Nonlinear theory of a correlated emission laser

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A nonlinear theory of correlated emission lasers is presented. It is shown that if three-level atoms in the V configuration are injected in a coherent superposition of the upper two states in a doubly resonant cavity, the diffusion coefficient for the relative phase vanishes under certain conditions.

I. INTRODUCTION

It is well known that the natural linewidth of a laser arises due to spontaneous-emission fluctuations.[1] The quantum noise is undesirable particularly in high-precision experiments where small changes of a physical quantity are converted into a phase shift (passive scheme) or frequency shift (active scheme). The procedure followed in such measurements is to place a lasing medium or sending laser light in a cavity whose optical path length is sensitive to the physical effect to be measured. The shift is then determined by heterodyning the light from this laser with that from a reference laser. Examples include gravitational wave detection,[2] laser gyroscope,[3] measurement of thermal expansion coefficients,[4] and tests of metric theories of gravitation.[5] Because of its potential applications to the above-mentioned problems, the idea of correlated spontaneous emission has drawn a great deal of interest.

Recently Scully showed that, in a correlated emission laser (CEL), it is possible to eliminate the quantum noise in the beat note by correlating the spontaneous emission events of two laser modes generated from three-level atoms inside a doubly resonant cavity.[6] It was shown that in a doubly resonant cavity, if the atoms are excited coherently to the upper two states, the spontaneous-emission events are strongly correlated under certain conditions which is indeed the case in the quantum-beat experiments and Hanle effect experiments.[7,8] In Ref. 9 a linear theory of the quantum-beat laser was given and it was shown that the relative phase-diffusion coefficient vanishes under certain detuning conditions.

The results of that paper can be summarized as follows. Consider the atomic system in Fig. 1. If the atoms are prepared in a coherent superposition of $|a\rangle$ and $|b\rangle$ via an external driving field ν_3, the difference of the corresponding phase $\phi_a - \phi_b$ is constant. The random phase ϕ_c cancels from the beat signal of the two spontaneously emitted fields thus leading to a nonfluctuating contribution to the beat note of the lasing modes. The physical condition under which the noise quenching occurs is that the field detunings from the corresponding atomic lines are equal to half the Rabi frequency of the driving field and they are much larger than the atomic decay constants. Subsequently a nonlinear theory of the quantum-beat laser was formulated.[10] In this paper the strong coupling of the upper states is taken to all orders and starting with the conditions derived in Ref. 9, it is shown that the quenching of the relative phase noise persists even above threshold.

However, instead of using the strong microwave signal to couple the upper two levels, one could pump the atoms in a coherent superposition of the upper levels to quench the relative phase noise between the two modes as, e.g., in the Hanle laser.[6] In another scheme, correlated emission could be achieved in a ring cavity through spatial modulation of the gain medium.[11]

In this paper we present a nonlinear theory of CEL. We follow the first scheme, i.e., we consider three-level atoms being pumped to a coherent superposition of upper two levels inside a doubly resonant cavity. Unlike Ref. 10, we do not impose any *a priori* conditions. In the resulting diffusion coefficient, which is complicated, various parameters such as gain coefficients, coupling constants, detuning, etc., can be chosen arbitrarily. In Sec. II we derive an equation of motion for the element of the reduced density matrix for the field modes. In Sec. III we calculate the diffusion coefficient for the relative phase angle between the two modes. Conditions can then be derived for the gain coefficients, coupling constants, detuning, and decay rate under which the diffusion coefficient vanishes. We also discuss one particular condition under which this coefficient vanishes completely.

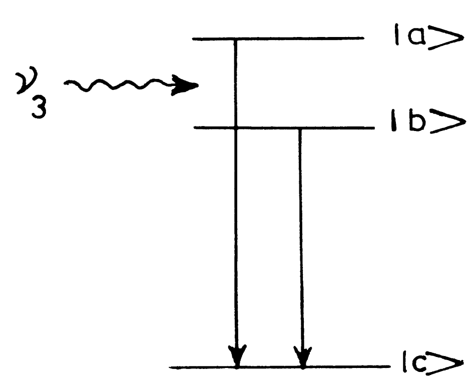

FIG. 1. Energy-level diagram for quantum-beat laser.

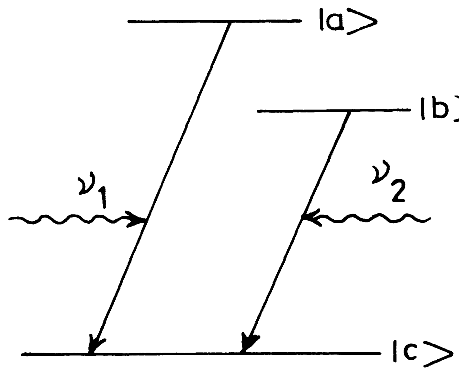

FIG. 2. Energy-level diagram for CEL.

