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Interaction of liquid crystals with electromagnetic fields: Mauguin theorem, angular momentum conservation, and optical Fréedericksz transitions in twisted nematic liquid crystals

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We study the propagation of an elliptically polarized light beam normally incident onto an arbitrarily oriented liquid crystal in steady-state conditions. The Euler-Lagrange equations for the molecular director and the equations describing the evolution of the beam polarization in the birefringent medium are derived from a unique variational principle, which is proved to be consistent with the geometric-optics approximation. The Hamiltonian formulation of the theory is studied in detail. The conservation of total angular momentum and total free energy in the process is derived from Noether's theorem, and the theory of the adiabatic invariants is used to obtain a new proof of Mauguin's theorem of crystal optics. The general analytical solution of the propagation problem is presented for the important case of pure twisted structures. It is proved that two particular solutions exist (called Mauguin's solutions) obeying Mauguin's theorem rigorously. Only these solutions may exhibit the occurrence of the optical Fréedericksz transition. In general, multiple optical thresholds are found. An analytical formula to obtain the thresholds is also derived.

I. INTRODUCTION

The study of the nonlinear interaction between intense optical fields and nematic liquid crystals (NLC) has received a great deal of renewed interest in recent years.[1] The existence of a characteristic threshold intensity, below which no molecular reorientation can be induced, was demonstrated both theoretically[2] and experimentally[3] for the case of linearly polarized light incident onto a homeotropically aligned sample. The underlying physical mechanism of such an effect, known as the optical Fréedericksz transition (OFT), is essentially the same as in the corresponding dc Fréedericksz transition.[4] The geometry dictates, in fact, that the polarization of the light beam remain linear in traversing the cell, even with molecular reorientation.

There are a number of other dc Fréedericksz transitions with different geometries to which one can also find an optical analogue. In most cases, however, the underlying physical mechanisms of the dc- and optical-field-induced transitions are very different, because the beam polarization varies in propagating through the medium. An example is the OFT in a planarly aligned nematic cell induced by a light beam linearly polarized in a direction perpendicular to the molecular alignment. A linearized theory for such a process shows that even the threshold behavior for the induced transition is characteristically unique and quite different from the dc analogue.[5] Clearly, an exact theory, accounting simultaneously for the optical-field-induced molecular reorientation and the change of the light polarization, would be desirable.

A rigorous study of the action of an optical field on a NLC is complicated because the field propagates in an inhomogeneous anisotropic medium and therefore the electromagnetic energy density, the Poynting vector, and the polarization state, all vary in space throughout the sample. Zel'dovich *et al.*[6] realized that the optical-field-induced molecular reorientation of the liquid crystal is to be determined consistently with Maxwell's equations. In the Zel'dovich approach, the Euler-Lagrange equations for the molecular director are obtained by minimizing the total free energy with the components of the optical electric field held constant. The solution of Maxwell's equations in the geometric-optics approximation (GOA) is then introduced back into these equations to obtain the final form of the differential equations governing the director distribution in the sample.

A variational principle avoiding the somewhat indirect method of Zel'dovich was proposed by Ong, who also criticized the Zel'dovich theory.[7] In Ong's approach, the GOA is introduced consistently in the free-energy density from the beginning. The intensity (average energy flux along the propagation direction) of the optical field enters Ong's theory as a fixed parameter. Although the approaches of Zel'dovich and Ong lead to the same final equations for the molecular director, the latter approach has the advantage of yielding a variational principle directly related to the equations that one wants to study. Both Zel'dovich's and Ong's approaches, however, have been carried out in simple geometries, where the polarization of the optical beam remains linear, even in the presence of the molecular reorientation. This excludes, for instance, twisted nematic structures (or cholesteric structures with large pitch), that may be also interesting for applications.

In this paper, a generalization of Ong's variational principle to arbitrarily distorted nematic structures and to arbitrary polarization states of the light beam is presented, for the case of normal incidence. The physical process is very complicated, in general, since all degrees of freedom (splay, twist, and bend) of the NLC are coupled together, with the parameters (e.g., Stokes' parameters) describing the polarization of the light. It would therefore be very useful to derive both the Euler-Lagrange equations for the director and the equations governing the polarization state of the optical field from a single variational principle so that one could then use the powerful tools of analytical mechanics to find conserved quantities or adiabatic invariants.

