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Causality in the Coulomb gauge: A direct proof

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A direct calculation of causality behavior in Coulomb-gauge solutions to the Maxwell equations is given, with an arbitrary time-dependent source.

The Coulomb gauge is commonly used in quantum electrodynamics owing to the simple quantization rules.[1,2] The results should be equivalent to Lorentz-gauge calculations unless gauge-dependent approximations are used. However, an obvious property of Lorentz-gauge calculations is that all solutions for fields and potentials can be expressed in terms of causal propagators. This is not the case in the Coulomb gauge, as the scalar potential is acausal. In fact, the vector potential is also acausal in the Coulomb gauge. It has an instantaneously propagating part which is often overlooked in conventional interpretations of the solutions.

We show in a direct calculation how these acausal potentials are able to generate causal **E** and **B** fields. An earlier calculation demonstrates this equivalence in the case of a sinusoidally varying source.[3] We note that it is difficult to distinguish advanced from retarded solutions with this type of source. An inverse Fourier transform is required to treat a nonsinusoidal current, with corresponding problems in proving the existence of such transforms. Our much simpler proof is valid for an arbitrary, bounded, time-varying current source. In addition, we are able to clearly identify the terms responsible for the apparently acausal behavior.

The proof proceeds from the usual Maxwell equations, with

$$\nabla\cdot\mathbf{E}=\rho/\epsilon\ , \tag{1a}$$

$$\nabla\cdot\mathbf{B}=0\ , \tag{1b}$$

$$\nabla\times\mathbf{B}=\mu(\mathbf{J}+\epsilon\dot{\mathbf{E}})\ , \tag{1c}$$

$$\nabla\times\mathbf{E}=-\dot{\mathbf{B}}\ . \tag{1d}$$

In the Coulomb gauge, the new potentials **A** and ϕ are introduced so that

$$\mathbf{E}=-\nabla\phi-\dot{\mathbf{A}}\ , \tag{2a}$$

$$\mathbf{B}=\nabla\times\mathbf{A}\ , \tag{2b}$$

$$\nabla\cdot\mathbf{A}=0\ . \tag{2c}$$

It is immediately obvious that ϕ satisfies Poisson's equation, since from the Coulomb-gauge restriction (2c) and (1a)

$$\nabla^2\phi=-\rho/\epsilon\ . \tag{3}$$

Solving, with boundary conditions vanishing as $|\mathbf{r}|\to\infty$,

$$\phi(\mathbf{x})=\int\frac{\rho(\mathbf{r}',t)}{4\pi\epsilon\,|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'\ . \tag{4}$$

Here we have used the notation $\mathbf{x}\equiv(\mathbf{r},t)$. This solution is acausal: ϕ depends on ρ without retardation. Next, from (1c),

$$\nabla\times(\nabla\times\mathbf{A})=\mu(\mathbf{J}-\epsilon(\nabla\dot{\phi}+\ddot{\mathbf{A}}))\ . \tag{5}$$

Applying the Coulomb-gauge equation again, we find[4]

$$\nabla^2\mathbf{A}-\epsilon\mu\ddot{\mathbf{A}}=-\mu\mathbf{J}+\mu\epsilon\nabla\dot{\phi}\ . \tag{6}$$

Solving with outgoing wave boundary conditions and a homogeneous term $\mathbf{A}_H(\mathbf{x})$,

$$\mathbf{A}(\mathbf{x})=\frac{\mu}{4\pi}\int\left[\frac{\delta(c[t-t']-|\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right]\times[\mathbf{J}(\mathbf{x}')-\epsilon\nabla\dot{\phi}(\mathbf{x}')]d^4\mathbf{x}'+\mathbf{A}_H(\mathbf{x})\ . \tag{7}$$

Our notation of $\nabla\dot{\phi}$ implies partial differentiation relative to both arguments. Clearly, the integral must include ϕ evaluated at positions for which $\mathbf{r}\approx\mathbf{r}'$, and hence $t\approx t'$. However, ϕ is acausal, as it depends instantaneously on changes in the charge density at distant locations from the location where ϕ is evaluated. Thus **A** is also acausal as it is a function of a source term which includes ϕ. This point is often ignored in elementary treatments of the Coulomb gauge. Note that $(\mathbf{J}-\epsilon\nabla\dot{\phi})$ is a transverse field, since by Eq. (3) and charge conservation

$$\nabla\cdot(\mathbf{J}-\epsilon\nabla\dot{\phi})=\nabla\cdot\mathbf{J}+\dot{\rho}=0\ . \tag{8}$$

We next wish to calculate the physical fields of the potentials in Eqs. (4) and (7). It is simplest to evaluate the **B** field initially. Dropping the homogeneous term,

$$\mathbf{B}(\mathbf{x})=\frac{\mu}{4\pi}\int\nabla_r\times\left[\frac{\delta(c[t-t']-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right]\times[\mathbf{J}(\mathbf{x}')-\epsilon\nabla\dot{\phi}(\mathbf{x})]d^4\mathbf{x}'\ . \tag{9a}$$

Hence, on partial integration with respect to $\mathbf{r}'$,

$$\mathbf{B}(\mathbf{x})=\frac{\mu}{4\pi}\int\left[\frac{\delta(c[t-t']-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right]\nabla_{\mathbf{r}'}\times\mathbf{J}(\mathbf{x}')d^4\mathbf{x}' . \tag{9b}$$

Here we have used the fact that the term in braces is a function of $(\mathbf{r}-\mathbf{r}')$ to exchange differentiation in $\mathbf{r}$ and $\mathbf{r}'$. The surface terms that occur in partial integration vanish provided $\mathbf{J}$ and ρ are bounded, since $\nabla\dot{\phi}\sim 1/|\mathbf{r}|^2$ at large $|\mathbf{r}|$. The term in $\dot{\phi}$ vanishes on taking the curl of a gradient. Thus only causal terms remain in the solution for the **B** field, even though it is obtained from an apparently acausal vector potential.

The **E** field must satisfy Eq. (1c), which imples that the solution can be written as

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t_0)+\frac{1}{\mu\epsilon}\int_{t_0}^{t}[\nabla\times\mathbf{B}(\mathbf{r},t')-\mu\mathbf{J}(\mathbf{r},t')]dt' . \tag{10}$$

This expresses the **E** field in terms of its initial value, together with causal fields evaluated at the same point and earlier times only. Thus we have demonstrated the causal behavior of the **E** field as well, so that each of the physical fields is causal.

We note that our arguments are applicable to both quantum and classical operator equations. In addition, they hold for either advanced or retarded solutions if the sign of the time arguments is reversed.

While the use of Coulomb gauge techniques is relatively widespread, the fact that both the potentials have acausal behavior is often ignored. These must always be combined to obtain causal physical behavior. That gauge transformations can alter the apparent causality properties of the potentials is a universal property of gauge field theories of all types. In fact there is never any guarantee of causality properties of the potentials, only of the observed fields. The usual procedure of defining causal Green's functions in terms of the potentials is only correct if it produces the correct causal behavior in the observables. This appears to be relevant to such questions as photon localization,[5] a topic of much current interest, and possibly to lattice gauge theory[6] as well.

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