frac{1}{2}(1-z)^{1/2}]\}^{-1}$
P	$(1-z)^{-7/4}+(1-z)^{-1}+(1-z)^{-1/4}+(1-\frac{1}{2}z)^{-5/4}$
Q	$[(1-z)^{-7/4}+(1-z)^{-1}+(1-z)^{-3/4}](1+z)^{1.5}+(1-\frac{1}{2}z)^{-5/4}$
R	$(\cos\sqrt{z}\,)^{1/2}+[\cos(\frac{1}{2}\sqrt{z}\,)]^{1/2}+e^{-z}$
T	$(\cos\sqrt{z}\,)^{1/2}\{1+[1-(\frac{7}{11})^2z]^{1/2}\}$
U	$(1-z)^{-7/4}+(1-z)^{-3/4}+(1+\frac{1}{3}z)^{-1.25}+e^{-z}$
V	$(1-z)^{-7/4}+(1-z)^{-5/4}+\left[\frac{1-\frac{1}{3}z}{1+\frac{1}{3}z}\right]^{1/2}$
W	$\left[\frac{1-z}{1+\frac{1}{2}z}\right]^{-7/4}+\left[\frac{1-z}{1+z}\right]^{-5/4}+\left[\frac{1+\frac{1}{5}z}{1-\frac{1}{3}z}\right]^{1/2}$

TABLE II. Properties of test functions.

Series	Global monodromic dimension	Disk of 1 singularity: Local monodromic dimension	Disk of 1 singularity: Separability	Disk of 2 singularities: Local monodromic dimension	Disk of 2 singularities: Separability	Comments
M	b	b	S	b	S	Baker Lubinsky theorem infinite number of sing.
A	$1+b$	$1+b$	S			$[-1/1;2;2]$
O	$1+b$	$1+b$	S	$1+b$	S	$[-1/0;1;2]$ and $[0/1;2;-1]$
J	2	$1+b$	S	2	S	$[-1/1;2;3]$
T	2	$1+b$	N	2	N	infinite number of sing.
K	$2+b$	$1+b$	S	$2+b$	S	
U	$2+b$	$1+b$	S	$2+b$	S	
R	$2+b$	$1+b$	N	$2+b$	N	infinite number of sing.
D	3	$1+b$	S	$2+b$	N	4 sing. total
H^*	3	$1+b$	S			line of sing. $\mathrm{Re}(x)=1$
E	3	$1+b$	N	$2+b$	N	
P	3	$2+b$	S	3	S	confluent $m=2$
V	3	$2+b$	S			confluent $m=2$
Q	3			$2+b$	S	confluent $m=2$ and sing. at -1
W	3			$2+b$	N	confluent $m=2$ and sing. at -1
L	$3+b$	$3+b$	S			confluent $m=3$

special cases. Specifically Nuttall has studied approximants defined by $(m-1)$st-order homogeneous differential equations to meromorphic functions with a finite number of singularities on an m-sheeted Riemann surface defined by an irreducible polynomial $R(y,z)$. Such functions can, for example, have $(z-z_0)^{1/m}$-type branch points. Of course, by Riemann's monodromy theorem (Sec. II), an mth order homogeneous differential equation with polynomial coefficients would exactly represent such a function. However, Nuttall's work illuminates the situation when such an exact representation is not possible. His principal results shows that almost all of the zeros of the polynomial coefficients for the unrestricted $[-1/n;\ldots;n]$ approximants lie on well-defined curves in this case. We use the notation degree -1 to denote a missing polynomial. It is tempting to suppose, and we investigate numerically, that a situation analogous to that for Padé approximants occurs. Namely, the extended complex plane is cut in such a way that the equation of the form defining the integral approximant could exactly represent the given function along all paths which do not touch any of the cuts. The diagonal integral approximants are then expected to converge, except perhaps for a set of points of small size in some sense, away from the cuts. The selection of the cut set from all possible such cut sets is expected to be minimal in some sense.

As a remark, we note that since Padé approximants (to convergent series) generally converge geometrically fast at regular points or even at poles (the reciprocal of the Padé approximant is meant at a pole) and much more slowly at branch points or cuts (which they cannot exactly represent), we expect the same type of rapid convergence at exactly representable points but warn that it is unlikely at points or cuts which cannot be exactly represented by the integral approximant being used.

In the case of integral approximants, the identification of a horizontal sequence with the disk convergence property is less simple. We have, however, isolated one property which aids in the analysis and appears to be essential to this question. It is the separation property.

Separation property. If a function $f(z)$, possibly multiform, can be written as $f(z)=f_i(z)+f_o(z)$, where for a disk $\mathcal{D}=\{z \mid |z| \le R\}$, $f_o(z)$ is analytic for all $z\in\mathcal{D}$ and every continuation of $f_i(z)$ is analytic for all (z) in the finite complex plane outside of $\mathcal{D}$, then $f(z)$ has the separation property with respect to $\mathcal{D}$.

The simplest nontrivial example of functions with the separation property is the class of meromorphic functions which have the separation property with respect to any disk D of finite radius R. For this class Baker and Lubinsky[9] have identified the horizontal sequences with the disk convergence property. To quote one example of their results, suppose $f(z)$ is meromorphic in $|z|<R$ with l poles of total multiplicity p. Then $[L/p-1;l-1;\ldots;l-1;l]$ converges geometrically in L and L goes to infinity on any compact subset of $|z|<R$ not including any pole of $f(z)$. The selection of the degrees of the finite polynomials is of key importance here to the proof of convergence. The degrees are chosen so any other set of polynomials of the same degrees which cancel out the singularities in $|z|<R$ are just a constant

TABLE III. Values of critical parameters for test functions with no confluent singularities.

