Classical inverse problem for finite scattering region

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It is shown that in classical mechanics for a finite scattering region a spherically symmetric potential can sometimes be determined from the deflection function for particles of a single energy E. Calculations for several examples such as a harmonic oscillator, a repulsive-attractive potential, and a reflection from a spherical mirror have been performed. Nonuniqueness is shown to occur in some cases.

I. INTRODUCTION

In the customary formulation of the classical scattering problem, the potential is assumed given and the deflection function (and the cross section) is calculated. In the inverse scattering problem, one assumes that the cross section (and the deflection function) is experimentally determined and the potential to be found (see Ref. 1 and the references therein). The question of how to construct the classical deflection function from the differential cross section of a fixed energy[2] is not the subject of our consideration.

The problem of determination of the potential from scattering data for an infinite scattering region has been studied in many articles and books.[1-3] For the finite scattering region Cuer[4] studied the class of transparent potentials. His examples show that in the inverse problem of fixed energy ambiguities in the classical limit exist.

In this paper we consider, in a general case, the scattering of the classical particle of energy E by a finite spherically symmetric scatterer. We suppose that the deflection function, as a function of the angle, is given and we calculate the potential inside the scattering region.

II. METHOD OF SOLUTION

Suppose that a free particle of kinetic energy E moves parallel to the x axis at a distance b (Fig. 1). When it penetrates in the spherical region of radius R it is subject to a conservative force field whose potential is V. We shall assume that V is rotationally invariant, so that it depends only on the magnitude of the distance r of the force center in the coordinate origin. After the scattering in the region of radius R, the particle moves freely again.

From classical mechanics, the deflection angle ψ is related to b and $V(r)$ by the equation

$$\psi(b)=\int_{r_m}^{R}\frac{b\,dr}{r^2\left[1-\frac{b^2}{r^2}-\frac{V(r)}{E}\right]^{1/2}}\,, \tag{2.1}$$

where r_m is the largest root of the denominator. It is convenient to introduce the new variable $x=b^{-2}$ so that $u=1/r$ regarding ψ as a function of x and V as function of u. Then (2.1) becomes

$$\psi(x)=\int_{1/R}^{um}\frac{du}{\left[x\left[1-\frac{V(u)}{E}\right]-u^2\right]^{1/2}}\,. \tag{2.2}$$

From Fig. 1 it is easy to find the connection between the deflection angle $\theta(b)$ and the angle $\psi(b)$

$$\psi(b)=\frac{\theta(b)+\pi}{2}-\alpha(b)\,, \tag{2.3}$$

where $\alpha(b)=\arcsin(b/R)$. We now define the functions $v(u)$ and $\omega(u)$ by

$$v(u)=1-\frac{V(u)}{E},\quad \omega=\frac{u^2}{v}\,. \tag{2.4}$$

In terms of v and ω (2.2) becomes

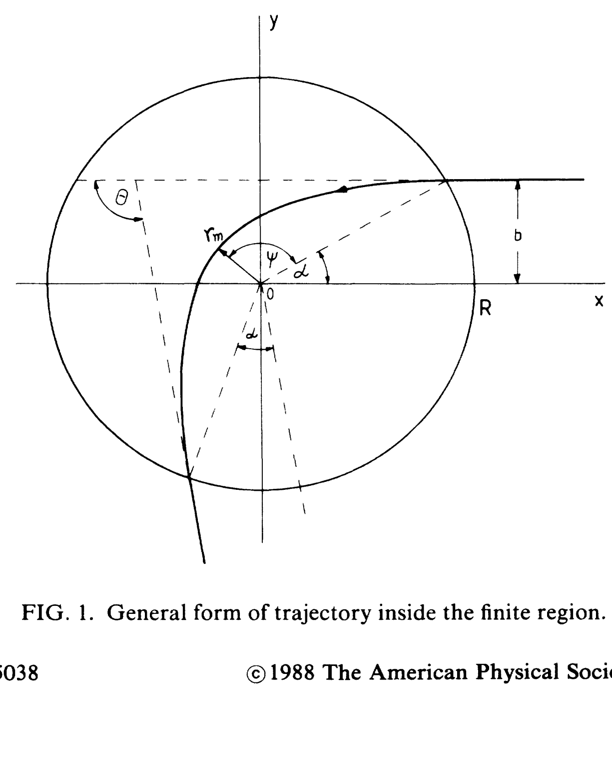

FIG. 1. General form of trajectory inside the finite region.