II. EQUATION OF MOTION FOR REDUCED DENSITY MATRIX FOR THE FIELD MODES

We consider the system of three-level atomic system shown in Fig. 2 inside a doubly resonant cavity interacting with a two-mode field being pumped in a coherent superposition of the upper two levels at rate r. The Hamiltonian for the system is

$$H=H_0+V\ , \tag{1}$$

where

$$H_0=\sum_i \hbar\omega_i\,|\,i\,\rangle\langle\,i\,|+\hbar\nu_1 a_1^\dagger a_1+\hbar\nu_2 a_2^\dagger a_2\ , \tag{2}$$

$$V=\hbar(\,g_1 a_1\,|\,a\,\rangle\langle\,c\,|+g_1^* a_1^\dagger\,|\,c\,\rangle\langle\,a\,|+g_2 a_2\,|\,b\,\rangle\langle\,c\,| +g_2^* a_2^\dagger\,|\,c\,\rangle\langle\,b\,|\,)\ . \tag{3}$$

Here $|\,a\,\rangle$ and $|\,b\,\rangle$ are the upper two levels and $|\,c\,\rangle$ is the ground level, g_1 and g_2 are the coupling constants for the transitions $|\,a\,\rangle\rightarrow|\,c\,\rangle$ and $|\,b\,\rangle\rightarrow|\,c\,\rangle$, respectively, and a_1 and a_2 ($a_1^\dagger$ and $a_2^\dagger$) are the destruction (creation) operators for the photons in the modes 1 and 2, respectively. The wave function $|\,\psi\rangle$ can be written as

$$|\,\psi\rangle=\sum_{n_1,n_2}(\,C_{a,n_1,n_2}\,|\,a,n_1,n_2\,\rangle+C_{b,n_1,n_2}\,|\,b,n_1,n_2\,\rangle +C_{c,n_1,n_2}\,|\,c,n_1,n_2\,\rangle)\ . \tag{4}$$

In our scheme the only states that couple together are the states 1, 2, and 3 defined as

$$|\,1\,\rangle=|\,a,n_1-1,n_2\,\rangle\ , \tag{5a}$$

$$|\,2\,\rangle=|\,b,n_1,n_2-1\,\rangle\ , \tag{5b}$$

$$|\,3\,\rangle=|\,c,n_1,n_2\,\rangle\ . \tag{5c}$$

The matrix element of the reduced density matrix for the field ρ_F is obtained by taking a trace of the atom-field density matrix over atomic variables, i.e.,

$$\langle\,n_1,n_2\,|\,\rho_F\,|\,n_1',n_2'\,\rangle =[\rho_{11'}]_{n_1\rightarrow n_1+1,n_1'\rightarrow n_1'+1} +[\rho_{22'}]_{n_2\rightarrow n_2+1,n_2'\rightarrow n_2'+1}+\rho_{33'}\ . \tag{6}$$

The Schrödinger equation for the matrix element $\langle\,n_1,n_2\,|\,\rho_F\,|\,n_1',n_2'\,\rangle$ is therefore

$$\langle\,n_1,n_2\,|\,\dot\rho_F\,|\,n_1',n_2'\,\rangle=-i/\hbar(\,V_{13}\rho_{31'}-\rho_{13'}V_{3'1'}\,)_{n_1\rightarrow n_1+1,n_1'\rightarrow n_1'+1} -i/\hbar(\,V_{23}\rho_{32'}-\rho_{23'}V_{3'2'}\,)_{n_2\rightarrow n_2+1,n_2'\rightarrow n_2'+1}-i/\hbar(\,V_{31}\rho_{13'}+V_{32}\rho_{23'}-\rho_{31'}V_{1'3'}-\rho_{32'}V_{2'3'}\,)\ . \tag{7}$$

In order to evaluate $\rho_{13'},\rho_{31'},\rho_{23'}\rho_{32'}$, we start with the Schrödinger equation

$$|\,\dot\psi(t)\,\rangle_I=-i/\hbar V_I\,|\,\psi(t)\,\rangle_I\ , \tag{8}$$

which gives

$$\dot C_{a,n_1-1,n_2}=-ig_1\sqrt{n_1}e^{i\Delta t}C_{c,n_1,n_2}\ , \tag{9a}$$

$$\dot C_{b,n_1,n_2-1}=-ig_2\sqrt{n_2}e^{i\Delta t}C_{c,n_1,n_2}\ , \tag{9b}$$

$$\dot C_{c,n_1,n_2}=-i(g_1^*\sqrt{n_1}C_{a,n_1-1,n_2}+g_2^*\sqrt{n_2}C_{b,n_1,n_2-1})e^{-i\Delta t}\ . \tag{9c}$$