Function	z_1	γ_1	A_1	$A_{1,0}$	z_2	γ_2	As_2	$A_{2,0}$
D	1.0	-1.5	i	1.923 35	-1.75	-1.25	0.988 950	0.710 615
E	1.0	-1.5	$1.837\,12i$	1.923 35	-1.75	-1.25	0.988 950	0.501 025
H^*	1.0	-1.5	i	0.632 158	$1\pm0.481\,575i$	-1.25	0.023 227 5 $\pm0.032\,275i$	$-1.596\,98$ $\pm1.989\,58i$
K	1.0	-1.5	i	0.847 513	-1.25	-1.25	1.321 71	3.786 64
M	2.467 40	-1	-2.0	0.202 692	22.2066	-1.0	-2.0	0.022 515 8
R	2.467 40	0.5	$0.564\,190i$	0.925 701	9.869 64	0.5	$0.398\,942i$	$i+$ $5.172\,32\times10^{-5}$
T	2.467 40	0.5	$0.5801\,192i$	0.0	2.469 39	0.5	$-0.016\,001\,0$	$0.025\,144\,5i$
U	1.0	-1.75	0.707 107 $+0.707\,107i$	1.065 83	-3.0	-1.25	3.948 22	20.5275

Function	z_3	γ_3	A_3	$A_{3,0}$
D	$\pm2i$	-1.25	$-0.382\,683$ $\pm0.923\,88i$	$-0.383\,200$ $\pm0.228\,982i$
E	±21	-1.25	$-0.382\,683$ $\pm0.923\,88i$	$-0.070\,037$ $\pm0.081\,091\,3i$
H^*				
K				
M	61.6850	-1.0	-2.0	0.008 105 69
R	22.2066	0.5	0.325 735	$0.707\,107i$ $+2.268\,78\times10^{-10}$
T	22.2066	0.5	0.325 735 $+2.501\,41i$	0.0

TABLE IV. Values of critical parameters for test functions with a confluent singularity.

Function	z_1	γ_1	γ_2	A_1	A_2	A_0	z_3	γ_3	A_3	$A_{3,0}$
L	1.0	−1.75	−1.0	0.707 107 + 0.707 107i	−1.0	0.367 879				
P	1.0	−1.75	−1.0	0.707 107 + 0.707 107i	−1.0	2.378 41	2.0	−1.25	−1.681 79 + 1.681 79i	0.414 214
Q	1.0	−1.75	−1.0	2.0−2.0i	−2.828 47	4.499 73	−1.0	1.5	1.391 91	0.602 401
V	1.0	−1.75	−1.25	0.707 107 + 0.707 107i	−0.707 107 + 0.707 107i	0.707 107	−3.0	−0.5	2.449 49	0.265 165
W	1.0	−1.75	−1.25	1.437 62 + 1.437 62i	−1.681 79 + 1.681 79i	1.341 64	−1.0	1.25	0.420 448	1.451 31

multiple of a particular one. Furthermore these polynomial degrees are chosen so that when such a set of polynomials cancels the singularity of $f(z)$ the polynomial multiplying the highest derivative vanishes only at the poles of $f(z)$.

More generally suppose $f(z)$ has the separation property with respect to $\mathcal{D}$, a finite number of singular points in the interior of $\mathcal{D}$, and none on the boundary of $\mathcal{D}$, and that all those singular points, plus the point at infinity for the $f_i(z)$ (defined by the separation property) are singular points of finite order (2.4). Suppose further that $f_i(z)$ is of exact monodromic dimension m. Under these hypothesis, the corollary to Riemann's monodromy theorem shows that there exist $A_j(z)$ polynomials, such that

$$\sum_{j=0}^{m} A_j(z) f_i^{(j)}(z) = 0 \ . \tag{3.4}$$

From (3.4) it follows directly, as the differential multiplier,

$$\left[\sum_{j=0}^{m} A_j(z) \frac{d^j}{dz^j} \right] , \tag{3.5}$$

TABLE V. Coefficients of the polynomials $P(z)$, $Q(z)$, $R(z)$, and $T(z)$ for approximants $[L/1;2;2]$, $L=0,1,2,3$ for test function 0. The coefficients are tabulated in ascending order.

	$L=0$	$L=1$	$L=2$	$L=3$
$P(z)$	1.0	1.0	1.0	1.0
	−0.6667	−0.6667	−0.6667	−0.6667
	−0.3333	−0.3333	−0.3333	−0.3333
$Q(z)$	41.8757	29.5511	11.1411	3.2624
	−29.6384	−21.4230	−9.1496	−3.8972
	−14.2363	−10.1281	−3.9915	−1.3653
$R(z)$	−7.6182	−5.5641	−2.4957	−1.1826
	−21.3545	−15.1922	−5.9872	−2.0479
$T(z)$	−14.2363	−10.1281	−3.9915	−1.3653
		0.0	0.0	0.0
			0.0	0.0
				0.0

