Finite-temperature field theory and quantum noise in an electrical network

T. Garavaglia*
Institiúid Árd-Léighinn Bhaile Átha Cliath, Baile Átha Cliath 4, Ireland
(Received 22 June 1987; revised manuscript received 14 January 1988)

Finite-temperature ($0 \leq T < \infty$) field (FTF) theory with an effective spectral Lagrangian density formulation is used to study quantum noise in an electrical network. Solutions for the finite second moments that satisfy the uncertainty principle bound are given for a dissipative quantum oscillator. A regularization method, based on the analysis of a semi-infinite low-pass filter, is employed, and it leads to results which differ from those of the Drude model. To illustrate the FTF method, an example is given using an ideal finite-temperature coherent state.

In this work, it is demonstrated how finite-temperature field (FTF) theory leads to a simple derivation of the fluctuation-dissipation theorem and the relation between thermal and quantum noise in an electrical network. When the dissipative elements are represented by semi-infinite filters, a procedure results that yields finite second moments for both charge and current as well as compatibility with the uncertainty bound. This method is demonstrated with an investigation of a series LRC dissipative quantum oscillator, and analytical results are given which illustrate the features of thermal and quantum noise. Other approaches to the dissipative quantum oscillator are discussed and reviewed in Refs. 1–5.

Unless regularized, the integral for the second moment of the current is divergent. An effective spectral Lagrangian density is employed which results from the analysis of a semi-infinite low-pass filter. This approach, described in Appendix A, results in both the introduction of dissipation through a frequency-dependent damping coefficient and a cutoff frequency which depends upon the quality factor for the oscillator. For contrast, expressions for the moments obtained from the frequently employed Drude regularized model are given. In addition, ensemble averages at finite temperature are performed with the aid of FTF methods. For illustration, examples using an ideal finite-temperature coherent state are given in Appendix B.

Here, I follow the notation and quantization procedure developed in Ref. 1; however, the present approach differs in that FTF's are introduced so as to include temperature dependence in the expressions for quantum noise. In addition, the ideal transmission lines are replaced with semi-infinite low-pass filters composed of repeated basic elements of inductance ($L_0 = L_T \Delta x$) and capacitance ($C_0 = C_T \Delta x$). The characteristic impedance of the filter,[6] obtained from (A1) and the values Z_1 and Z_2 used to obtain (A6),

$$Z_0(a,b) = i\omega L_0/2 + (L_0/C_0 - \omega^2 L_0^2/4)^{1/2} \tag{1}$$

implies a maximum frequency above which the voltage wave becomes damped. Below this frequency the voltage in the nth element of the filter is

$$V(n) = \exp(ikn\,\Delta x)V(0)\ , \tag{2}$$

where $k/2 = \omega/2v(\omega)$ relates the phase velocity $v(\omega) = R(\omega)/L_T$, the wave number k, and the real part $R(\omega)$ of the characteristic impedance (1).

At frequency ω the effective Lagrangian density for the LCR network is

$$\mathcal{L}_\omega(Q_\omega, \partial_x Q_\omega, \partial_t Q_\omega, \beta) = \frac{\delta(x)}{2}\left[L\left[\frac{\partial}{\partial t}Q_\omega(x,t,\beta)\right]^2 - Q_\omega^2(x,t,\beta)/C\right] + L_T\frac{H(x)}{2}\left[\left[\frac{\partial}{\partial t}Q_\omega(x,t,\beta)\right]^2 - v^2(\omega)\left[\frac{\partial}{\partial x}Q_\omega(x,t,\beta)\right]^2\right]\ , \tag{3}$$

where $\delta(x)$ and $H(x)$ are, respectively, the Dirac and Heaviside distributions, and $Q_\omega(x,t,\beta)$ ($\beta = 1/k_B T$ and k_B is Boltzmann's constant) is the charge density. The postulates of quantization[7] suggest for the spectral integrals of the charge density and its conjugate momentum $\Phi_\omega(x,t,\beta)$ the commutation relation

$$[Q(x,t,\beta), \Phi(x',t,\beta)] = i\hbar\delta(x - x') \tag{4a}$$

where the spectral integral is defined as

$$Q(x,t,\beta) = \int_0^{\omega_{\max}} Q_\omega(x,t,\beta)\,d\omega\ . \tag{4b}$$