$$\psi(x)=\int_{1/R^2}^{x}\frac{g(\omega)d\omega}{(x-\omega)^{1/2}}\ , \tag{2.5}$$

where

$$g(\omega)=\frac{1}{v^{1/2}}\frac{du}{d\omega}\ . \tag{2.6}$$

Equation (2.5) may be considered to be an integral equation for the determination of $g(\omega)$. Since it is of Abel type[5] it can be solved explicitly with the result

$$g(\omega)=\frac{1}{\pi}\frac{d}{d\omega}\int_{1/R^2}^{\omega}\frac{\psi(x)dx}{(\omega-x)^{1/2}}\ . \tag{2.7}$$

Upon integration and differentiation in the expression (2.7) we obtain the function $g(\omega)$ explicitly. Then we determine the potential $V(r)$ from the differential equation (2.6).

III. HARMONIC OSCILLATOR

Let us now consider the deflection function

$$\theta(b)=\arcsin\left[\frac{b}{R}\right]\ . \tag{3.1}$$

Then, using the variable x instead of b, from (2.3) we obtain

$$\psi(x)=\frac{\pi}{2}-\tfrac{1}{2}\arcsin\left[\frac{1}{Rx^{1/2}}\right]\ , \tag{3.2}$$

and from (2.7) the expression for $g(\omega)$ becomes

$$g(\omega)=\frac{1}{2\left[\omega-\frac{1}{R^2}\right]^{1/2}}-\frac{1}{2\pi}\frac{d}{d\omega}I(\omega)\ , \tag{3.3}$$

where

$$I(\omega)=\int_{1/R^2}^{\omega}\frac{\arcsin\left[\frac{1}{Rx^{1/2}}\right]}{(\omega-x)^{1/2}}dx\ . \tag{3.4}$$

To evaluate the integral $I(\omega)$, we introduce a new variable $y=1/Rx^{1/2}$. Then we have

$$I(\omega)=\frac{\pi}{R}\left[\frac{1}{a^2}-1\right]-\frac{2a}{R}I(a,-\tfrac{1}{2})\ , \tag{3.5}$$

where $a=1/R\omega^{1/2}$ and

$$I(a,-\tfrac{1}{2})=\int_a^1\frac{\arccos y\ dy}{y^2(y^2-a^2)^{1/2}}\ . \tag{3.6}$$

The integral (3.6) has been calculated in Appendix B [see formulas (B12)]. Using this result we obtain

$$I(\omega)=\pi\left[\omega-\frac{1}{R^2}\right]^{1/2}-\pi\omega^{1/2}+\frac{\pi}{R}\ , \tag{3.7}$$

and

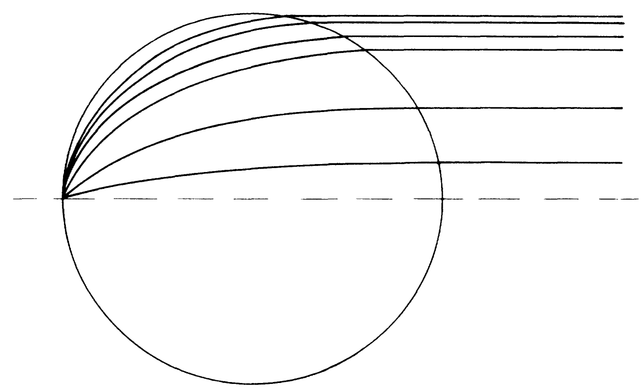

FIG. 2. Luneburg lens.

$$g(\omega)=\frac{1}{4\left[\omega-\frac{1}{R^2}\right]^{1/2}}+\frac{1}{4\omega^{1/2}}\ . \tag{3.8}$$