applied to $f_o(z)$ does not change the radius of convergence, that

$$\sum_{j=0}^{m} A_j(z) f^{(j)}(z) = \phi(z) \ , \tag{3.6}$$

where $\phi(z)$ is analytic in $\mathcal{D}$. It is no loss of generality if we assume that of all possible sets of $A_j(z)$ for which (3.4) holds, we have chosen that one for which $A_m(z)$ is of minimum degree. $A_m(z)$ cannot be identically zero because $f_i(z)$ is of monodromic dimension m. For this choice $A_m(z)$ is essentially unique, i.e., all such $A_m(z)$ are just constant multiples of each other. For suppose there were two such $B_m(z)$ and $C_m(z)$ not constant multiples of each other. Then we can form a linear combination of $B_m(z)$ and $C_m(z)$ of lower degree in z by cancelling the highest power of z and leaving a new set of polynomials for which (3.4) holds with $A_m(z)$ of lower degree. But this result is contrary to the hypothesis that we had chosen that set of $A_j(z)$ for which $A_m(z)$ is of minimum degree. Therefore we conclude that it is essentially unique. If we now normalize it as we please, say, the largest coefficient of any power of z in $A_m(z)$ is unity, then, by the remark after the proof of our disk monodromy theory (Sec. II), the other $A_j(z)$ and $\phi(z)$ are uniquely fixed. For this set of degrees of the polynomials we say therefore that (3.5) is an essentially unique differential multiplier.

Baker[4] has proven that if $f(z)$ satisfies an equation of the form (3.6) with an essentially unique differential multiplier, with $\phi(z)$ an entire function, then the $[L/M_0;\ldots;M_m]$ converge to $f(z)$ in the limit $L\to\infty$ where the polynomial degrees M_i are those appropriate to the essentially unique differential multiplier. His proof can be transcribed to the present case, except we now need the estimate that the coefficients of the Taylor series of $\phi(z)$ are of the order R^{-n}. If $r<R$ is the largest magnitude of the location of any singular point of $f_i(z)$ in D then the difference between the appropriately normalized, polynomial coefficients of $[L/M_0;\ldots;M_m]$ and the $A_j(z)$ of (3.7) is of the order,

$$L^{\text{fixed power}}(r/R)^L \ ,$$

which goes to zero as $L\to\infty$. The convergence of the inhomogeneous part follows as well. Thus we conclude the following theorem.

Theorem (separation property). Under the hypothesis stated before Eq. (3.4), (i) there exists an essentially unique differential multiplier for disk $\mathcal{D}$, whose coefficients are polynomials of degrees $M_0, M_1, \ldots, M_m$, and (ii) the integral approximants $[L/M_0;\ldots;M_m]$ converge as $L \to \infty$ to $f(z)$ on compact subsets of $\mathcal{D}$ not containing singular points of $f(z)$.

By virtue of this de Montessus-type theorem and the Baker-Lubinsky[9] results, when $f(z)$ has the separation property, we obtain convergence in a disk by letting the degree of the inhomogeneous term approach infinity while the other polynomial degrees remain fixed and finite. The appropriate degrees can be determined by a systematic scan with M_m increasing steadily since if the M_j, $j<m$, are too large the high-order coefficients will converge to zero. With a limited number of coefficients, as is usually the case in practice, it may be difficult to carry out this procedure.

Finally, we consider the case when the separation property does not hold. In this case, the representation in a disk is given by the disk monodromy theorem, Eq. (2.8). If the monodromic dimension is m in the disk then it may be possible to represent the function as an analytic background plus a singular part of monodromic dimension $(m-1)$. In this case the horizontal sequence is $[L/L;\cdots;L;M_{m-1}]$ with M_{m-1} fixed and $L \to \infty$. If the above-mentioned reduction of the monodromic dimension by the inclusion of an analytic background term does not take place for the particular example, then the horizontal sequence $[-1/L;\ldots;L;M_m]$ will be required.

It is important to remember here, as throughout our discussion, that convergence of horizontal sequences of approximants can only be expected if the differential multiplier, plus the inhomogeneous term when present, is essentially unique. Even so, we have only demonstrated convergence when the separation property holds, but do investigate it numerically for some test functions which do not have this property. If one tries to apply a sequence of the sort used for functions with the separation property to those without it, one cannot expect to find convergence beyond the first singularity. As an example let $x=z-z_0$, where z_0 is the nearest singularity to the origin of $f(z)$. Suppose further that the dominant part of the high-order coefficients of the expansion in z come

TABLE VI. Estimates of critical parameters at first singularity for test functions with no confluent term, from horizontal and diagonal sequences of approximants.

Test		Horizontal sequence				Diagonal sequence			
function	n	z_1	γ_1	A_1	$A_{1,0}$	z_1	γ_1	A_1	$A_{1,0}$
			$[L/1;2;3]$						
D	18	2.8	1.5	0.7	0.3	4.3	3.3	2.3	0.8
	28	5.7	4.0	2.6	0.7	7.0	5.2	4.4	3.2
	40	7.9	6.2	4.5	3.0	12.0	9.7	7.3	5.1
	50	11.1	9.4	8.5	6.5	13.3	10.9	9.8	7.3
			$[L/4;4;5]$						
E	18	3.5	2.1	0.0	0.0	4.0	2.6	1.4	0.3
	28	6.5	4.7	3.8	2.1	6.7	5.0	4.7	3.2
	40	8.6	6.6	5.7	3.5	10.4	8.3	7.2	4.9
	50	9.6	7.2	6.4	3.6	15.2	12.8	9.6	8.0
			$[L/4;4;5]$						
H^*	18	1.9	0.9	0.0	0.0	1.9	1.0	0.0	0.0
	28	2.2	0.8	0.0	0.0	3.0	1.7	0.2	0.0
	40	3.1	1.3	0.5	0.0	5.9	4.1	3.4	1.8
			$[L/2;2;3]$						
K	18	7.7	5.1	4.2	2.3	6.0	4.6	3.8	2.4
	28	14.2	12.5	9.6	8.3	11.6	9.7	8.8	6.7
			$[L/0;0;1]$						
M	18	12.5	11.0	11.5	9.0	23	21.4	10.6	8.9
			$[L/1;2;3]$						
R	18	3.2	1.3	0.0	0.0	4.4	2.4	2.3	3.2
	28	8.2	5.9	5.7	6.9	9.6	7.6	7.6	8.4
	40	14.9	12.1	11	11.2	13.9	10.6		
			$[L/4;4;5]$						
T	18	5.5	3.9			4.0	0.3	2.7	
	28	6.5	3.3	4.8		6.5	3.3	4.8	
			$[L/1;2;3]$						
U	18	6.5	5.0	4.2	2.4	8.3	6.2	5.5	3.0
	28	16.2	14.5	10.5	8.9	12.6	10.7	10.5	7.1