A derivation similar to that in Ref. 1 yields the spectral Langevin equation

$$\int_0^{\omega_{\max}}\Big[L\ddot{Q}_\omega(t,\beta) + R(\omega)\dot{Q}_\omega(t,\beta) + [Q_\omega(t,\beta)/C] - 2R(\omega)\dot{Q}_\omega^{\rm in}(t,\beta)\Big]d\omega = 0\ , \tag{5}$$

where $Q_\omega(t,\beta)=Q_\omega(0,t,\beta)$ and where dissipation is represented by a spectral damping coefficient

$$\gamma(\omega)=\frac{R(\omega)}{L}=\frac{R\nu}{L\Lambda}\Big/\tan^{-1}\left[\frac{(\nu/\Lambda)}{[1-(\nu/\Lambda)^2]^{1/2}}\right], \qquad (6)$$

with $R=(L_0/C_0)^{1/2}$, $\nu=\omega/\omega_0$, $\Lambda=2Q_0C/C_0$, $\omega_0=(LC)^{-1/2}$, and $Q_0=L\omega_0/R$. Here ω_0 is the natural frequency of the LC system, Q_0 the quality factor, and $\Lambda\omega_0$ the highest frequency passed by the filter.

Before proceeding, it is of interest to discuss the nature of the model associated with the effective Lagrangian density (3) and its associated Langevin equation. The Hamiltonian obtained from (3) consists at each frequency ω of two conservative parts. The first part which results upon integration over x with the Dirac distribution produces the Hamiltonian for the LC oscillator that can receive and transmit energy of different frequencies into the filter. The second part models effectively the dispersive wave nature of the filter. The overall energy is conserved in the sense that any energy transferred from the oscillator is transmitted down the filter without attenuation when the frequency is below the cutoff; however, above this frequency, the energy is reflected back into the oscillator. It is a result of the discontinuous coupling with the Heaviside distribution that the dissipative nature of the problem appears in (5). When (5) is evaluated as a matrix element between coherent states, the resulting expression for the classical limit of a dissipative oscillator with spectral damping appears.

The normalized Heisenberg FTF

$$Q(t,\beta)=\left[\frac{\hbar}{2\pi K_2(Q_0,0)}\right]^{1/2}\int_0^{\Lambda\omega_0}d\omega[\omega R(\omega)]^{1/2}\left[\frac{iA^{\text{in}}(\omega)e^{-i\omega t}}{(\omega^2L-1/C)+i\omega R(\omega)}+\text{H.c.}\right] \qquad (7)$$

is obtained from (4b) and (5) when $Q(0,t,\beta)$ is expanded in terms of in-field creation and annihilation operators associated with the wave equation found from the second part of the effective Lagrangian density (3). Following the methods described in Appendix B, the boson in-field operator is expressed in terms of the FTF operators[8] as

$$A^{\text{in}}(\omega)=[1+f(\beta)]^{1/2}A(\omega,\theta)+f^{1/2}(\beta)\tilde{A}^{\dagger}(\omega,\theta)\ , \qquad (8)$$

where $\sinh(\theta)=f^{1/2}(\beta)$. The matrix element of the number operator

$$N^{\text{in}}(\omega)=A^{\dagger\text{in}}(\omega)A^{\text{in}}(\omega)$$

between finite-temperature vacuum states, needed to evaluate ensemble averages, is

$$(\beta 0\,|\,N^{\text{in}}\,|\,0\beta)=f(\beta)=1/[\exp(\hbar\omega\beta)-1]\ . \qquad (9)$$