Now, substituting (3.8) and (2.4) in (2.5) yields

$$\frac{du}{u d\omega}=\frac{1}{4\omega}+\frac{1}{4\left[\omega\left[\omega-\frac{1}{R^2}\right]\right]^{1/2}}\ . \tag{3.9}$$

The solution of the differential equation (3.9) is

$$Cu^2=\omega^{1/2}\left[\omega^{1/2}+\left[\omega-\frac{1}{R^2}\right]^{1/2}\right]\ , \tag{3.10}$$

where C is an arbitrary constant. Choosing $C=1$ and returning to variable v the expression (3.10) becomes

$$\frac{1}{R^2\omega}=1-(1-v)^2\ . \tag{3.11}$$

Taking into consideration that $v=1-V(r)/E$, we obtain the potential of the harmonic oscillator

$$\frac{V(r)}{E}=\begin{cases}\left[\frac{r}{R}\right]^2-1, & r\le R\\ 0, & r>R\ .\end{cases} \tag{3.12}$$

The above potential (3.12) focuses all incoming particles into one point. This kind of refractor in optics is known as a Luneburg lens[6] (see Fig. 2). In fact, it is a sphere with radial-dependent index of refraction

$$n(r)=\left[2-\left[\frac{r}{R}\right]^2\right]^{1/2}\ . \tag{3.13}$$

IV. REPULSIVE-ATTRACTIVE POTENTIAL

We choose the deflection function in the form

$$\theta(b)=\pi-4\arccos\left[\frac{b}{R}\right]\ , \tag{4.1}$$

which in the interval $[0,R]$ has negative and positive values (Fig. 3). From (2.3) one can calculate

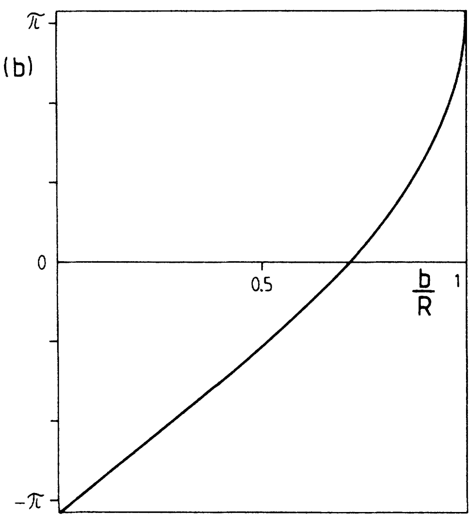

FIG. 3. Shape of the deflection function given by expression (4.1).

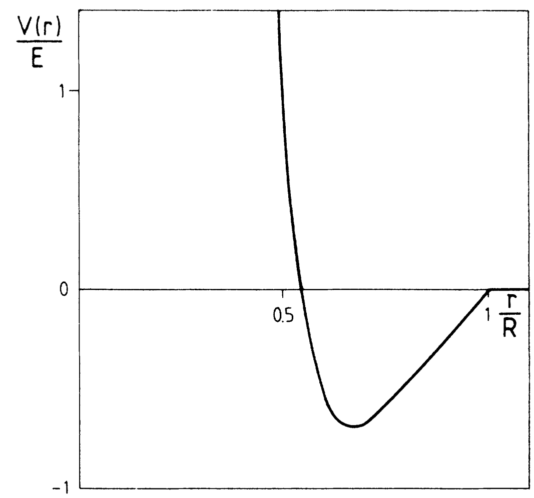

FIG. 4. Shape of the repulsive-attractive potential given by expression (4.6).

$$\psi(x)=\arcsin\left[\frac{1}{Rx^{1/2}}\right] . \tag{4.2}$$

Inserting this function in (2.7) and making the integration, we obtain

$$g(\omega)=\frac{1}{2}\frac{1}{\left[\omega-\frac{1}{R^2}\right]^{1/2}}-\frac{1}{\pi}\frac{d}{d\omega}\left[\frac{2}{R^2\omega^{1/2}}I(a,-\tfrac{1}{2})\right] , \tag{4.3}$$

where $I(a,-\frac{1}{2})$ is given in (3.6). Using the result (B11), the expression (4.3) becomes

$$g(\omega)=\frac{1}{2}\left[\frac{1}{\left[\omega-\frac{1}{R^2}\right]^{1/2}}-\frac{1}{\omega^{1/2}}\right] . \tag{4.4}$$