TABLE VII. Estimates of critical parameters at second singularity for test functions with no confluent term, from horizontal and diagonal sequences of approximants.

Test function	n	Horizontal sequence z_2	γ_2	A_2	$A_{2,0}$	Diagonal sequence z_2	γ_2	A_2	$A_{2,0}$
			[L/4;4;5]						
D	18	2.7	1.2	0.6	0.0	3.9	3.1	3.0	1.0
	28	4.1	2.4	2.0	0.3	5.3	3.5	2.9	1.4
	40	5.6	3.6	2.8	0.8	8.8	6.9	6.2	4.4
	50	3.7	1.8	3.0	1.6	11.1	8.9	7.8	5.9
			[L/4;4;5]						
E	18	1.6	0.3	0.0	0.0	1.7	0.6	0.0	0.0
	28	3.0	1.2	0.6	0.0	4.3	2.5	1.8	0.2
	40	4.4	2.5	1.8	0.2	7.0	4.7	3.7	1.5
	50	4.5	2.5	1.8	0.0	7.3	5.0	4.3	1.8
			[L/4;4;5]						
H^*	28	0.7	0.0	0.0	0.0	1.0	0.0	0.0	0.0
	40	1.3	0.0	0.0	0.0	3.6	1.7	1.4	1.0
	50	2.8	1.4	0.0	0.0	5.6	3.4	2.6	2.4
			[L/2;2;3]						
K	18	4.0	2.4	1.9	1.0	3.0	1.1	0.3	0.0
	28	12.5	10.8	9.5	8.7	9.0	7.0	6.4	5.0
	40	21.9	20.0	9.5	9.0	14.0	11.8	9.4	9.6
			[L/1;1;2]						
M	18	3.6	2.2			7.5	6.0		
	24	5.4	3.9			8.4	6.9		
			[L/4;4;5]						
T	18	1.2	0.0			2.9	0.0		
	28	3.8	0.0			3.8	0.0		
			[L/1;2;3]						
U	18	2.2	1.3	0.3	0.1	1.5	0.3	0.0	0.0
	28	5.2	3.5	3.0	2.5	3.9	2.2	1.6	1.2
	40	5.8	3.9	3.3	2.6	8.0	5.6	4.9	3.8
			[L/1;2;3]						
R	40	0.8	1.0						
	50	2.2	1.3						

from the expansion of

$$f(z)=ax^{-\gamma}+bx^{-\gamma+1}+cx^{-\gamma+2}+\cdots\ . \tag{3.7}$$

If we try to analyze (3.7) with the [L/1;2] approximants we need to solve for the limit $L\to\infty$ of the coefficients in

$$(Ax+Bx^2)y'+(C+Dx)y=\sum_{j=0}^{L}E_jx^j=E_(x)\ . \tag{3.8}$$

These coefficients can be computed to be (normalizing $A=1$)

$$B=(b^2-2ac)/ab,\quad C=\gamma,\quad D=\gamma B-b/a\ , \tag{3.9}$$

which locates a second zero of the coefficient of y' at $z=z_0-1/B$ which, in general, has nothing to do with the second singular point. Of course, this calculation is not correct if $b=0$. It does, however, illustrate how pro-

TABLE VIII. Estimates of critical parameters at third singularity for test functions with no confluent terms, from horizontal and diagonal sequences of approximants.

Test function	n	Horizontal sequence z_3	γ_3	A_3	$A_{3,0}$	Diagonal sequence z_3	γ_3	A_3	$A_{3,0}$
			[L/3;3;4]						
M	18	0.8	0.4			1.8	0.5		
	28					2.4	1.7		
			[L/4;4;5]						
T	18	1.0	0.0			2.0	0.0		
	28	2.1	0.7			2.4	0.0		
	35	2.7	1.5			4.3	0.0		

TABLE IX. Estimates of critical parameters for test functions with a dominant confluent singularity, from horizontal and diagonal sequences of approximants. The quoted values of the confluent critical exponents γ_1,γ_2 were computed from (2.33).