The Heisenberg field (7) has been normalized using the integral $K_2(Q_0,0)$, defined in (13), so as to satisfy (4a) in the form

$$[Q(t,\beta),L\dot{Q}(t,\beta)]=i\hbar \qquad (10)$$

which implies for the variances the uncertainty inequality

$$\sigma(Q,\beta)\sigma(L\dot{Q},\beta)\geq\hbar/2\ , \qquad (11)$$

where these variances are defined similarly to the one in (B9). Using (7)–(9), the second moments become the fluctuation-dissipation theorem results[9] modified by spectral damping,

$$\sigma^2(Q,z)=(\beta 0\,|\,Q^2(t,\beta)\,|\,0\beta)=\frac{\hbar K_1(Q_0,z)}{L\omega_0 2K_2(Q_0,0)}\ ,$$
$$\sigma^2(L\dot{Q},z)=(\beta 0\,|\,[L\dot{Q}(t,\beta)]^2\,|\,0\beta)=\frac{\hbar L\omega_0 K_3(Q_0,z)}{2K_2(Q_0,0)}\ , \qquad (12)$$

$$K_m(Q_0,z)=\int_0^{\Lambda}\frac{\nu^m\coth(\nu/z)d\nu}{\pi Q_0(\nu)\{(\nu^2-1)^2+[\nu/Q_0(\nu)]^2\}}\ , \qquad (13)$$

where

$$Q_0(\nu)=(Q_0\Lambda/\nu)\tan^{-1}\{\nu/\Lambda/[1-(\nu/\Lambda)^2]^{1/2}\}\ .$$

The results for the zero-temperature ($z=2k_BT/\hbar\omega_0=0$) vacuum are found with the replacement $\coth(\nu/z)\rightarrow 1$.

In the large $Q_0(R\rightarrow 0)$ limit, one finds the LC oscillator result from (12) and (13) since $2K_2(Q_0,0)\rightarrow 1$ and $K_1(Q_0,z)\rightarrow K_3(Q_0,z)\rightarrow\coth(1/z)/2$. The zero-temperature vacuum results are

$$2\pi Q_0K_3(Q_0,0)=\tfrac{1}{2}\ln[(\Lambda^2-1)^2+(\Lambda/Q_0)^2]+(1-1/2Q_0^2)2\pi Q_0K_1(Q_0,0)\ , \qquad (14)$$

$$2\pi Q_0K_1(Q_0,0)=\frac{Q_0^2}{[1-(2Q_0)^2]^{1/2}}\left[\ln\left[\frac{1-2Q_0^2+[1-(2Q_0)^2]^{1/2}}{1-2Q_0^2-[1-(2Q_0)^2]^{1/2}}\right]-\ln\left[\frac{(\Lambda^2-1)2Q_0^2+1+[1-(2Q_0)^2]^{1/2}}{(\Lambda^2-1)2Q_0^2+1-[1-(2Q_0)^2]^{1/2}}\right]\right]_{0<Q_0<1/2}\ , \qquad (15a)$$

$$2\pi K_1(\tfrac{1}{2},0)\approx 2[1-1/(\Lambda^2+1)]\ , \qquad (15b)$$

$$2\pi Q_0K_1(Q_0,0)\approx\frac{Q_0^2}{[(2Q_0)^2-1]^{1/2}}2[\phi(Q_0,0)-\phi(Q_0,\Lambda)]_{1/2<Q_0}\ , \qquad (15c)$$

with

$$\tan\phi(Q_0,\Lambda)=[(2Q_0)^2-1]^{1/2}/[(\Lambda^2-1)2Q_0^2+1]\ .$$

The phase angle in (15c) is such that $\pi/2 \le \phi(Q_0,0) \le \pi$ and $0 \le \phi(Q_0,\Lambda)$. In the limit $Q_0 \to \infty$, $K_1(Q_0,0) \to K_3(Q_0,0) \to \frac{1}{2}$, one finds the coherent-state equality for (11). Numerical results for (12) show the equipartition of energy behavior expected at high temperature, the clearly defined transition region between pure thermal and pure quantum noise, and values for the dimensionless variance in Q which for small values of Q_0 and low temperature are less than the standard quantum limit indicating the presence of a squeezed quantum state.[10]