According to the previous result, the differential equation (2.6) has the solution

$$(Cu-1)^2=1-\frac{1}{R^2\omega} . \tag{4.5}$$

Choosing the integrated constant $C=R$ and returning again to previous variables, we obtain the following potential (see Fig. 4):

$$\frac{V(r)}{E}=\begin{cases}1-\left[\frac{R}{r}\right]^3\left[2-\frac{R}{r}\right], & r\le R\\ 0, & r>R .\end{cases} \tag{4.6}$$

The trajectory of the particle inside the scattering region after particle has reached its minima can be found from the formula

$$\phi(r,b)=3\alpha(b)-\int_r^R\frac{b\,dr}{r^2\left[1-\frac{b^2}{r^2}-\frac{V(r)}{E}\right]^{1/2}} . \tag{4.7}$$

Inserting the potential (4.6) into (4.7) and making a simple calculation, we obtain

$$\phi(r,b)=3\arcsin\left[\frac{b}{R}\right]+\arcsin\left[1-\left[\frac{b}{R}\right]^2\right]^{1/2}-\arcsin\left[\frac{1-\frac{r}{R}\left[\frac{b}{R}\right]^2}{\left[1-\left[\frac{b}{R}\right]^2\right]^{1/2}}\right] , \tag{4.8}$$

or

$$\frac{r}{R}=\left[\frac{R}{b}\right]\left\{1-\left[1-\left[\frac{b}{R}\right]^2\right]^{1/2}\cos[\phi-2\alpha(b)]\right\} . \tag{4.9}$$

For this particular case we can compute the form of caustics. From the relation $\partial\phi/\partial b=0$, where ϕ is given by (4.8), we get

$$\frac{b}{R}=\left[\frac{5r-2-[(5r-2)^2-3r^2(2r-1)(5-2r)]^{1/2}}{3r^2}\right]^{1/2} . \tag{4.10}$$

Now (4.10) and (4.8) determine the function $\phi(r)$ which describes the forbidden region for the particles (Fig. 5).

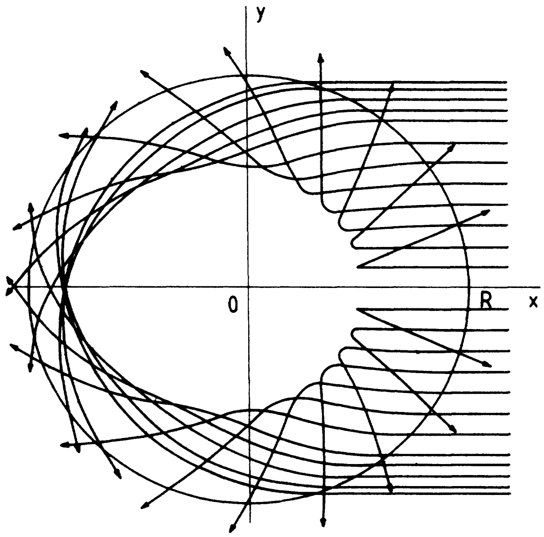

FIG. 5. Trajectories of particle in the repulsive-attractive potential (4.6). The envelope of trajectories is caustics given by (4.10) and (4.8).

