Structure of clusters generated by spatio-temporal intermittency and directed percolation in two space dimensions

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Within the sustained regimes of spatio-temporal intermittency occurring in coupled map lattices and directed percolation with two space dimensions, fluctuating clusters are shown to be well described by ordinary percolation concepts and critical exponents.

Following the success of dynamical systems theory in describing deterministic chaotic phenomena in strongly confined situations, much work is now devoted to the study of spatially extended systems, aiming at a better understanding of the nature of turbulence. In this field, many "intermediate" models in one space dimension (1D) e.g., chains of coupled maps) have been introduced and phenomenologically explored recently.[1] A scenario specific to weakly confined situations has been shown to occur in various one-dimensional systems[2(a),2(b)] or quasi-dimensional systems such as Rayleigh-Bénard convection in an annulus.[2(c),2(d)] It describes the *transition* from a "laminar" (regular) state possibly unstable to localized perturbations of finite amplitude to sustained regimes of *spatio-temporal intermittency* where disorder occurs both in space and time.

Previous work[2(a),2(b)] was devoted to a test, in the one-dimensional case, of Pomeau's initial conjecture[3] that the transition to spatio-temporal intermittency is equivalent to a phase transition of the directed percolation type, indicating thus a path from turbulence in large deterministic systems to critical phenomena in statistical mechanics.

Here we concentrate on the regimes of sustained spatio-temporal disorder exhibited by the two-dimensional (2D) systems defined below.

The first one is a 2D square lattice of maps coupled by diffusion. The local evolution law f (the elementary map) is designed to fulfill the minimal requirements for exhibiting spatio-temporal intermittency when coupled in an array.[2(b)] It reads

$$f(X)=\begin{cases} rX \quad \text{if } X\epsilon[0,\tfrac{1}{2}] \\ r(1-x) \quad \text{if } X\epsilon[\tfrac{1}{2},1] \\ X \quad \text{if } X>1,\ r>2 \end{cases}$$

(only the case $r=3$ is considered in this work). The coupled map lattice (CML) can be expressed as

$$\begin{aligned} X_{i,j}^{n+1}=f(X_{i,j}^{n})+\tfrac{1}{4}\epsilon[f(X_{i-1,j}^{n})+f(X_{i+1,j}^{n}) \\ +f(X_{ij-1}^{n})+f(X_{i,j+1}) \\ -4f(X_{i,j}^{n})] \,, \end{aligned}$$

where subscripts denote the spatial position of the sites, the superscripts the time, and ϵ is the coupling strength. The local evolution law consists of a chaotic repellor ($X<1$) connected to a continuum of fixed points ($X>1$). All sites whose local state is smaller (larger) than one are then said to be turbulent (laminar). This system, in the infinite size limit, possesses a critical coupling ϵ_c marking the condition of propagation of disorder (turbulent sites). For $\epsilon>\epsilon_c$ sustained regimes of spatio-temporal intermittency are observed whereas the homogeneous laminar state is the asymptotic state for $\epsilon<\epsilon_c$.

Following the analogy developed in the 1D case, we study the above CML in parallel with 2D directed bond percolation (DP), considered here as a typical probabilistic cellular automation with two local states one of which is absorbing.[4] Although emerging from many different fields (flows in porous media, epidemics, forest fires, ...), directed percolation is considered here in connection with the problem of the transition to turbulence. As in the one-dimensional case,[2(b)] active sites are said to be "turbulent." Also, the existence of an absorbing state is crucial to model the metastability of the "laminar" state in the deterministic systems. Furthermore, the system can be viewed as a dynamical and irreversible process mapped onto a 2D space. For numerical simplicity, we use a body-centered cubic (bcc) lattice (two space dimensions plus time) for which the state of a site is determined by four equivalent "parents." The only parameter is the probability p for a bond to be active (opened) and the automation is fully defined by a set of five elementary probabilities corresponding to all possible configurations of a four-site neighborhood in a two-state medium with the symmetries of the lattice. The transition is characterized by the existence of a percolation threshold at $p=p_c$ $\simeq 0.287$. For $p>p_c$, and in the limit of infinite systems, disorder (active states) propagates to infinity in time and space, giving rise to sustained regimes similar to those of spatio-temporal intermittency. For $p<p_c$ any arbitrary initial condition evolves to the homogeneous absorbing state, disorder being only transient.

As in one dimension, the transitions of both systems have indeed many common features and are best described in the framework of critical phenomena. In this Brief Report, we will focus not on the transition point[5] but on the spatial structure of the clusters generated by the two processes.