For simplicity, we have taken the detunings to be equal, i.e.,

$$\Delta_1=\Delta_2=\Delta\ ,$$

where

$$\Delta_1=\omega_a-\omega_c-\nu_1\ , \tag{10a}$$

$$\Delta_2=\omega_b-\omega_c-\nu_2\ . \tag{10b}$$

Since we assume the initial atomic state to be a superposition of upper two levels, the initial state of the system is

$(C_a|a\rangle + C_b|b\rangle)$, where C_a and C_b are the atomic probability amplitudes associated with the levels $|a\rangle$ and $|b\rangle$. On solving the set of equations (9a)–(9c) for an initial time t_0 and including the level decays in the usual way, we obtain

$$C_{a,n_1-1,n_2}(t)=\frac{e^{-1/2(\gamma+i\Delta)(t-t_0)}}{|g_1|^2n_1+|g_2|^2n_2}\times[(|g_2|^2n_2e^{-(i\Delta/2)(t-t_0)}+|g_1|^2n_1\{\cos[\beta(t-t_0)]-(i\Delta/2\beta)\sin[\beta(t-t_0)]\})\times C_{a,n_1-1,n_2}(t_0)-g_1g_2^*\sqrt{n_1n_2}(e^{-(i\Delta/2)(t-t_0)}-\{\cos[\beta(t-t_0)]-(i\Delta/2\beta)\sin[\beta(t-t_0)]\})C_{b,n_1,n_2-1}(t_0)], \quad (11a)$$

$$C_{b,n_1,n_2-1}(t)=\frac{e^{-1/2(\gamma+i\Delta)(t-t_0)}}{|g_1|^2n_1+|g_2|^2n_2}\times[-g_1^*g_2\sqrt{n_1n_2}(e^{-(i\Delta/2)(t-t_0)}-\{\cos[\beta(t-t_0)]-(i\Delta/2\beta)\sin[\beta(t-t_0)]\})\times C_{a,n_1-1,n_2}(t_0)+(|g_1|^2n_1e^{-(i\Delta/2)(t-t_0)}+|g_2|^2n_2\{\cos[\beta(t-t_0)]-(i\Delta/2\beta)\sin[\beta(t-t_0)]\})C_{b,n_1,n_2-1}(t_0)], \quad (11b)$$

$$C_{c,n_1,n_2}(t)=-\frac{i\sin[\beta(t-t_0)]}{\beta}e^{-1/2(\gamma+i\Delta)(t-t_0)}[g_1^*\sqrt{n_1}C_{a,n_1-1,n_2}(t_0)+g_2^*\sqrt{n_2}C_{b,n_1,n_2-1}(t_0)], \quad (11c)$$

where

$$\beta=(\Delta^2/4+|g_1|^2n_1+|g_2|^2n_2)^{1/2}. \quad (12)$$

For simplicity, we have taken the decay rates for the three levels to be equal, i.e., $\gamma_a=\gamma_b=\gamma_c=\gamma$.

We can now determine $\rho_{13'}$ by summing the contribution $C_{a,n_1-1,n_2}(t)C^*_{c,n'_1,n'_2}(t)$ of all atoms which are injected at random initial times t at rate r in a coherent superposition of the upper levels, i.e.,

$$\rho_{13'}=r\int_{-\infty}^{t_0}dt_0C_{a,n_1-1,n_2}(t)C^*_{c,n'_1,n'_2}(t). \quad (13)$$

On substitution from Eqs. (11a) and (11c) we obtain

$$\rho_{13'}=\frac{ir}{|g_1|^2n_1+|g_2|^2n_2}\left[g_1\sqrt{n'_1}\left[\frac{|g_2|^2n_2}{L_{n_1,n_2}}+\frac{|g_1|^2n_1M_{n_1,n_2}}{D_{n_1,n_2}}\right]|C_a|^2\langle n_1-1,n_2|\rho_F|n'_1-1,n'_2\rangle -g_1^2g_2^*\sqrt{n_1n'_1n_2}\left[\frac{1}{L_{n_1,n_2}}-\frac{M_{n_1,n_2}}{D_{n_1,n_2}}\right]C_a^*C_b\langle n_1,n_2-1|\rho_F|n'_1-1,n_2\rangle +g_2\sqrt{n'_2}\left[\frac{|g_2|^2n_2}{L_{n_1,n_2}}+\frac{|g_1|^2n_1M_{n_1,n_2}}{D_{n_1,n_2}}\right]C_aC_b^*\langle n_1-1,n_2|\rho_F|n'_1,n'_2-1\rangle -g_1|g_2|^2\sqrt{n_1n_2n'_2}\left[\frac{1}{L_{n_1,n_2}}-\frac{M_{n_1,n_2}}{D_{n_1,n_2}}\right]|C_b|^2\langle n_1,n_2-1|\rho_F|n'_1,n'_2-1\rangle\right], \quad (14)$$