V. SPHERICAL MIRROR

Let us consider the following problem. How do we determine the potential of a finite range which scatters the particle in a way equivalent to a reflection from a spherical mirror? We start with the reflection of ray from the concave side of the sphere. Multiple reflections are not taken into consideration. From Fig. 6 is clear that

$$\theta(b)=2\arccos\left[\frac{b}{R}\right]. \tag{5.1}$$

Now using (5.1) in (2.3) yields

$$\psi(b)=2\arccos\left[\frac{b}{R}\right]. \tag{5.2}$$

Inserting this expression into (2.7) and repeating the procedure used in thus preceding sections we get

$$g(\omega)=\frac{1}{\omega^{1/2}}, \tag{5.3}$$

and solving the differential equation (2.6) (with the integrated constant $C=1/R$) we obtain the potential

$$\frac{V(r)}{E}=\begin{cases}1-\dfrac{R}{r}, & r\le R\\ 0, & r>R\,.\end{cases} \tag{5.4}$$

The trajectory of the particle which is a subject of the potential (5.4) can be obtained from the formula

$$\phi(r,b)=\alpha(b)-\int_r^R \frac{dr}{r\left[\dfrac{Rr}{b^2}-1\right]^{1/2}}. \tag{5.5}$$

Then

$$\phi(r,b)=\frac{\pi}{2}+\arccos\left[\frac{b}{R}\right]-2\arccos\left[\frac{b}{(Rr)^{1/2}}\right], \tag{5.6}$$

or

$$r=\frac{2b^2}{R[1-\cos(\phi+\alpha)]}. \tag{5.7}$$

Several examples of trajectories as illustration of formula (5.7) are given in Fig. 7.

Let us now consider the reflection of rays from the convex side of the sphere. The problem is how to deter-

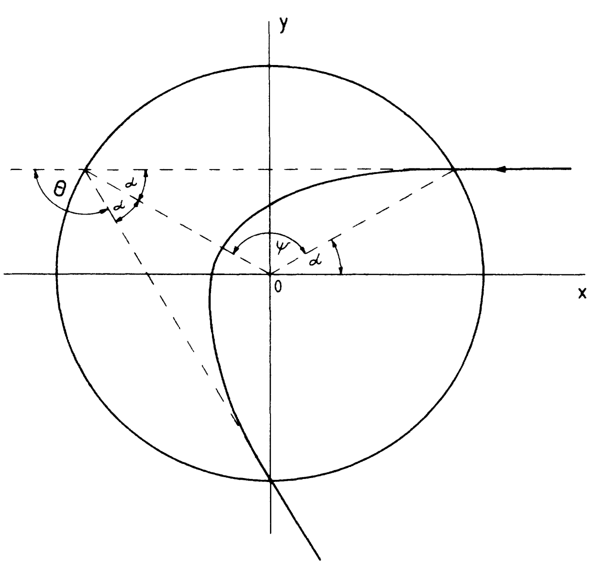

FIG. 6. General form of trajectory for the concave spherical mirror reflection.

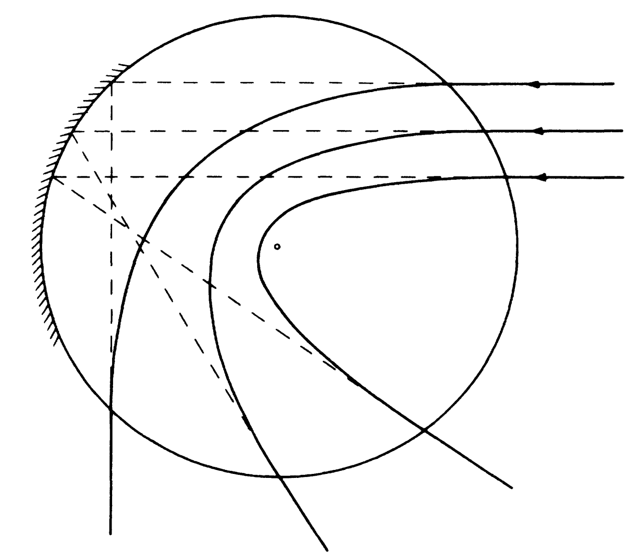

FIG. 7. Trajectories for concave spherical mirror reflection for different values of the impact parameter; $b=0.3, 0.5, 1/\sqrt{2}$.