FIG. 1. Typical snapshots of the spatial structure of directed percolation after the transient period following random initial conditions. Similar pictures are obtained for the CML described in the text. Sites in the absorbing-laminar state are black. The size of the lattice is $N=50$. The boundary conditions are periodic. (a) $p=0.290$ (just above the percolation threshold). (b) $p=0.345$ (in the region described in the text). (c) $p=0.400$ (above this region).

Above the threshold, and when starting from not too particular initial conditions such as a homogeneous state, both systems reach stationary regimes of spatio-temporal intermittency with well-defined statical properties.

The mean fraction of turbulent-active sites f_t goes to zero when approaching the threshold from above.[5] Far away from this region, f_t is close to one and the laminar-absorbing clusters are small. For intermediate values of f_t, turbulent and laminar sites are in roughly equal proportions, giving rise to very intricate instantaneous spatial structures (see Fig. 1).

Let us now consider this in terms of *ordinary percolation*: At each time step, the spatial structure is viewed as a particular realization of a site percolation problem on a 2D square lattice with unknown correlations. Near the intermittency threshold, laminar-absorbing sites always percolate and turbulent-active sites form small clusters. On the contrary, there are only small clusters of laminar sites and turbulent sites always percolate for the large values of f_t observed far away from the intermittency threshold. Continuing the parallel investigation of our two systems, we now focus on the precise description of what happens in the intermediate region using the framework of finite-size scaling theory.[6]

Let us consider an ordinary site percolation problem on a 2D lattice governed by a concentration parameter a. We note $R=R(a,L)$ the probability of existence, at concentration a, of a percolating cluster on a lattice of linear dimension L. Sites of the same type are said to be connected when they can be joined by a path of similar sites following the grid lines of the transverse 2D square lattice. (On a finite lattice a cluster is said to percolate if there is at least one cluster connecting the "top" and the "bottom" of the array). For an infinite lattice, $R(a)$ is a step function: $R=0$ below and $R=1$ above a_c, the percolation threshold. We have numerically estimated R_1 (R_2), the probability for absorbing-laminar (active-turbulent) sites to percolate for DP and the CML described above.[7]

Figure 2 shows the variation of R_1 and R_2 with p for different lattice sizes L in the case of directed percolation on a body-centered cubic lattice. Results are similar for the CML defined above. The curve in each set is clearly approaching a step function when L increases. Two thresholds are thus defined, one for the percolation of absorbing-laminar sites, and one for the percolation of active-turbulent sites (see Table I). This is a usual feature of ordinary site percolation problems, for which the two

TABLE I. Summary of the values of the measured exponents for both systems at both thresholds. The exact values for ordinary percolation are given for comparison. The thresholds and the ν exponents are estimated from sets of curves similar to those of Fig. 2 for values of L between 50 and 400. The precision is 10^{-3} for the thresholds, 3×10^{-2} for ν. The error bars on the thresholds set the precision of the corresponding concentrations to 5×10^{-3}. Exponents ρ were determined by varying L between 25 and 400 and by calculating the mean size of the largest cluster over 2×10^4 iterations after a transient of 2×10^4 iterations starting from random initial conditions. Exponents τ were calculated on a 400×400 lattice by cumulating the cluster sizes distribution over 2×10^3 iterations, also after a transient period following random initial conditions. The estimated precision on these exponents is 2×10^{-2}.

System	Site type	Threshold	Concentration	ν	$1/\rho$	$1/\rho_{per.}$	τ	τ_{eff}
DP	absorbing	$p=0.315$	0.526	1.34	1.95	1.91	1.96	2.05
	active	$p=0.345$	0.581	1.33	1.92	1.90	1.97	2.07
CML	laminar	$\epsilon=0.173$	0.570	1.33	1.92	1.89	1.88	2.05
	turbulent	$\epsilon=0.186$	0.533	1.36	1.93	1.90	1.92	2.06
	Exact values for ordinary percolation			$\frac{4}{3}$		1.896	2.055	