where

$$M_{n_1,n_2}=\gamma^2-i\gamma\Delta-|g_1|^2(n_1-n'_1)-|g_2|^2(n_2-n'_2), \quad (15)$$

$$L_{n_1,n_2}=\gamma^2+i\gamma\Delta+|g_1|^2n'_1+|g_2|^2n'_2, \quad (16)$$

$$D_{n_1,n_2}=\gamma^4+\gamma^2\Delta^2+2\gamma^2[|g_1|^2(n_1+n'_1)+|g_2|^2(n_2+n'_2)]+[|g_1|^2(n_1-n'_1)+|g_2|^2(n_2-n'_2)]^2. \quad (17)$$

Similarly,

$$\rho_{23'}=\frac{ir}{|g_1|^2n_1+|g_2|^2n_2}\left[-|g_1|^2g_2\sqrt{n_1n_1'n_2}\left[\frac{1}{L_{n_1,n_2}}-\frac{M_{n_1,n_2}}{D_{n_1,n_2}}\right]|C_a|^2\langle n_1-1,n_2|\rho_F|n_1'-1,n_2'\rangle\right.$$
$$+g_1\sqrt{n_1'}\left[\frac{|g_1|^2n_1}{L_{n_1,n_2}}+\frac{|g_2|^2n_2M_{n_1,n_2}}{D_{n_1,n_2}}\right]C_a^*C_b\langle n_1,n_2-1|\rho_F|n_1'-1,n_2'\rangle$$
$$-g_1^*g_2^2\sqrt{n_1n_2n_2'}\left[\frac{1}{L_{n_1,n_2}}-\frac{M_{n_1,n_2}}{D_{n_1,n_2}}\right]C_aC_b^*\langle n_1-1,n_2|\rho_F|n_1',n_2'-1\rangle$$
$$\left.+g_2\sqrt{n_2'}\left[\frac{|g_1|^2n_1}{L_{n_1,n_2}}+\frac{|g_2|^2n_2M_{n_1,n_2}}{D_{n_1,n_2}}\right]|C_b|^2\langle n_1,n_2-1|\rho_F|n_1',n_2'-1\rangle\right]. \quad (18)$$

Also

$$\rho_{31'}=[\rho_{13'}]^*_{n_1\leftrightarrow n_1',n_2\leftrightarrow n_2'}, \quad (19)$$

$$\rho_{32'}=[\rho_{23'}]^*_{n_1\leftrightarrow n_1',n_2\leftrightarrow n_2'}, \quad (20)$$

Henceforth, a complex conjugate would also imply the interchange $n_1\leftrightarrow n_1'$ and $n_2\leftrightarrow n_2'$. On substitution from Eqs. (14)–(20) in Eq. (7) we obtain

$$\langle n_1,n_2|\dot\rho_F|n_1',n_2'\rangle$$
$$=\left\{-r\left[\frac{|g_1|^2}{D_{n_1+1,n_2}}\left[\tfrac{1}{2}|g_1|^2(n_1-n_1')^2+\tfrac{1}{2}\gamma^2(n_1+n_1'+2)+i\gamma\Delta n_1\right.\right.\right.$$
$$\left.+|g_2|^2(n_1+1)n_2+|g_2|^2(n_1'+1)n_2\frac{N_{n_1+1,n_2}}{L_{n_1+1,n_2}}\right]|C_a|^2$$
$$+\frac{|g_2|^2}{D_{n_1,n_2+1}}\left[\tfrac{1}{2}|g_2|^2(n_2-n_2')^2+\tfrac{1}{2}\gamma^2(n_2+n_2'+2)+i\gamma\Delta n_2\right.$$
$$\left.\left.+|g_1|^2(n_2+1)n_1+|g_1|^2(n_2'+1)n_1\frac{N_{n_1,n_2+1}}{L_{n_1,n_2+1}}\right]|C_b|^2\right]\langle n_1,n_2|\rho_F|n_1',n_2'\rangle$$
$$+\frac{r}{D_{n_1+1,n_2}}\left[g_1g_2^*\sqrt{(n_1+1)n_2}\left[2|g_1|^2(n_1'+1)-[\gamma^2+i\gamma\Delta+|g_1|^2(n_1+1)\right.\right.$$
$$\left.+|g_2|^2n_2]+|g_1|^2(n_1'+1)\frac{N_{n_1+1,n_2}}{L_{n_1+1,n_2}}-|g_2|^2n_2'\frac{N^*_{n_1+1,n_2}}{L^*_{n_1+1,n_2}}\right]$$
$$\times C_a^*C_b\langle n_1+1,n_2-1|\rho_F|n_1',n_2'\rangle$$
$$+|g_1|^2|g_2|^2\sqrt{(n_1+1)(n_1'+1)n_2n_2'}\left[1+\frac{N_{n_1+1,n_2}}{L_{n_1+1,n_2}}\right]|C_b|^2$$
$$\left.\times\langle n_1+1,n_2-1|\rho_F|n_1'+1,n_2'-1\rangle\right]$$