mine the attractive potential which produces the same effect? From Fig. 6 it is obvious that

$$\theta(b)=\pi+2\arcsin\left[\frac{b}{R}\right], \tag{5.8}$$

and $\psi(x)=\pi$. Inserting these relations into (2.7) yields

$$g(\omega)=\frac{1}{\left[\omega-\frac{1}{R}\right]^{1/2}}, \tag{5.9}$$

and from (2.6) we obtain

$$\frac{V(r)}{E}=\begin{cases}1-\dfrac{4R^3}{r(R+r)^2}, & r\le R\\ 0, & r>R\ .\end{cases} \tag{5.10}$$

A simple calculation gives the trajectory of the particle inside the sphere (Fig. 8)

$$\theta(r,b)=\arcsin\left[\frac{b}{R}\right]+\arcsin\left[\frac{2R^3-b^2(R+r)}{2R^2(R^2-b^2)^{1/2}}\right] -\arcsin\left[\frac{2R^2r-b^2(R+r)}{2rR(R^2-b^2)^{1/2}}\right]. \tag{5.11}$$

In the above-mentioned examples we met with the so-called orbiting scattering. The characteristics of this kind of motion is that there are nonunique solutions of the inverse scattering problem. Indeed, if instead of functions (5.1) and (5.8) we use the functions

$$\begin{aligned}\theta(b)&=2\arccos\left[\frac{b}{R}\right]+2n\pi\ ,\\ \theta(b)&=\pi+2\arcsin\left[\frac{b}{R}\right]+2n\pi\ ,\end{aligned} \tag{5.12}$$

where $n=1,2,\ldots,$ we obtain the same effect of reflection. In this case the particle inside the sphere makes many loops and it produces many different potentials.

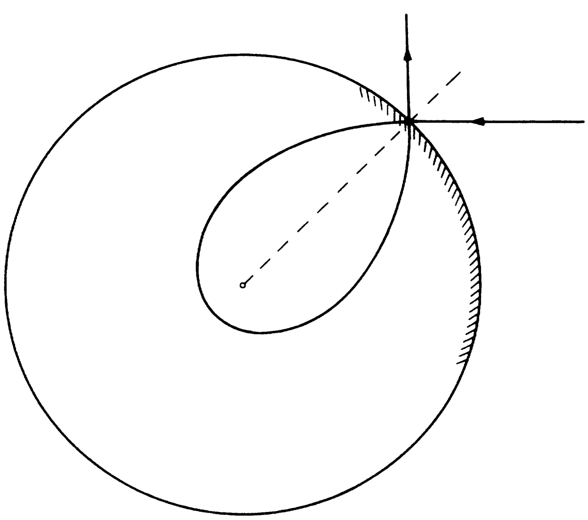

FIG. 8. Trajectory for convex spherical mirror reflection ($b=1/\sqrt{2}$).

The above case is typical example of ambiguities in the determination of potential from scattering data using the fixed energy method. In quantum scattering theory it has been a subject of investigation by many authors (Newton,[7] Sabatier,[8–10] and Chadan[11,12]). This fact is linked with the interpolation of phase shifts in the semiclassical inverse method.

The same phenomenon also appears in classical scattering theory for a class of potentials which gives a classical deflection function equal to zero (mod$2n\pi$) (transparent potentials) (Cuer[4]). We might think that ambiguities in the classical inverse problem at fixed energy exist in all cases of finite scatterer.

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APPENDIX A

The aim of this appendix is evaluation of the integral

$$J(a,\alpha)=\frac{2}{\pi}\int_a^1(x^2-a^2)^\alpha\arccos x\,dx\ . \tag{A1}$$

Then

$$J(a,0)=\frac{2}{\pi}[-a\arccos a+(1-a^2)^{1/2}] \tag{A2}$$

and

$$\frac{\partial J(a,\alpha)}{\partial a}=-\alpha J(a,\alpha-1) \tag{A3}$$

for $\alpha>0$. From this, using the method of mathematical induction, we obtain

$$\begin{aligned}J(a,n)&=n!\int_{a^2}^1\int_{x_1}^1\cdots\int_{x_{n-1}}^1 J(t,0)dt\,dx_{n-1}\cdots dx_1\\ &=n\int_{a^2}^1(t-a^2)^{n-1}J(t,0)dt\ .\end{aligned} \tag{A4}$$