The paper is organized as follows. In Sec. II our variational approach is presented. In Sec. III the evolution of the polarization of the light beam in the medium is studied and the relevant equations are derived. In Sec. IV the conservation of the total (elastic plus optical) free-energy and total angular momentum is derived from the Hamiltonian version of the theory. In Sec. V Mauguin's theorem of crystal optics is proved using the theory of the adiabatic invariants. In Sec. VI the case of pure twist is studied and the general elliptic-integral solution is presented. Finally, in Sec. VII the occurrence of the optical Fréedericksz transition is investigated and the multiple thresholds are calculated.

II. VARIATIONAL APPROACH

Let us consider a NLC cell of thickness d confined between the planes $z=0$ and $z=d$ of a Cartesian system of coordinates. In the cell, the average molecular orientation is described by the director $\hat{\mathbf{n}}=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The molecular orientation is not uniform, in general. We assume, therefore, $\theta=\theta(z)$ and $\phi=\phi(z)$, neglecting the dependence on the x and y coordinates. We shall consider in this paper the equilibrium orientation of the director, so that the angles θ and ϕ are independent of time. In each plane $z=$const, the optical axis is directed parallel to $\hat{\mathbf{n}}(z)$. A monochromatic beam of frequency ω is normally incident on the cell along z. The polarization state of the input beam may be arbitrary. Because of the sample birefringence, the polarization of the beam changes as it propagates through the cell. Since the medium is transparent, however, the beam intensity I, defined as the z component of the average Poynting vector, remains constant.

The NLC is assimilated to a slowly varying positive uniàxial medium, so that $\theta(z)$ and $\phi(z)$ vary appreciably over a length much greater than the optical wavelength λ. Consistency with the GOA requires, however, that the polarization state of the light must also change slowly throughout the medium. This implies in turn that the birefringence of the medium must be low. The characteristic length l_c over which the polarization state changes appreciably is, in fact, $l_c \simeq \lambda/[\bar{n}(\theta)-n_o]$, where $\bar{n}(\theta)$ and n_o are the refractive indices of the extraordinary and ordinary wave in the NLC. The condition $l_c \gg \lambda$ then implies

$$\bar{n}(\theta)-n_o \ll 1 \; . \tag{1}$$

We shall assume that this condition is satisfied throughout the medium.

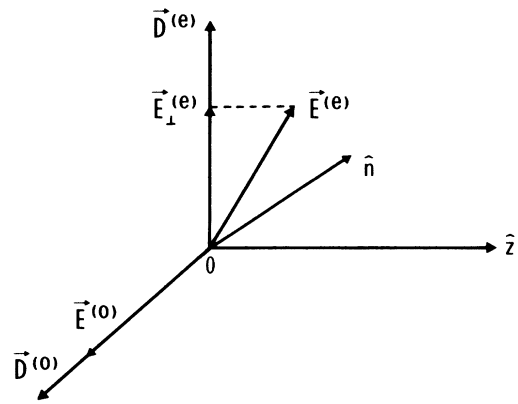

FIG. 1. Directions of the fields $\mathbf{E}^{(o)}$, $\mathbf{D}^{(o)}$, $\mathbf{E}^{(e)}$, and $\mathbf{D}^{(e)}$ and of the molecular director $\hat{n}$ in the liquid crystal. The vectors $\mathbf{E}^{(e)}$, $\mathbf{D}^{(e)}$, and $\hat{\mathbf{n}}$ are are coplanar. The z axis is directed along the propagation direction of the optical beam.