$$+\frac{r}{D_{n_1,n_2+1}}\left[g_1^*g_2\sqrt{n_1(n_2+1)}\left[2|g_2|^2(n_2'+1)-[\gamma^2+i\gamma\Delta+|g_1|^2n_1+|g_2|^2(n_2+1)]\right.\right.$$

$$\left.-|g_1|^2n_1'\frac{N^*_{n_1,n_2+1}}{L^*_{n_1,n_2+1}}+|g_2|^2(n_2'+1)\frac{N_{n_1,n_2+1}}{L_{n_1,n_2+1}}\right]C_aC_b^*$$

$$\times\langle n_1-1,n_2+1|\rho_F|n_1',n_2'\rangle$$

$$+|g_1|^2|g_2|^2\sqrt{n_1n_1'(n_2+1)(n_2'+1)}\left[1+\frac{N_{n_1,n_2+1}}{L_{n_1,n_2+1}}\right]|C_a|^2$$

$$\left.\times\langle n_1-1,n_2+1|\rho_F|n_1'-1,n_2'+1\rangle\right]$$

$$+\frac{\gamma^2r}{D_{n_1,n_2}}(|g_1|^2\sqrt{n_1n_1'}\;|C_a|^2\langle n_1-1,n_2|\rho_F|n_1'-1,n_2'\rangle$$

$$+g_1^*g_2\sqrt{n_1n_2'}C_aC_b^*\langle n_1-1,n_2|\rho_F|n_1',n_2'-1\rangle$$

$$+g_1g_2^*\sqrt{n_1'n_2}C_a^*C_b\langle n_1,n_2-1|\rho_F|n_1'-1,n_2'\rangle$$

$$\left.+|g_2|^2\sqrt{n_2n_2'}\;|C_b|^2\langle n_1,n_2-1|\rho_F|n_1',n_2'-1\rangle)\right\}+\text{c.c.}\;, \tag{21}$$

with

$$N_{n_1,n_2}=2\gamma^2+|g_1|^2(n_1-2n_1')+|g_2|^2(n_2-2n_2')\;. \tag{22}$$

In the above equation wherever $n_1(n_2)$ is shifted it is implicit that $n_1'(n_2')$ is also shifted.

Equation (21) gives the time evolution of the elements of the reduced density matrix for the field modes 1 and 2 in the gain mechanism. In addition, the following terms should be included which correspond to the cavity losses:

$$\langle n_1,n_2|\dot\rho_F|n_1',n_2'\rangle_{\text{loss}}=-\tfrac{1}{2}[C_1(n_1+n_1')+C_2(n_2+n_2')]\langle n_1,n_2|\rho_F|n_1',n_2'\rangle$$

$$+C_1[(n_1+1)(n_1'+1)]^{1/2}\langle n_1+1,n_2|\rho_F|n_1'+1,n_2\rangle$$

$$+C_2[(n_2+1)(n_2'+1)]^{1/2}\langle n_1,n_2+1|\rho_F|n_1',n_2'+1\rangle\;, \tag{23}$$

where C_1 and C_2 are the loss coefficients for modes 1 and 2, respectively. As would be seen in Sec. III, the cavity loss terms do not contribute to the diffusion constant of the relative phase angle.