In conclusion, it is interesting to compare the above results with those of the Drude regularized model which is described in Ref. 4. In this model the damping coefficient is replaced with

$$\gamma(\omega) = (R/L)/(1 - i\nu/\Lambda_D) , \tag{16}$$

where the dimensionless Drude frequency $\Lambda_D = \omega_D/\omega_0$ is large compared to 1. For this model the range of integration extends to infinity and the denominator function in (13) is replaced with

$$\pi Q_0 \{(\nu^2-1)^2 + (\nu/Q_0)^2[1 + (Q_0/\Lambda_D)(\nu^2-1)]^2\} . \tag{17}$$

Numerical results for this model differ mostly from those of the former model for small values of the quality factor. In addition, for large but finite Λ_D the integral $K_3(Q_0,z)$ remains convergent in the Drude model. However, for the filter model this integral goes to the ideal transmission line divergent result in the limit as $C_0 = C_T \Delta x \to 0$. It should be emphasized that in the regularization method proposed the occurrence of a cutoff frequency for an infinite series of coupled oscillators has a well-established history in the theory of lattice dynamics.[11] Below this frequency, waves can be transmitted without attenuation through the lattice. In addition to the discussions related to the history of wave propagation in one-dimensional harmonic lattices with nearest-neighbor interactions, the analogy between electrical and mechanical lattices and discussions of the dispersive nature of wave propagation can be found in the book by Brillouin.[12]

Another closely related model reported recently[13] treats the dissipative oscillator in such a way that the spectral integrals which represent the variances remain finite even when the range of integration is extended to infinity. In this model a local oscillator is coupled harmonically to a finitely extended one-dimensional string (continuum transmission line). For any finite coupling strength, even in the limit of an infinite string, the fluctuation moments remain finite when the spectral integration range extends to infinity. A logarithmic divergence appears in the variance for the momentum when the coupling becomes rigid. It is this coupling which determines the location of the cutoff frequency of the Lorentzian factor.

APPENDIX A: FILTER REGULARIZATION

It is shown in this Appendix how the analysis of a semi-infinite low-pass filter suggests an effective spectral Lagrangian density and regularization procedure. The impedance $Z_0(a,b)$ may be written as

$$Z_0 = Z_1 + 1/(1/Z_2 + 1/Z_0) , \tag{A1}$$

which can be solved for Z_0. The voltage V_n and the current I_n for the nth loop are related by

$$V_n = I_n Z_1 + V_{n+1} , \tag{A2}$$

so that $V_{n+1} = \alpha V_n$ with $\alpha = (Z_0 - Z_1)/Z_0$ and

$$V_n = I_n Z_0 = \alpha^n V_0 . \tag{A3}$$

The current loop equations are[6]

$$I_n Z_2 - I_{n-1} Z_2 + I_n Z_1 + I_n Z_2 - I_{n+1} Z_2 = 0 . \tag{A4}$$

This set of equations is seen to be equivalent to (A2) when it is recognized that

$$V_n = I_{n-1} Z_2 - I_n Z_2 . \tag{A5}$$

With the choice $Z_1 = i\omega L_0$, $Z_2 = 1/i\omega C_0$, and $\omega(L_0 C_0)^{1/2} = 2\nu/\Lambda$, one finds

$$\alpha = e^{i\delta} = \frac{\pm[1-(\nu/\Lambda)^2]^{1/2} - i\nu/\Lambda}{\pm[1-(\nu/\Lambda)^2]^{1/2} + i\nu/\Lambda} , \tag{A6}$$