Therefore, we assume that

$$J(a,\alpha)=\alpha\int_{a^2}^1(t-a^2)^{\alpha-1}J(t,0)dt\ , \tag{A5}$$

or

$$J(a,\alpha)=\frac{2\alpha}{\pi}\int_{a^2}^1(t-a^2)^{\alpha-1}[(1-t)^{1/2} -t^{1/2}\arccos t^{1/2}]dt\ . \tag{A6}$$

Using a new variable $t=x^2$, from (A6) we obtain the following recursive relation:

$$J(a,\alpha)=-\frac{2a^2\alpha}{1+2\alpha}J(a,\alpha-1)+\frac{2\alpha}{\pi(1+2\alpha)}\int_{a^2}^{1}(x-a^2)^{\alpha-1}(1-x)^{1/2}dx \ . \tag{A7}$$

By the replacement $t=(x-a^2)/(1-a^2)$, one can calculate that

$$\int_{a^2}^{1}(x-a^2)^{\alpha-1}(1-x)^{1/2}dx=(1-a^2)^{\alpha+1/2}B(\alpha,\tfrac{3}{2}) \ , \tag{A8}$$

where $B(\alpha,\frac{3}{2})$, is the beta function. Then the expression (A7) becomes

$$J(a,\alpha)=-\frac{2a^2\alpha}{2\alpha+1}J(a,\alpha-1)+\frac{2\alpha(1-a^2)^{\alpha+1/2}}{2\alpha+1}B(\alpha,\tfrac{3}{2}) \ . \tag{A9}$$

Using the identity

$$\frac{B(\alpha+1,\frac{1}{2})}{2\alpha+2k+1}=B(\alpha+k,\tfrac{3}{2})\frac{2\alpha+3}{2\alpha+2}\frac{2\alpha+5}{2\alpha+4}\cdots\frac{2\alpha+2k-1}{2\alpha+2k-2} \ , \tag{A10}$$

we can calculate the sum

$$\sum_{k=0}^{\infty}J(a,\alpha+k)p(\alpha,a,k) \ , \tag{A11}$$

where

$$p(\alpha,a,k)=\begin{cases}\dfrac{(-1)^{k+1}}{a^{2k}}\dfrac{2\alpha+3}{2\alpha+2}\dfrac{2\alpha+5}{2\alpha+4}\cdots\dfrac{2\alpha+2k+1}{2\alpha+2k}, & k=1,2,\ldots,\\ 1, \quad k=0 \ . \end{cases} \tag{A12}$$

As result of summation (A11), we get

$$J(a,\alpha)=\frac{a^{2\alpha+1}}{\pi}B(\alpha+1,\tfrac{1}{2})\times\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2\alpha+k+1}\left[\frac{1}{a^2}-1\right]^{\alpha+k+1/2} \ . \tag{A13}$$

From the last expression it is easy to rewrite $J(a,\alpha)$ in the simple integral form, suitable for numerical calculations

$$J(a,\alpha)=\frac{a^{2(2\alpha+1)}}{2\pi^{1/2}}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})}\int_0^{1/a^2-1}\frac{t^{\alpha+1/2}}{1+t}dt \tag{A14}$$

for $\alpha>-1$. Some particular values of $J(a,\alpha)$ are

$$\begin{aligned}
&J(a,-\tfrac{1}{2})=-\ln a \ ,\\
&J(a,0)=\frac{2}{\pi}[(1-a^2)^{1/2}-a\arccos a] \ ,\\
&J(a,\tfrac{1}{2})=\frac{a^2}{2}\ln a-\frac{a^2}{4}+\frac{1}{4} \ ,\\
&J(a,1)=\frac{4}{3\pi}\left[\frac{(1-a^2)^{1/2}}{3}(1-4a^2)+a^3\arccos a\right] \ ,\\
&J(a,\tfrac{3}{2})=\tfrac{3}{32}[(1-a^2)(1-3a^2)-4a^4\ln a] \ .
\end{aligned} \tag{A15}$$