Test function	n	Horizontal sequence z_1	γ_1	γ_2	A_1	A_2	A_0	Diagonal sequence z_1	γ_1	γ_2	A_1	A_2	A_0
				$[L/3;3;4]$									
L	18	4.1	0.0	0.0	0.0	0.0		2.5	1.8	0.0	0.5	0.0	
	28	5.3	2.3	1.7	1.8	1.3		5.0	2.3	2.3	2.1	0.8	
	40	5.9	2.9	1.3	2.3	0.5		5.8	2.3	2.1	2.0	0.4	
	50	6.1	3.1	1.5	2.5	0.6		6.4	2.9	1.8	2.2	1.8	
				$[L/4;4;5]$									
P	18	4.5	0.0	0.0	0.0	0.0	0.0	4.5	0.0	0.0	0.0	0.0	0.0
	28	5.2	1.8	0.6	1.2	0.0	0.0	5.6	3.0	1.0	2.0	0.3	0.0
	40	5.8	3.0	1.2	2.4	0.4	0.0	6.0	2.0	1.2	2.2	0.0	0.0
	50	6.0	3.3	1.4	3.0	0.6	0.0	6.3	2.0	2.0	2.3	0.0	0.0
				$[L/3;4;5]$									
Q	18	3.8	0.0	0.0	0.0	0.0	0.0	4.2	0.0	0.0	0.0	0.0	0.0
	28	4.6	2.0	1.2	1.5	0.0	0.5	5.2	2.2	1.1	1.8	0.4	0.3
	40	5.1	2.5	1.4	2.0	0.4	0.6	5.6	4.6	3.3	2.8	0.0	0.0
	50	5.3	3.2	2.1	2.6	1.2	1.0	9.9	8.2	5.8	2.8	0.0	0.0
				$[L/0;1;2]$									
V	18	3.4	0.0	0.0	0.0	0.0	0.0	3.7	1.6	0.9	0.0	0.0	0.0
	28	3.7	2.5	2.3	1.6	1.2	2.4	4.2	3.3	2.5			
	40	8.0	7.4	7.0	4.0	4.2	4.2	6.5	5.0	3.8			
	50	12.4	11.6	11.2	6.3	6.4	4.2	8.1	6.6	5.3	4.9	4.4	2.4
				$[L/3;4;5]$									
W	18	3.5	0.0	0.0	0.0	0.0	0.0	3.5	0.0	0.0	0.0	0.0	0.0
	28	3.8	2.9	2.3	1.7	1.7	0.6	4.1	4.5	2.4	2.1	2.4	0.4
	40	4.4	2.7	2.0				5.3	5.5	4.0	3.4	3.2	1.3
	50	7.8	6.6	5.7	3.7	4.0	3.6	8.6	7.1	5.9	3.7	4.0	3.4

foundly the separation property affects the structure of the function.

We remark that Baker[4] has proven a convergence uniqueness theorem which implies $y \rightarrow f$ as L goes to infinity. The conditions of this theorem are met in the case (3.9) for $|z|<|z_0|$, $z \neq z_0 - 1/B$ and, in fact, convergence fails for $|z|>|z_0|$ because of the divergence of $E_L(x)$ unless very special cancellations occur. A further note of caution is that while the horizontal sequences we suggest above for functions without the separation property correctly mimic the representation (2.8), it is not *a priori* obvious that the degrees of the polynomials mimicking the infinite series should all go to infinity at the same rate.

IV. ANALYSIS OF TEST FUNCTIONS

In order to test the effectiveness of our proposed method of series analysis we have applied it to a number of test functions, shown in Table I. Some of these test

TABLE X. Estimates of critical parameters at second singularity for test functions with a dominant confluent singularity, from horizontal and diagonal sequences of approximants.

Test function	n	Horizontal sequence z_3	γ_3	A_3	$A_{3,0}$	Diagonal sequence z_3	γ_3	A_3	$A_{3,0}$
			$[L/3;4;5]$						
Q	18	5.2	3.9	3.5	5.4	3.6	2.3	1.8	3.8
	28	9.5	6.3	5.5	9.0	7.2	5.3	4.6	7.5
	40	8.6	6.6	5.9	8.9	13.0	10.9	9.7	13.1
	50	9.6	7.4	6.6	9.9	16.6	14.5	11.5	13.3
			$[L/3;4;5]$						
W	18	2.9	1.6	1.3	3.1	3.3	2.0	1.9	3.5
	28	5.2	3.5	5.0	5.5	6.3	4.7	4.8	6.7
	40	6.3	4.3	4.4	6.7	9.0	6.8	6.4	9.2
	50	10.2	8.1			13.1	10.8		

functions have been introduced previously[13] to serve as test cases for methods of series analysis, while others are introduced here for the first time.

Tables II–IV list some of the properties of these test functions. The notation is as follows. For functions with no confluent singularities, in the vicinity of the nth singularity, we write

$$f(z)\sim A_n(z-z)^{\gamma_n}+A_{n,0} \tag{4.1}$$

and we seek to estimate the critical parameters z_n, γ_n, A_n, and $A_{n,0}$ for the first and second, and sometimes the third, singularities. For functions with a leading confluent singularity we write

$$f(z)\sim A_1(z-z_1)^{\gamma_1}+A_2(z-z_1)^{\gamma_2}+A_0 \tag{4.2}$$

near the confluent singularity, and

$$f(z)\sim A_3(z-z_3)^{\gamma_3}+A_{3,0} \tag{4.3}$$

near the second singularity.

For purposes of analysis the functions have been expanded to as many as 50 terms. The series have been then analyzed, in the manner described in Sec. II, to estimate the critical parameters. There is, of course, a vast amount of data and it is only possible to report key results. These are given in Tables V–XIII. A few words about these tables are in order. Firstly, rather than reporting the values of the critical parameters themselves, we choose to tabulate the quantities

$$\epsilon=-\log_{10}\left|\frac{p-p_{\text{exact}}}{p_{\text{exact}}}\right| , \tag{4.4}$$

where p is the estimate of the critical parameter. This quantity ϵ is the number of significant figures correctly estimated. We give the results from a particular horizontal sequence, as specified, and from unconstrained diagonal sequences. Missing entries indicate that the particular approximant was defective, a feature which also occurs in standard Padé analysis, or that the computer program failed in some way. Another potential problem, particularly with the longer series, is round-off error. The apparent deterioration of some estimates with increasing series length is probably due to this cause.