where

$$\tan(\delta/2) = \mp(\nu/\Lambda)/[1-(\nu/\Lambda)^2]^{1/2} . \tag{A7}$$

Introducing the definitions $n\delta = -k(\omega)x$, $x = n\Delta x$, $L_0 = L_T \Delta x$, $C_0 = C_T \Delta x$, and $k(\omega) = \omega/v(\omega)$, it follows with $V_0 = E_0 \exp(-i\omega t)$ that

$$V_\omega(x,t) = E_0 \exp\{i[k(\omega)x - \omega t]\} , \tag{A8}$$

which is a solution of the wave equation

$$[\partial_{xx} - v(\omega)^{-2}\partial_{tt}]V_\omega(x,t) = 0 . \tag{A9}$$

The speed $v(\omega)$ is found from (A7) using $\delta = -\omega\Delta x/v(\omega)$, and the maximum frequency passed is found from $\omega/\omega_0 = \Lambda$. Above this frequency, δ is no longer real and the voltage does not propagate down the filter.

The charge in the nth loop of dimension Δx is related to the voltage by

$$V_\omega(x,t) = \partial_x Q_\omega(x,t)\Delta x/C_0 , \tag{A10}$$

where the charge density $Q_\omega(x,t)$ satisfies (A9). The effective spectral Lagrangian density for the charge density is the second term in (3). The action for the effective Lagrangian density (3) is used to find the Euler-Lagrange equation. From the part of the effective Lagrangian density (3) associated with the distribution $H(x)$, one obtains

$$\int_0^{\omega_0\Lambda} d\omega[\partial_{xx} - v^{-2}(\omega)\partial_{tt}]Q_\omega(x,t) = 0 , \tag{A11}$$

where

$$Q_\omega(x,t) = q_\omega \exp\{i[k(\omega)x \pm \omega t]\} . \tag{A12}$$

The spectral sum of this solution is (4b), and q_ω is found from the quantization condition (4a) imposed on the noninteracting fields propagated in the filter. It should be emphasized that this procedure requires C_0 to remain small but nonzero and that (A9) and (A11) are approximations for small Δx to the exact finite difference equations which are found when ∂_x represents finite difference partial differentiation.

APPENDIX B: FINITE-TEMPERATURE FIELD METHODS

The finite-temperature ensemble averages which give the moments (12) are performed using thermofield operators.[8] To illustrate this approach, I evaluate the variances for the operators P and Q, and the mean value of the energy for a simple harmonic oscillator of natural frequency ω_0 using an ideal finite-temperature coherent state. The Hamiltonian for this system is

$$2H/\hbar\omega_0 = 2A^\dagger A + 1 = P^2 + Q^2 , \tag{B1}$$

where $i2^{1/2}P = A - A^\dagger$, and $2^{1/2}Q = A + A^\dagger$. The finite-temperature operators $A(\theta)$ and $\tilde{A}(\theta)$ are obtained from the commuting zero-temperature boson operators A and $\tilde{A}$ which satisfy

$$[A, A^\dagger] = [\tilde{A}, \tilde{A}^\dagger] = 1 . \tag{B2}$$

These finite-temperature operators are found upon using the Bogoliubov transformations

$$A(\theta) = G^\dagger(\theta) A G(\theta) , \; \tilde{A}(\theta) = G^\dagger(\theta) \tilde{A} G(\theta) , \tag{B3}$$

where the unitary operator $G(\theta)$ is

$$G(\theta) = \exp[\theta(A\tilde{A} - A^\dagger \tilde{A}^\dagger)] , \tag{B4}$$

with $\sinh\theta = f^{1/2}(\beta)$ of (9).