For normal incidence, in each plane $z=$const, the electric fields $\mathbf{E}^{(e)}$ and $\mathbf{E}^{(o)}$ and the electric inductions $\mathbf{D}^{(e)}$ and $\mathbf{D}^{(o)}$ of the extraordinary and ordinary waves are directed as shown in Fig. 1. The total electromagnetic energy density w is the sum of the energy density of the two waves:

$$w=\frac{1}{8\pi}[n_o^2\,|\mathbf{E}^{(o)}|^2+\bar{n}^2(\theta)\,|\mathbf{E}_\perp^{(e)}|^2] \; , \tag{2}$$

where

$$\bar{n}(\theta)=\frac{n_e n_o}{(n_e^2\cos^2\theta+n_o^2\sin^2\theta)^{1/2}} \; , \tag{3}$$

and $\mathbf{E}_\perp^{(e)}=\mathbf{E}^{(e)}-\hat{\mathbf{z}}(\hat{\mathbf{z}}\cdot\mathbf{E})$. In Eq. (3), n_e and n_o denote the extraordinary and ordinary indices of the material, respectively.

Similarly, the total intensity I is the sum of the average z components of the Poynting vectors $\mathbf{S}^{(o)}$ and $\mathbf{S}^{(e)}$ of the two waves:

$$I=\frac{c}{8\pi}[n_o\,|\mathbf{E}^{(o)}|^2+\bar{n}(\theta)\,|\mathbf{E}_\perp^{(e)}|^2] \; . \tag{4}$$

We describe the polarization of the optical field by the Stokes parameter $S_0=|E_x|^2+|E_y|^2=|\mathbf{E}^{(o)}|^2+|\mathbf{E}^{(e)}|^2$, $S_1=|E_x|^2-|E_y|^2$, $S_2=2\,\mathrm{Re}(E_x^*E_y)$, and $S_3=2\,\mathrm{Im}(E_x^*E_y)$. We introduce also the ellipticity e of the polarization ellipse $e=S_3/S_0$ and the angle $\psi=\tan^{-1}(S_2/S_1)$ that the ellipse's major axis forms with the x axis.

Observing that $E_\perp^{(e)}=E_x\cos\phi+E_y\sin\phi$ and using the identity

$$\begin{aligned}|\mathbf{E}_\perp^{(e)}|^2&=\tfrac{1}{2}[S_0+S_1\cos(2\phi)+S_2\sin(2\phi)]\\&=\tfrac{1}{2}S_0\{1+(1-e^2)^{1/2}\cos[2(\psi-\phi)]\}\; ,\end{aligned} \tag{5}$$

we can write the total electromagnetic energy density as

$$w=(I/c)\frac{n_o^2+\frac{1}{2}[\bar{n}^2(\theta)-n_o^2]\{1+(1-e^2)^{1/2}\cos[2(\psi-\phi)]\}}{n_o+\frac{1}{2}[\bar{n}(\theta)-n_o]\{1+(1-e^2)^{1/2}\cos[2(\psi-\phi)]\}} . \tag{6}$$

In deriving the last term in Eq. (5) we used the well-known formulas

$$s_1=(1-e^2)^{1/2}\cos(2\psi) ,$$
$$s_2=(1-e^2)^{1/2}\sin(2\psi) , \tag{7}$$
$$s_3=e ,$$

relating Stokes's parameters to the polarization ellipticity e and the ellipse orientation angle ψ.

In the limit of low birefringence [see Eq. (1)], w simplifies to

$$w=(In_o/c)+(I/2c)[\bar{n}(\theta)-n_o] \times\{1+(1-e^2)^{1/2}\cos[2(\psi-\phi)]\} . \tag{8}$$

Following Ong,[7] we take as a variational function the thermodynamic potential density $\tilde{F}$, defined by $\tilde{F}=F_0-w$, where F_0 is the elastic free-energy density of the NLC, given by

$$F_0=\tfrac{1}{2}(k_{22}\sin^2\theta+k_{33}\cos^2\theta)\sin^2\theta\,(d\phi/dz)^2 +\tfrac{1}{2}(k_{11}\sin^2\theta+k_{33}\cos^2\theta)(d\theta/dz)^2 , \tag{9}$$

and w is given by Eq. (8). In Eq. (9), k_{11}, k_{22}, and k_{33} denote the splay, twist, and bend elastic constants of the liquid crystal, respectively.

We assume that the equilibrium orientation of the molecular director in the presence of an intense optical field is given by the functions $\theta(z)$ and $\phi(z)$ for which

$$\int_0^d \tilde{F}\,dz=\int_0^d (F_0-w)dz=\text{minimum} . \tag{10}$$

The minimum is intended with respect to all varied functions $\theta(z)$ and $\phi(z)$ assuming fixed values at the end points $z=0$ and $z=d$ and for a fixed intensity I and polarization state (e,ψ) of the optical field.