Test functions A, O, and J are exactly represented by low-order approximants, as indicated in Table II. In particular, the function A satisfies the equation

$$(1-z)(\tfrac{5}{2}-z)f''-(\tfrac{11}{4}+2z-z^2)f'+(\tfrac{3}{2}z-\tfrac{21}{4})f=0$$

and thus the approximant $[-1/1;2;2]$ is exact. Function J is represented exactly by the approximant $[-1/1;2;3]$. Our computer code obtains these known exact solutions, providing at least a partial check on its correctness. Other approximants, e.g., $[L/-1;0;1]$ for A and $[L/1;2;2]$ for J, perform well, not only for the nearest singularity at $z=1$ but also for the second singularity at $z=-1.25$. Test function O is interesting since it satisfies both

$$2(3+z)(1-z)f_O'-(1+3z)f_O-2=0$$

and

$$2(3+z)(1-z)f_O''-(5+7z)f_O'-3f_O=0$$

and hence can be represented exactly by approximants $[-1/0;1;2]$ and $[0/1;2;-1]$. Taking an arbitrary linear combination of these two equations shows that there is not a unique differential multiplier for the approximant $[0/1;2;2]$. In Table V we illustrate the results obtained by our machine calculation for this case.

Test function M has an infinite number of simple poles at $z=(n+\frac{1}{2})^2\pi^2$ and can be used to illustrate the Baker-Lubinsky[9] theorems on convergence of horizontal sequences. Our results in Tables VI and VII show that the horizontal sequence $[L/0;0;1]$ converges rapidly at the nearest singularity, while the sequence $[L/1;1;2]$ also gives the second singularity to 4 or 5 figure accuracy with 24 terms.

The test functions K and U both have the separation property and have monodromic dimension 2 plus an analytic background term. We have used a variety of horizontal and diagonal sequences and all do very well for both the first and second singularities (Tables VI and VII). We expect a double zero at the point $z=1$ in $P(z)$ for test function U as can be seen by a simple calculation.

The two test functions T and R are both nonseparable and have an infinite number of singularities. Both series provide a severe challenge for any analysis method. The first singularity is obtained quite accurately for R by both horizontal and diagonal sequences, but no sequence is able to consistently detect the second singularity to any reasonable accuracy. Function T has a "pseudo-confluent" singularity at $z=\frac{11}{7}$, near the dominant singularity at $\pi/2$. Beyond the leading singularity the results are erratic, with the diagonal unconstrained sequences doing best. Results are given in Tables VI–VIII.

The next group of functions D, H^*, and E have monodromic dimension 3. Functions D and H^* are separable for the first singularity while E is not. Horizontal sequences converge well for the first singularity for D but the nearby second and third singularities greatly slow the convergence for H^*. Surprisingly, we find equally good convergence for E. Fair convergence is obtained for the second singularities of D and E but results for H^* are rather poor. For these three functions, for both the first and second singularities, the diagonal approximants are significancy better than the horizontal ones, as seen in Tables VI and VII.

The final group of test functions P, V, Q, W, and L all have a confluent singularity at $z=1$. The separable cases P, Q, and V, are all exactly representable as $L\to\infty$. The convergence here is fairly good for P and Q, and very good for V. For the nonseparable case W the convergence is better than for P and Q but poorer than V. For L the convergence is similar to that for P and Q even though the local monodromic dimension exceeds that of the approximant. The diagonal approximants are significantly better than the horizontal ones for Q, worse for V, and about the same for L, P, and W. The second singularity in Q and W is at the same distance from the origin as the first one. Convergence of the horizontal se-

TABLE XI. Comparison of exponent estimates for confluent and pseudoconfluent singularity from the approximate formula (2.33) and the contour method.

Test		Horizontal sequence				Diagonal sequence			
		γ_1		γ_2		γ_1		γ_2	
function	n	Approx.	Contour	Approx.	Contour	Approx.	Contour	Approx.	Contour
L	18					1.8	1.9	0.0	0.3
	28	2.3	2.8	1.7	2.4	2.3	3.0	2.3	2.1
	40	2.9	3.3	1.3	1.4	2.3	5.4	2.1	2.1
	50	3.1	3.5	1.5	1.5	2.9	4.0	1.8	2.0
P	18								
	28	1.8	2.0	0.6	0.7	3.0	2.7	1.0	1.0
	40	3.0	3.0	1.2	1.2	2.0	3.0	1.2	1.0
	50	3.3	3.4	1.4	1.4	2.0	5.7	2.0	2.7
Q	18								
	28	2.0	2.2	1.2	1.1	2.2	2.5	1.1	1.1
	40	2.5	2.8	1.4	1.4	4.6	4.9	3.3	3.6
	50	3.2	3.5	2.1	2.1	8.2	8.2	5.8	
T	18					3.6	3.9		
	28					3.8	7.4		
	40	4.2	5.3			4.6	9.2		
	50	4.0	5.7			5.1	9.2		
V	18					1.6	1.2	0.9	0.5
	28	2.5	2.5	2.3	2.3	3.3		2.5	
	40	7.4	7.4	7.0	7.0	5.0		3.8	
	50	11.6	10.5	11.2	11.2	6.6	6.6	5.3	5.3
W	18								
	28	2.9	3.0	2.3	2.3	4.5	4.3	2.4	2.4
	40	2.7		2.0		4.9	4.9	4.0	4.0
	50	6.6	6.6	5.7	5.7	7.1	7.1	5.9	5.9

TABLE XII. Comparison of unconstrained and constrained diagonal approximants for test functions with no confluent term.