The ideal finite-temperature coherent state is generated from the vacuum state $|0)\otimes|\tilde{0})$ upon the application of the operator $K(\alpha,\theta)$ which is defined in terms of the coherent-state operator $D(\alpha)$,[14] and the adjoint of the finite-temperature operator $G(\theta)$ as $K(\alpha,\theta) = G^\dagger(\theta) D(\alpha)$ so that the ideal finite-temperature coherent state becomes

$$|\alpha,\beta) = K(\alpha,\theta)|0)\otimes|\tilde{0}) . \tag{B5}$$

The operator $D(\alpha)$, where α is a complex number, is defined in terms of the boson operators A and $A^\dagger$ as

$$D(\alpha) = \exp(\alpha A^\dagger - \alpha^* A) . \tag{B6}$$

The ensemble average of an operator evaluated between ideal finite-temperature coherent states is found upon forming the matrix element of the operator using the state $|\alpha\beta)$ to write

$$(\beta\alpha|O(A,A^\dagger)|\alpha\beta) = (\tilde{0}|\otimes(0|K^\dagger(\alpha,\theta) O(A,A^\dagger) K(\alpha,\theta)|0)\otimes|\tilde{0}) . \tag{B7}$$

Since $K(\alpha,\theta)$ is unitary, these matrix elements can be evaluated upon using the transformation

$$K^\dagger(\alpha,\theta) A K(\alpha,\theta) = D^\dagger(\alpha) A(\theta) D(\alpha) = \cosh(\theta)(A+\alpha) + \sinh(\theta)\tilde{A}^\dagger , \tag{B8}$$

its adjoint, and the fact that both A and $\tilde{A}$ annihilate the zero-temperature vacuum state $|0)\otimes|\tilde{0})$. The variance in an operator $O(A,A^\dagger)$ is defined by the equation

$$\sigma^2(O)_{\alpha\beta} = (\beta\alpha|O^2|\alpha\beta) - (\beta\alpha|O|\alpha\beta)^2 . \tag{B9}$$

Following the method above, one obtains the results

$$\begin{aligned} (\beta\alpha|Q|\alpha\beta) &= 2^{-1/2}(\alpha+\alpha^*)[1+f(\beta)]^{1/2} , \\ (\beta\alpha|P|\alpha\beta) &= -i2^{-1/2}(\alpha-\alpha^*)[1+f(\beta)]^{1/2} , \\ (\beta\alpha|Q^2|\alpha\beta) &= f(\beta) + \tfrac{1}{2} + (\beta\alpha|Q|\alpha\beta)^2 , \\ (\beta\alpha|P^2|\alpha\beta) &= f(\beta) + \tfrac{1}{2} + (\beta\alpha|P|\alpha\beta)^2 , \\ (\beta\alpha|H/\hbar\omega_0|\alpha\beta) &= [1+f(\beta)]|\alpha|^2 + f(\beta) + \tfrac{1}{2} . \end{aligned} \tag{B10}$$

The results for the various special cases are found when either θ or α has the value zero. The variances for both Q and P are

$$\sigma^2(Q)_{\alpha\beta} = \sigma^2(P)_{\alpha\beta} = \tfrac{1}{2}\coth(1/z) . \tag{B11}$$

This is the finite-temperature result which gives the zero-temperature standard quantum limit $\sigma(Q) = \sigma(P) = 2^{-1/2}$. As demonstrated in Ref. 14, a coherent state can be produced from the radiation generated by an external electric current which radiates photons according to a Poisson distribution. The ideal finite-temperature coherent state (B5) is an eigenstate of the annihilation operator $A(\theta)$, and it can be viewed as representing the photon state resulting from a classical current which generates an ideal coherent state and which is placed in a heat bath of temperature T. If the operators which appear in the definition of $K(\alpha,\theta)$ are applied in reverse order, the resulting state $|\beta\alpha)$ represents a physically more realistic state in which a classical current is initially coupled to a heat bath. States of this type have been considered in Ref. 15 as signal plus noise states. When these states are used, the same result (B11) for the variances is found; however, new expressions are found for (B10). These are easily found from the former results with the replacement $\alpha \rightarrow \alpha/[1+f(\beta)]^{+1/2}$.

*Also at the Institiúid Teicneolaíochta Bhaile Átha Cliath, Baile Átha Cliath 4, Ireland.

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