Mean-field exponents for self-organized critical phenomena

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The mean-field critical exponents, $\tau=\frac{5}{2}$, $\sigma=\frac{1}{2}$, $\gamma=\beta=1$, etc., for the self-organized critical state are derived viewing the self-organized critical state as a critical branching process.

Recently, Bak *et al.*[1] described a critical phenomenon, occurring in a class of dissipative coupled systems, dynamically triggered by temporal fluctuations. One example is a sand pile, where sand is added randomly. The sand pile is "stable" if the slope everywhere is less than or equal to a critical slope, and the dynamically stationary state is obtained at the "critical" point, where the growth of the pile exactly balances the creation of avalanches, i.e., where the sand added equals the sand sliding off the pile. Since this state is an attractor of an *intrinsic* dynamics, it is denoted a *self-organized critical state.* The self-organized critical state has power-law distributions for the avalanches, both in lifetime and size.

In this paper the self-organized critical state is described by a *critical branching process.* From this picture the "mean-field" exponents for the self-organized critical state emerge naturally. To be specific, consider the branching process shown in Fig. 1 by which in each "generation" an individual is replaced by zero, one, or two descendants, and denote the corresponding probabilities by C_0, C_1, and C_2. From one generation to the next, the number of individuals in average increases by a factor C_1+2C_2. At criticality, where the "family" barely survives,

$$C_0=C_2=(1-C_1)/2\ . \tag{1}$$

The view that a self-organized state can be regarded as a critical branching process allows a determination of mean-field exponents. In general, one branching process in a "sand-pile" model may trigger another.[2] However,

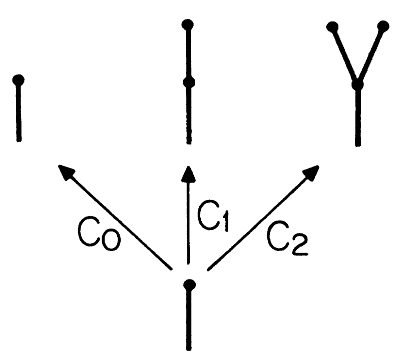

FIG. 1. Branching process describing the mean-field dynamics of the "sand pile."

in sufficient high-dimensional space the overlap of these processes will reduce to finite times, and the scaling exponents will not be influenced. Assuming the *direction* of this triggering to be random, the critical dimension is $D=4$.[3] Above this dimension the scaling exponents can be derived from a *single* branching process.

To extract the critical exponents τ, σ, γ, β, etc.,[1] the *generating function*

$$\mathcal{C}(x)=C_0+C_1x+C_2x^2 \tag{2}$$

is introduced. The probability $D(s)$ that a branching process creates a tree with *exactly* s "individuals" is for large s,[4]

$$D(s)=\left[\frac{\mathcal{C}(\sqrt{C_0/C_2})}{4\pi C_2}\right]^{1/2} a^{-s}s^{1-\tau}\ , \tag{3a}$$

where $\tau=\frac{5}{2}$ and

$$a=\frac{\sqrt{C_0/C_2}}{\mathcal{C}(\sqrt{C_0/C_2})} \tag{3b}$$

equals 1 at criticality.

Close to criticality, i.e., for $|C_0-C_2|\ll C_0$, expansion of Eq. (3b) yields

$$a\simeq 1+\frac{1}{4C_0}(C_0-C_2)^2\ . \tag{4}$$

Since $a^{-s}\simeq 1-(a-1)s$, the cutoff s_c in size s is

$$s_c\propto|C_0-C_2|^{-1/\sigma}\ , \tag{5}$$

where $\sigma=\frac{1}{2}$. From Eqs. (3a) and (5) the critical exponent γ for the "susceptibility"

$$\chi=\sum_s sD(s)\propto|C_0-C_2|^{-\gamma} \tag{6}$$

is determined to be $\gamma=(3-\tau)/\sigma=1$. Also, the critical exponent β for survival or "overflow" when $C_2>C_0$,

$$j=1-\sum_s D(s)\propto(C_2-C_0)^{\beta} \tag{7}$$

is obtained to be $\beta=(\tau-2)/\sigma=1$.

The lifetime distribution $D(t)$ can be calculated as well, assuming that the lifetime is related to the number of generations l of the tree by random walk, i.e., $t\propto l^z$, where $z=2$. The probability $q(l)$ that a branching process stops *at or prior* to the generation l is given by recurrence,[4]

$$q(l)=\mathcal{C}(q(l-1)) \ . \tag{8}$$

At criticality the fixed point $x^*=1$ of $\mathcal{C}(x)$ is barely stable; the derivative $\mathcal{C}'(x^*)=1$. By expansion, for large l,

$$q(l) \simeq q(l-1)+C_0[q(l-1)-1]^2 \ . \tag{9}$$

The probability $D(l)$ that a process stops *exactly* at the generation l is

$$D(l)=q(l)-q(l-1) \simeq q'(l) \simeq C_0[q(l)-1]^2 \ . \tag{10}$$

Thus $q(l) \simeq 1-(C_0 l)^{-1}$ and

$$D(t \propto l^2)=D(l)=\frac{1}{C_0} l^{-2} \propto t^{\phi-2} \ , \tag{11}$$

where $\phi=1$. Notice that the probability $1-j$ of eventual extinction when $C_2 > C_0$ is given by the stable fixed point $x^*=C_0/C_2$ of $\mathcal{C}(x)$ [Eq. (8)]. Consequently, $j=(C_2-C_0)/C_2$, and $\beta=1$ as found by summation above.

In conclusion, the theory of branching processes has been utilized to understand the mean-field behavior of self-organized critical phenomena. From this we can also understand how the dimensionality influences on the actual values of the exponents; for example, the overlap of branching processes decreases the value of τ that accordingly is smaller in two dimensions than in three. Finally, the picture predicts an upper critical dimension $D=4$, above which the mean-field exponents are exact.

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[1]P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); C. Tang and P. Bak, *ibid.* **60**, 2347 (1988); J. Stat. Phys. (to be published).

[2]Activating a coupling (growth bond) does not "kill" it.

[3]N. Jain and S. Orey, Isr. J. Math. **6**, 379 (1968).

[4]See, e.g., T. E. Harris, *The Theory of Branching Processes* (Springer, Berlin, 1963), p. 32.