Test		Unconstrained sequence								Constrained sequence							
function	n	z_1	γ_1	A_1	$A_{1,0}$	z_2	γ_2	A_2	$A_{2,0}$	z_1	γ_1	A_1	$A_{1,0}$	z_2	γ_2	A_2	$A_{2,0}$
D	18	4.3	3.3	2.3	0.8	3.9	3.1	3.0	1.0	3.4	2.1	1.4	0.3	1.8	0.8	0.7	0.2
	28	7.0	5.2	4.4	3.2	5.3	3.5	2.9	1.4	7.2	5.2	3.7	2.0	5.7	3.6	2.7	1.4
	40	12.0	9.7	7.3	5.1	8.8	6.9	6.2	4.4	10.5	8.3	7.2	5.2	8.5	6.6	5.9	4.0
	50	13.3	10.9	9.8	7.3	11.1	8.9	7.8	5.9	13.1	10.7	9.8	7.3	11.4	9.1	7.4	5.8
E	18	4.0	2.6	1.4	0.3	1.7	0.6	0.0	0.0	3.9	2.4	1.6	0.3	0.8	0.0	0.0	0.0
	28	6.7	5.0	4.7	3.2	4.3	2.5	1.8	0.2	6.6	4.9	4.0	2.0	4.2	2.4	1.7	0.4
	40	10.4	8.3	7.2	4.9	7.0	4.7	3.7	3.5	11.3	9.2	8.4	5.9	6.4	4.4	4.2	2.0
	50	15.2	12.8	9.6	8.0	7.3	5.0	4.3	1.8	13.8	11.4	9.5	7.8	6.6	6.4	3.7	1.6
H^*	18	1.9	1.0	0.0	0.0					1.8	0.7	0.0	0.0				
	28	3.0	1.7	0.2	0.0	1.0	0.0	0.0	0.0	2.3	0.8	0.0	0.0	0.8	0.0	0.0	0.0
	40	5.9	4.1	3.4	1.8	3.6	1.7	1.4	1.0	5.8	3.7	2.9	0.4	3.7	2.1	1.7	1.1
	50	7.2	5.2	3.7	1.1	5.6	3.4	2.6	2.4	7.0	4.8	4.0	1.2	5.2	3.0	3.0	2.0
K	18	6.0	4.6	3.8	2.4	3.0	1.1	0.3	0.0	4.0	2.1	0.0	0.0	1.8	1.1	0.0	0.0
	28	11.6	9.7	8.8	6.7	9.0	7.0	6.4	5.0	11.3	9.4	8.5	6.3	7.8	5.9	5.1	1.7
	40	18.0	15.8	9.6	8.3	14.0	11.8	9.4	9.6	18.1	15.9	9.6	8.3	13.9	11.6	9.5	8.7
M	18	23.0	21.4	10.6	8.9	7.5	6.0			22.7	21.1	10.6	8.4	7.4	5.7		
R	18	4.4	2.4	2.3	3.2					4.1	2.2	2.1	3.0				
	28	9.6	7.6	7.6	8.4					8.8	6.4						
	40	13.9	10.6							10.6	7.9						
	50	14.7	12.1							12.7	10.0						
U	18	8.3	6.2	5.5	3.0	1.5	0.3	0.0	0.0	6.4	4.8	4.4	1.9	0.9	0.0	0.0	0.0
	28	12.6	10.7	10.5	7.1	3.9	2.2	1.6	1.2	13.7	11.7	10.3	8.2	3.8	1.9		
	40	20.5	18.3	10.5	8.9	8.0	5.6	4.9	3.8	20.6	18.3	10.5	8.9	5.7	3.6		

TABLE XIII. Comparison of unconstrained and constrained diagonal approximants for test functions with confluent singularities. The quoted values of the confluent critical exponents γ_1,γ_2 were computed from (2.33).

Test function	n	z_1	γ_1	γ_2	A_1	A_2	A_0	z_3	γ_3	A_3	$A_{3,0}$
					Unconstrained sequence						
L	18	2.5	1.8	0.0	0.5	0.0					
	28	5.0	2.3	2.3	2.1	0.8		1.1	0.0		
	40	5.8	2.3	2.1	2.0	0.4		2.4	0.0		
	50	6.4	2.9	1.8	2.2	1.2		1.8	0.0		
P	18	4.5	0.0	0.0	0.0	0.0	0.0	2.6	0.0		
	28	5.6	3.0	1.0	2.0	0.3	0.0	3.0	2.8		
	40	6.0	2.0	1.2	2.2	0.0	0.0	3.8	3.6		
	50	6.3	2.0	2.0	2.3	0.0	0.0	6.3	6.2		
Q	18	4.2	0.0	0.0	0.0	0.0	0.0	3.6	2.3	1.8	3.8
	28	5.2	2.2	1.1	1.8	0.4	0.3	7.2	5.3	4.6	7.5
	40	5.6	4.6	3.3	2.8	0.0	0.0	13.0	10.9	9.7	13.1
	50	9.9	8.2	5.8	2.8	0.0	0.0	16.6	14.5	11.5	13.2
T	18	4.0	3.6	0.0	2.7			2.9	0.0		
	28	6.5	3.8	0.0	4.8			3.8	0.0		
	40	8.0	4.6	4.7	5.0			7.7	4.6		
	50	8.4	5.1	5.1	5.0			7.6	4.2		
V	18	3.7	1.6	0.9	0.0	0.0	0.0	3.0	1.2	1.0	0.0
	28	4.2	3.3	2.5				6.7	4.6		
	40	6.5	5.0	3.8				10.0	7.6	7.1	5.6
	50	8.1	6.6	5.3	4.9	4.4	2.4	10.0	7.6	7.2	5.8
W	18	3.5	0.0	0.0	0.0	0.0	0.0	3.3	2.0	1.3	3.5
	28	4.1	4.5	2.4	2.1	2.4	0.4	6.3	4.7	4.8	6.7
	40	5.3	5.5	4.0	3.4	3.2	1.3	9.0	6.8	6.4	9.2
	50	8.6	7.1	5.9	3.7	4.0	3.4	13.1	10.8		