III. DIFFUSION CONSTANT FOR THE RELATIVE PHASE

We can now derive the Fokker-Planck equation for the coherent state representation for the field $P(\alpha_1,\alpha_2)$ which is defined by

$$\rho_F=\int d^2\alpha_1\int d^2\alpha_2P(\alpha_1,\alpha_2)|\alpha_1,\alpha_2\rangle\langle\alpha_1,\alpha_2|\;. \tag{24}$$

Here $|\alpha_1,\alpha_2\rangle$ is a coherent state which is an eigenstate of the destruction operators a_1,a_2 with eigenvalues α_1 and α_2, respectively. α_1 and α_2 are complex numbers.

Since we are interested in the relative phase angle between the two modes, we define

$$\alpha_1=r_1e^{i\theta_1},\quad \alpha_2=r_2e^{i\theta_2}\;, \tag{25a}$$

with $r_i^2=\bar n_i$ $(i=1,2)$ as the mean number of photons in a particular mode and

$$\theta=\theta_1-\theta_2\;,\qquad \mu=\frac{\theta_1+\theta_2}{2}\;. \tag{25b}$$

Here we assume that the mean number of photons in mode 1 and 2 is very large and hence amplitude fluctuations can be neglected. We also assume that the atoms have initially been prepared in a coherent superposition of the upper two states, i.e.,

$$C_a=\frac{1}{\sqrt2}e^{i\phi},\quad C_b=\frac{1}{\sqrt2}e^{-i\phi}\;. \tag{26}$$

Transforming the Fokker-Planck equation into new variables is somewhat lengthy but straightforward. Apart from other terms, the terms that interest us are $D(\theta)$, $D(\mu,\theta)$, and $D(\mu)$ which are the coefficients of the second-order derivatives $\partial^2 P/\partial^2\theta$, $\partial^2 P/\partial\mu\partial\theta$, and $\partial^2 P/\partial^2\mu$, respectively. Following the same procedure as in Ref. 9, we obtain the following values for the above-mentioned coefficients:

$$D(\theta)=D_1(\theta)+D_2(\theta)\ , \tag{27a}$$

$$D(\mu,\theta)=D_1(\mu,\theta)+D_2(\mu,\theta)\ , \tag{27b}$$

$$D(\mu)=D_1(\mu)+D_2(\mu)\ , \tag{27c}$$

with

$$\begin{aligned} D_1(\theta)=\frac{1/16}{(\gamma^2+\Delta^2+4|g_1|^2\bar{n}_1+4|g_2|^2\bar{n}_2)}\Bigg\{ & \frac{A_{11}|g_2|^2}{\bar{n}_1\bar{n}_2}(\bar{n}_1-\bar{n}_2)^2+2(A_{11}|g_1|^2+A_{22}|g_2|^2) \\ & +\gamma^2\left[\left[\frac{A_{11}}{\bar{n}_1}+\frac{A_{22}}{\bar{n}_2}\right]-2\frac{A_{12}}{\sqrt{\bar{n}_1\bar{n}_2}}e^{i(2\phi-\theta)}\right] \\ & +\frac{A_{12}}{\sqrt{\bar{n}_1\bar{n}_2}}e^{i(2\phi-\theta)}[|g_1|^2(\bar{n}_2-3\bar{n}_1)+|g_2|^2(\bar{n}_1-3\bar{n}_2)]\Bigg\}+\text{c.c.}\ , \end{aligned} \tag{28a}$$

$$\begin{aligned} D_2(\theta)= & \frac{-1/16}{(\gamma^2+\Delta^2+4|g_1|^2\bar{n}_1+4|g_2|^2\bar{n}_2)(\gamma^2-i\gamma\Delta+|g_1|^2\bar{n}_1+|g_2|^2\bar{n}_2)} \\ & \times\Bigg\{\frac{A_{11}|g_2|^2}{\bar{n}_1\bar{n}_2}\times[|g_1|^2(\bar{n}_1^3+2\bar{n}_1^2\bar{n}_2+\bar{n}_1\bar{n}_2^2-\bar{n}_1^2-\bar{n}_1\bar{n}_2+\bar{n}_2^2) \\ & \qquad +|g_2|^2(\bar{n}_2^3+2\bar{n}_1\bar{n}_2^2+\bar{n}_1^2\bar{n}_2+\bar{n}_1^2-\bar{n}_1\bar{n}_2-\bar{n}_2^2)-2\gamma^2(\bar{n}_1+\bar{n}_2)^2] \\ & -A_{12}\frac{|g_1|^2}{\sqrt{\bar{n}_1\bar{n}_2}}e^{i(2\phi-\theta)}[|g_1|^2(4\bar{n}_1^2+4\bar{n}_1\bar{n}_2+\bar{n}_1+6\bar{n}_2)-|g_2|^2(3\bar{n}_1^2+2\bar{n}_1\bar{n}_2-\bar{n}_2^2+3\bar{n}_1+5\bar{n}_2) \\ & \qquad -2\gamma^2(\bar{n}_1+\bar{n}_2)] \\ & +A_{21}|g_2|^2e^{-i(2\phi-\theta)}[|g_1|^2(3\bar{n}_2+2\bar{n}_1\bar{n}_2-\bar{n}_1^2+5\bar{n}_1+3\bar{n}_2)-|g_2|^2(4\bar{n}_2^2+4\bar{n}_1\bar{n}_2+6\bar{n}_1+\bar{n}_2) \\ & \qquad +2\gamma^2(\bar{n}_1+\bar{n}_2)]\Bigg\}+\text{c.c.}\ , \end{aligned} \tag{28b}$$