The Euler-Lagrange equations resulting from principle (10) read

$$d/dz\,[(k_{22}\sin^2\theta+k_{33}\cos^2\theta)\sin^2\theta(d\phi/dz)] +(I/c)[\bar{n}(\theta)-n_o](1-e^2)^{1/2}\sin[2(\psi-\phi)]=0 , \tag{11a}$$

$$[k_{33}-(k_{33}-k_{11})\sin^2\theta](d^2\theta/dz^2) -(k_{33}-k_{11})\sin\theta\cos\theta(d\theta/dz)^2 -\sin\theta\cos\theta(k_{33}-2(k_{33}-k_{22})\sin^2\theta](d\phi/dz)^2 +(I/2c)\bar{n}\,'(\theta)\{1+(1-e^2)^{1/2}\cos[2(\psi-\phi)]\}=0 , \tag{11b}$$

with $\bar{n}\,'=d\bar{n}(\theta)/d\theta$.

Equations (11) must be completed with the equations governing the polarization state of the optical field. In the cases where the polarization remains linear, Eqs. (11) coincide with the equations already reported in the literature.[1-3,6-8]

III. EVOLUTION OF THE POLARIZATION STATE

The evolution of the polarization state in weakly inhomogeneous birefringent media was studied by many authors (see, e.g., the bibliographies on Refs. 9 and 10). In these works, the basic equations governing the polarization of the light were derived from Maxwell's equations in the GOA. In the case of nonabsorbing uniaxial media such as NLC, these equations may be cast in the form of the precession equation[11,12]

$$d\mathbf{s}/dz=\boldsymbol{\Omega}\times\mathbf{s} , \tag{12}$$

where $\mathbf{s}=(s_1,s_2,s_3)$, $s_j=S_j/S_0$ $(j=1,2,3,)$ are the reduced Stokes parameters and $\boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega}=(\omega/c)[\bar{n}(\theta)-n_o](\cos(2\phi), \sin(2\phi), 0) . \tag{13}$$

It is worth noting that Eqs. (12) and (13) can be derived from the thermodynamic potential F [see Eq. (15), below], provided one considers F as a Hamiltonian function, having ψ as generalized coordinate and

$$l_z=-(I/\omega)e \tag{14}$$

as conjugate momentum. As a matter of fact, l_z is the average angular momentum carried by the optical beam along the propagation direction. The photon angular momentum l_z and the angle ψ, yielding the orientation of the polarization ellipse, appear, therefore, as conjugate variables in the Hamiltonian function $\tilde{F}$.

The thermodynamic potential $\tilde{F}$ behaves as a Lagrangian function with respect to the coordinates θ and ϕ and as a Hamiltonian function with respect to the coordinate ψ and the conjugate momentum l_z. In analytical mechanics, a function behaving as $\tilde{F}$ is known as a Routh function.[13]

Taking $\tilde{F}$ as Hamiltonian, one easily obtains Hamilton's equations for the angle ψ and the ellipticity e:

$$d\psi/dz=\partial\tilde{F}/\partial l_z =-(\omega/2c)[\bar{n}(\theta)-n_o][e/(1-e^2)^{1/2}] \times\cos[2(\psi-\phi)] , \tag{15}$$

$$de/dz=(\omega/I)\partial\tilde{F}/\partial\psi =(\omega/c)[\bar{n}(\theta)-n_o](1-e^2)^{1/2}\sin[2(\psi-\phi)] . \tag{16}$$

The equivalence between Eqs. (15) and (16) and Eqs. (12) and (13) is easily recognized using relations (7).

Equations (11a), (11b), and (15) and (16) form a complete set of differential equations that can be solved for $\theta(z)$, $\phi(z)$, $e(z)$, and $\psi(z)$, once appropriate boundary conditions are given. In Sec. VI, a particular analytical solution of these equations is presented.