Test function	n	z_1	γ_1	γ_2	A_1	A_2	A_0	z_3	γ_3	A_3	$A_{3,0}$
					Constrained sequence						
L	18	3.8	0.0	0.0				0.0	0.0		
	28	5.2	1.0	0.7	1.3	0.3		0.0	0.0		
	40	6.2	2.9	1.6				1.2	0.0		
	50	6.7	2.8	1.8	2.8	0.0		2.0	0.0		
P	18	4.4	0.0	0.0	0.0	0.0	0.0	3.7	0.0		
	28	5.6	3.0	1.0	2.0	0.0	0.0	3.9	2.6		
	40	6.0	2.0	1.2	2.2	0.0	0.0	3.9	2.0		
	50	6.0	2.1	2.0	2.3	0.6	0.0	4.0	2.7		
Q	18	3.9	0.0	0.0	0.0	0.0	0.0	4.5	3.3	2.8	4.7
	28	4.6	1.9	1.4	1.5	0.3	0.8	8.0	6.3	5.6	8.3
	40	5.5	4.2	3.0	2.8	0.2	0.0	12.7	10.6	9.8	12.8
	50	6.7	8.0	6.0	2.8	0.0	0.0	17.0	14.5	12.8	12.8
T	18	5.2	0.0	0.0	0.0			0.0	0.0		
	28	6.7	4.4	0.0	4.9			3.8	0.0		
	40	5.0	1.8	1.9	3.7			4.7	2.3		
	50	5.0	2.0	0.6	3.0			2.8	1.0		
V	18	3.8	0.0	0.0	0.0	0.0	0.0	4.0	2.4	2.2	1.0
	28	4.2	3.3	2.5				6.7	4.6	4.3	3.0
	40	5.8	4.4	3.1				10.6	8.2		
	50	7.9	6.1	5.0	4.8	4.1	2.2	9.9	7.5	7.0	5.7
W	18	3.5	0.0	0.0	0.0	0.0	0.0	3.1	1.9	1.6	3.3
	28	4.4	2.5	1.8	1.6	1.3	0.3	5.4	3.5		
	40	5.3	4.9	4.0	3.4	3.4	3.5	7.7	5.6		
	50	8.1	6.6	5.4	3.7	3.9	2.8	13.4	10.9		

quences is very good, and that of the diagonal sequences even better. Our results for these cases are given in Tables IX and X.

In Table XI we compare the accuracy of the contour method and the approximate formula (2.33) for the two exponents at the confluent singularity for functions L, P, Q, V, and W as well as for T, which has a pseudoconfluent singularity. Results are generally similar but the contour method is significantly better for function T.

In Table XII we compare the results obtained from unconstrained and constrained diagonal approximants for the functions D,E,H^*,K,M,R, and U. As pointed out in Sec. III imposition of the constraint (3.3) makes the approximants form invariant with respect to Euler transformations. For the constrained approximants sequence we have studied not only the invariant but also some nearly invariant approximants, i.e., $[N+n/N+m;N+\dot{3};N+4]$ with $n,m=0,\pm 1$. There is little appreciable difference between constrained and unconstrained approximants for the functions. Table XIII presents the corresponding results for the confluent or pseudoconfluent functions L, P, Q, T, V, and W. The function W is regular at $z=\infty$ and one might suppose that it would benefit from the use of the constraint, which puts $z=\infty$ on a par with any other regular point. Nevertheless, neither W nor any of the other functions show better convergence for the constrained sequences than for the unconstrained. Baker[14] has made some comparisons of the results with those of other methods.

A number of general conclusions can be stated, based on our analysis of the test functions. (i) Integral approximants can be highly successful in the analysis of power series for quite complicated functions. (ii) In most cases diagonal sequences appear to converge more rapidly than horizontal sequences, although horizontal sequences do as well or better for a few of the test functions. (iii) A comparison between constrained and unconstrained diagonal sequences reveals no systematic differences. Therefore we can conclude that the constraint is of no practical value. (iv) For functions with confluent singularities our contour method often gives better results than the approximate formula, although in most cases the difference is not great. We find that the best estimate of the critical point is from the singularity with the dominant exponent.

We have also compared the effectiveness of our method with the recurrence relation method of Rehr, Joyce, and Guttmann.[2] In most cases the Hermite-Padé approximants do somewhat better, but again the differences are not great.

We are currently using the method developed in this paper to analyze some of the series which have been derived for various lattice models in critical phenomena. This will be reported in a future paper.

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