$$\begin{aligned} D_1(\mu,\theta)=\frac{1/16}{[\gamma^2+\Delta^2+4|g_1|^2\bar{n}_1+4|g_2|^2\bar{n}_2]}\Bigg[& \frac{A_{11}|g_2|^2}{\bar{n}_1\bar{n}_2}(\bar{n}_2^2-\bar{n}_1^2)+2(A_{11}|g_1|^2-A_{22}|g_2|^2) \\ & +\frac{A_{12}}{\sqrt{\bar{n}_1\bar{n}_2}}e^{i(2\phi-\theta)}(|g_1|^2\bar{n}_2-|g_2|^2\bar{n}_1) \\ & +\gamma^2\left[\frac{A_{11}}{\bar{n}_1}-\frac{A_{22}}{\bar{n}_2}\right]\Bigg]+\text{c.c.}\ , \end{aligned} \tag{29a}$$

$$\begin{aligned} D_2(\mu,\theta)= & \frac{1/16}{(\gamma^2+\Delta^2+4|g_1|^2\bar{n}_1+4|g_2|^2\bar{n}_2)(\gamma^2-i\gamma\Delta+|g_1|^2\bar{n}_1+|g_2|^2\bar{n}_2)} \\ & \times\Bigg[\frac{A_{11}|g_2|^2}{\bar{n}_1\bar{n}_2}[|g_1|^2(\bar{n}_1^3-\bar{n}_1\bar{n}_2^2-\bar{n}_1^2+3\bar{n}_1\bar{n}_2-\bar{n}_2^2)+|g_2|^2(\bar{n}_1^2\bar{n}_2-\bar{n}_2^3+\bar{n}_1^2+\bar{n}_2^2-3\bar{n}_1\bar{n}_2) \\ & \qquad +2\gamma^2(\bar{n}_2^2-\bar{n}_1^2)] \\ & +\frac{A_{12}}{\sqrt{\bar{n}_1\bar{n}_2}}|g_1|^2e^{i(2\phi-\theta)}[|g_1|^2(4\bar{n}_1\bar{n}_2+6\bar{n}_2)+|g_2|^2(3\bar{n}_1^2+\bar{n}_2^2-2\bar{n}_2)-2\gamma^2\bar{n}_2] \\ & -\frac{A_{21}}{\sqrt{\bar{n}_1\bar{n}_2}}|g_2|^2e^{-i(2\phi-\theta)}[|g_1|^2(\bar{n}_1^2+3\bar{n}_2^2-2\bar{n}_1)+|g_2|^2(4\bar{n}_1\bar{n}_2+6\bar{n}_1)-2\gamma^2\bar{n}_1]\Bigg]+\text{c.c.}\ , \end{aligned} \tag{29b}$$

$$D_1(\mu)=\frac{1/64}{(\gamma^2+\Delta^2+4|g_1|^2\bar{n}_1+4|g_2|^2\bar{n}_2)}$$
$$\times\left\{\frac{A_{11}|g_2|^2}{\bar{n}_1\bar{n}_2}(\bar{n}_1+\bar{n}_2)^2+2(A_{11}|g_1|^2+A_{22}|g_2|^2)+\gamma^2\left[\left(\frac{A_{11}}{\bar{n}_1}+\frac{A_{22}}{\bar{n}_2}\right)+\frac{2A_{12}}{\sqrt{\bar{n}_1\bar{n}_2}}e^{i(2\phi-\theta)}\right]\right.$$
$$\left.+\frac{A_{12}}{\sqrt{\bar{n}_1\bar{n}_2}}e^{i(2\phi-\theta)}[|g_1|^2(3\bar{n}_1+\bar{n}_2)+|g_2|^2(\bar{n}_1+3\bar{n}_2)]\right\}+\text{c.c.}\ , \tag{30a}$$