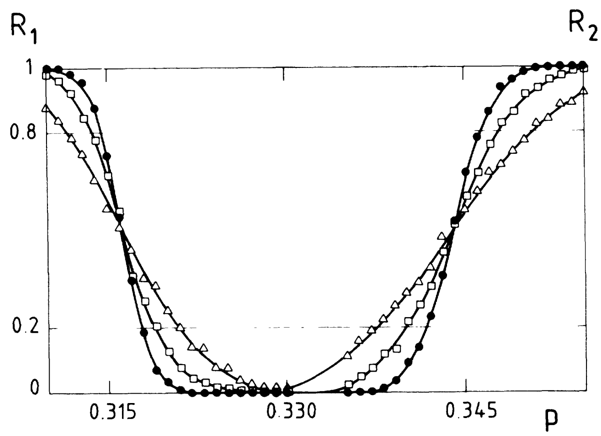

FIG. 2. Directed percolation: variation with p and for different values of the size L, of R_1 (R_2), the probability of the absorbing-laminar (active-turbulent) sites to percolate. The results are similar for the CML described in the text. The estimated asymptotic position of the inflexion point for large sizes was used to determine the thresholds. Δ, the quantity used to calculate the exponent ν, was measured on similar curves for both systems and both thresholds by determining the p interval delimited by the lines $R=0.2$ and 0.8. Each point was calculated during a simulation of 2000 iterations after a transient period. Sizes $L=50$, 71, 100, 141, 200, and 400 were investigated but only $L=50$ (△), $L=100$ (□), and $L=200$ (◊) are represented.

thresholds are related through duality. However, Table I shows that the systems studied here both display critical concentrations for the two thresholds which are not complementary of each other. This lack of duality is not surprising since the two types of sites (active-absorbing or turbulent-laminar) are not equivalent in the dynamical processes at the origin of the clusters, introducing local correlations in contrast with the case of plain geometrical percolation.

Scaling theory allows us to calculate critical exponents. Firstly, the width Δ of the transition region between small and large R scales as $L^{1/\nu}$ in ordinary percolation. This result is independent of the precise definition of Δ and we simply choose there the p interval between $R=0.2$ and 0.8 (see Fig. 2). For both systems and both thresholds we indeed find a good power law with an exponent close to the exact value of ordinary percolation (see Table I). Secondly, we considered the scaling of the size of the largest cluster S with L at threshold. For ordinary percolation $S \propto L^{1/\rho}$ with $1/\rho = d_f$, the fractal dimension in mass of the percolating cluster. The condition $a=a_c$ does not have to be fulfilled exactly, provided that a is close enough and above a_c such that the linear dimension of the lattice remains much smaller than the correlation length.[6]

This variation of S with L has been investigated for DP and the CML at both threshold values together with that of S_{per}, the size of the largest cluster when it actually percolates. Although the scaling is very good in all cases, the measured exponent for the variation of S is always slightly larger than the exact value for ordinary percolation (see Table I). However, this value is recovered with a good precision for the scaling of S_{per} (exponent ρ_{per}). This is probably due to the fact that the periodic boundary conditions used in the numerical simulations are not taken into account when determining the cluster sizes.

Finally, we measured a third exponent by determining the distribution of cluster sizes at the threshold. For both systems and both thresholds we also find very good power laws with τ exponents close to but smaller than the exact value for ordinary percolation. This is explained by a finite-size effect and the exact value is recovered when a parameter value close to the threshold is chosen such that percolation occurs at all timesteps ($R \simeq 1$) (see exponents τ_{eff} in Table I). To select an *effective threshold* value at a given size L is a usual procedure[6] when trying to determine exponents defined only in the infinite-size limit such as τ.

In conclusion, the spatial structures of the clusters generated by spatio-temporal intermittency and directed percolation can be described in terms of ordinary percolation. This is not too surprising since this analogy occurs in a region far away from the intermittency threshold where no critical dynamics is expected. In particular the correlation length and time are certainly very small so that the systems have almost no memory of their "history." The spatial structure is then essentially random (except for small scales where, for example, a checkerboard pattern can be rather frequent) with the relative proportion of sites fixed by the parameter (p or ϵ); this is of course the usual definition of an ordinary percolation problem. Yet, the local correlations in space and time introduced by the dynamics break the duality expected in ordinary percolation and could even, for some particular cases, alter the analogy put forward here.

Nevertheless, the spatial structures generated by two systems studied here exhibit the exponents of ordinary percolation which provides a promising tool to analyze the statistical properties of spatio-temporal patterns. For example, the behavior of the backbone of these percolating clusters quickly evolving in time could exhibit interesting dynamics. The emergence of "critical properties" in a noncritical region may enhance the physical interest of the phenomenon: both the deterministic and the probabilistic systems are viewed as dynamical processes on a 2D space where turbulent patches compete with laminar domains. Halfway between turbulence and statistical mechanics, they generate time-dependent self-similar structures in a simple and "natural" manner.

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[6]See, e.g., D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, London, 1985).

[7]We first checked that the spatial structures were not simply equivalent to ordinary percolation problem samplings realized at each timestep with an effective concentration equal to the instantaneous concentration of active-turbulent sites. As a matter of fact, at a given parameter value in the region of interest, there is no correlation between the fact that the system percolates or not and its instantaneous concentration. Furthermore, this quantity has normal fluctuations which go to zero as the size increases.

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