APPENDIX B

Let us consider the following integral:

$$I(a,\alpha)=\int_a^1\frac{(x^2-a^2)^\alpha}{x^2}\arccos x \, dx \ . \tag{B1}$$

It is obvious that

$$I(a,\alpha+1)=J(a,\alpha)-a^2I(a,\alpha) \ , \tag{B2}$$

where $J(a,\alpha)$ is the integral in (A1). Therefore

$$I(a,\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{a^{2k}}J(a,\alpha+k-1) \ . \tag{B3}$$

Inserting (A13) in (B3) we obtain

$$I(a,\alpha)=\frac{a^{2\alpha-2}}{4}\sum_{k=1}^{\infty}B(\alpha+k,\tfrac{1}{2})\times\sum_{j=1}^{\infty}\frac{(-1)^{k+j}}{\alpha-\frac{1}{2}+k+j}\times\left[\frac{1}{a^2}-1\right]^{\alpha-1/2+k+j} \ , \tag{B4}$$

or in the integral form

$$I(a,\alpha)=\frac{a^{2\alpha-1}}{4}\times\int_0^{1/a^2-1}\frac{1}{1+t}\sum_{k=0}^{\infty}(-1)^{k-1}B(\alpha+k,\tfrac{1}{2})\times t^{\alpha-1/2+k}dt \ . \tag{B5}$$

Taking into account that

$$B(\alpha+k,\tfrac{1}{2})=B(\alpha,\tfrac{1}{2})\frac{\alpha+k+1}{\alpha+\frac{1}{2}+k-1}\frac{\alpha+k-2}{\alpha+\frac{1}{2}+k-2}\cdots\frac{\alpha}{\alpha+\frac{1}{2}} \ , \tag{B6}$$

and putting (B6) in (B5) we have the following series:

$$F(t)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\alpha+k-1}{\alpha+\frac{1}{2}+k-1}\cdots\frac{\alpha}{\alpha+\frac{1}{2}}t^{\alpha-1/2+k} \ . \tag{B7}$$

The function $F(t)$ satisfies the differential equation

$$F'(t)+\frac{1}{2(t+1)}F(t)-\frac{\alpha t^{\alpha-1/2}}{t+1}=0\ , \qquad \text{(B8)}$$

which has the solution

$$F(t)=\frac{\alpha}{(t+1)^{1/2}}\int_0^t \frac{x^{\alpha-1/2}}{(x+1)^{1/2}}dx,\quad \alpha>-\tfrac{1}{2}\ . \qquad \text{(B9)}$$

From the above considerations it follows that

$$I(a,\alpha)=\frac{a^{2\alpha-1}}{2}B(\alpha,\tfrac{1}{2})\left[\int_0^{1/a^2-1}\frac{x^{\alpha-1/2}}{(x+1)}dx - a\int_0^{1/a^2-1}\frac{x^{\alpha-1/2}}{(x-1)^{1/2}}dx\right],\quad \alpha>-\tfrac{1}{2}\ . \qquad \text{(B10)}$$

Simple calculation of integral (B10) for $\alpha=0$ and $\alpha=\frac{1}{2}$ gives

$$I(a,0)=\frac{1}{a}\arccos a-\ln\left[\frac{1+(1-a^2)^{1/2}}{a}\right],$$
$$I(a,\tfrac{1}{2})=\frac{\pi}{2}(a-1-\ln a)\ . \qquad \text{(B11)}$$

The other values of the integral $I(a,\alpha)$ can be obtained from the expressions (A15) and the difference equation (B2), and are

$$I(a,-\tfrac{1}{2})=\frac{\pi(1-a)}{2a^2}\ ,$$
$$I(a,1)=\frac{2}{\pi}(1-a^2)^{1/2}-a\left[\frac{2}{\pi}+1\right]\arccos a+a^2\ln\left[\frac{1+(1-a^2)^{1/2}}{a}\right],$$
$$I(a,\tfrac{3}{2})=\frac{a^2}{2}(1+\pi)\ln a+\frac{a^2}{2}(\pi-2)-\frac{\pi a^3}{2}+\tfrac{1}{4}\ . \qquad \text{(B12)}$$

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