$$D_2(\mu)=\frac{1/64}{(\gamma^2+\Delta^2+4|g_1|^2\bar{n}_1+4|g_2|^2\bar{n}_2)(\gamma^2-i\gamma\Delta+|g_1|^2\bar{n}_1+|g_2|^2\bar{n}_2)}$$
$$\times\left[\frac{A_{11}|g_2|^2}{\bar{n}_1\bar{n}_2}[|g_1|^2(2\bar{n}_1^2\bar{n}_2-\bar{n}_1\bar{n}_2^2-\bar{n}_1^3+5\bar{n}_1\bar{n}_2+\bar{n}_1^2-\bar{n}_2^2)\right.$$
$$+|g_2|^2(2\bar{n}_1\bar{n}_2^2-\bar{n}_1^2\bar{n}_2-\bar{n}_2^3+5\bar{n}_1\bar{n}_2-\bar{n}_1^2+\bar{n}_2^2)+2\gamma^2(\bar{n}_1-\bar{n}_2)^2]$$
$$+\frac{A_{12}|g_1|^2}{\sqrt{\bar{n}_1\bar{n}_2}}e^{i(2\phi-\theta)}[|g_1|^2(4\bar{n}_1\bar{n}_2-4\bar{n}_1^2+6\bar{n}_2-\bar{n}_1)+|g_2|^2(\bar{n}_2^2-3\bar{n}_1^2+2\bar{n}_1\bar{n}_2+3\bar{n}_1+\bar{n}_2)$$
$$+2\gamma^2(\bar{n}_1-\bar{n}_2)]$$
$$+\frac{A_{21}|g_2|^2}{\sqrt{\bar{n}_1\bar{n}_2}}e^{-i(2\phi-\theta)}[|g_1|^2(\bar{n}_1^2-3\bar{n}_2^2+2\bar{n}_1\bar{n}_2+\bar{n}_1+3\bar{n}_2)+|g_2|^2(4\bar{n}_1\bar{n}_2-4\bar{n}_2^2+6\bar{n}_1-\bar{n}_2)$$
$$\left.+2\gamma^2(\bar{n}_2-\bar{n}_1)]\right]+\text{c.c.} \tag{30b}$$

In the above equations the gain coefficients A_{ij}'s are defined as

$$A_{ij}=\frac{2rg_ig_j^*}{\gamma^2}\ . \tag{31}$$

Note that for $g_2=0$, the above expressions reduce to the well-known results of Scully-Lamb theory. It should also be pointed out here that the cross-gain terms, i.e., the terms proportional to A_{12} and A_{21} have a phase dependence and arise due to the coherent pumping.

It is clear that when the coupling constants g_1 and g_2 are real and equal, i.e., $g_1=g_2=g$ and the cavity losses for both modes are equal, the equation of motion for the density matrix ρ_F [cf. Eq. (21)] becomes symmetric in n_1 and n_2 and hence $\bar{n}_1=\bar{n}_2=\bar{n}$. Under these conditions, the diffusion coefficient takes the form

$$D(\theta)=G[1-\cos(2\phi-\theta)]\ , \tag{32}$$

where

$$G=\frac{1/8}{(\gamma^2+\Delta^2+8|g|^2\bar{n})}$$
$$\times\left[\left(\frac{\gamma^2A}{\bar{n}}+2A|g|^2\right)\right.$$
$$\left.-A|g|^2\left(\frac{|g|^2(4\bar{n}-1)-4\gamma^2}{\gamma^2-i\gamma\Delta+2|g|^2\bar{n}}\right)\right]+\text{c.c.} \tag{33}$$

It has been shown by Schleich and Scully[12] that the solution of a general Fokker-Planck equation with the diffusion coefficient as in Eq. (32) peaks around a particular value θ_0. Hence for a particular choice of the phase $2\phi=\theta_0$, the diffusion coefficient vanishes. Under the same conditions $D(\mu,\theta)$ also vanishes. It is also possible to obtain other set of conditions under which the diffusion coefficient vanishes.

IV. CONCLUSION

In conclusion, in this paper we have presented a nonlinear theory of correlated emission laser to all orders in coupling constants. The two modes are coupled by initially preparing three level atoms in a coherent superposition of the upper two levels inside a doubly resonant cavity. We have calculated the diffusion coefficient for the relative phase angle between the two modes for arbitrary values of gain coefficients, coupling constants, detunings, and decay rates. We also discussed one of the conditions under which the diffusion coefficient vanishes, i.e., the spontaneous-emission events from the upper two levels to the lower level are strongly correlated. In this general framework, it is possible to obtain other sets of conditions on the laser parameters under which the CEL action takes place.

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