IV. HAMILTONIAN FORMALISM AND CONSERVED QUANTITIES

In order to find conserved quantities for the set formed by Eqs. (11) and (15), it is convenient to put the theory into a fully Hamiltonian form. The Hamiltonian H, corresponding to the Routh function $\tilde{F}$, is given by

$$H(p_\theta,\theta;p_\phi,\phi;-l_z,\psi) = H_0(p_\theta,\theta;p_\phi,\phi)+w(\theta,\phi;-l_z,\psi) , \quad (17)$$

where w is the electromagnetic energy density (7) expressed as a function of l_z by means of Eq. (14) and the Hamiltonian H_0 [corresponding to the elastic free energy (9)], is given by

$$H_0=\tfrac{1}{2}(k_{22}\sin^2\theta+k_{33}\cos^2\theta)\sin^2\theta]^{-1}p_\phi^2 + \tfrac{1}{2}(k_{11}\sin^2\theta+k_{33}\cos^2\theta)^{-1}p_\theta^2 . \quad (18)$$

Notice that in the Hamiltonian H the momentum conjugate to the coordinate ψ is $-l_z=+(I/\omega)e$. Since H is independent of z explicitly, its value E is a conserved quantity along with the solutions of Eqs. (11) and (15). The quantity $H=E=\text{const}$ has a simple physical meaning. The Hamiltonian H_0, in fact, has the same numerical value of the elastic free energy F_0. The total Hamiltonian H is therefore equal to $F=F_0+w$, i.e., to the *total* free-energy density F in the sample. We then conclude that the equilibrium state of the whole system (NLC plus optical field) is characterized by a uniform distribution of the total free energy (elastic plus optical) throughout the sample. This interesting result could hardly be obtained from Eqs. (11) and (15) without using the Hamiltonian approach.

The Hamiltonian H depends on the difference $\psi-\phi$ only and therefore it is invariant with respect to the transformation $\phi\rightarrow\phi+\delta;\psi\rightarrow\psi+\delta$ (δ arbitrary). By Noether's theorem, this implies the conservation of the sum of the momenta p_ϕ and $-l_z$, conjugate to ϕ and ψ, respectively:

$$p_\phi+(-l_z)=(k_{22}\sin^2\theta+k_{33}\cos^2\theta)\sin^2\theta(d\phi/dz) +(I/\omega)e=\text{const} . \quad (19)$$

The conservation law (19) has also a simple physical meaning. The momentum p_ϕ, in fact, is the negative of the angular momentum flux carried by the elastic forces in the NLC along the positive z direction.[14] From Eq. (14), we see that, therefore, relation (19) expresses the conservation of the total (elastic plus optical) angular momentum flux along the beam propagation direction.

V. MAUGUIN THEOREM

The Hamiltonian approach leads also to a new proof of Mauguin's theorem of crystal optics.[14] Let us consider a twisted uniaxial structure as it could be obtained, for example, by arranging a stack of N birefringent plates with their optical axes at angles ϕ_i $(i=1,2,\dots,N)$ with respect to the fixed x axis. The number N of plates is assumed to be very high and its optical thickness very small, so that the discrete distribution of the ϕ_i can be approximated by a continuous function $\phi(z)$. The function $\phi(z)$ may be arbitrary, but it is assumed to vary very slowly over the optical wavelength.

A linearly polarized light beam impinges normally onto the stack with its polarization parallel (or perpendicular) to the optical axis of the first plate, located at the plane $z=0$. Then, Mauguin's theorem states that the polarization of the beam follows adiabatically the direction of the local optical axis in the plates. In particular, the beam will emerge linearly polarized still parallel (or perpendicular) to the direction of the optical axis of the last plate of the stack. An alternative formulation of this theorem is to say that, if $\Omega\cdot s=\pm1$ [cf. Eq. (13)] at $z=0$, then $\Omega\cdot s$ will remain ±1 throughout the whole stack. Although originally formulated in the framework of crystal optics, Mauguin's theorem finds applications also in the optics of liquid crystals.[15]

Let us consider now an inhomogeneous uniaxial medium having its local optical axis directed along the unit vector $\hat{\mathbf{n}}(z)$. Let $\theta(z)$ and $\phi(z)$ be the polar angles of $\hat{\mathbf{n}}$. The functions $\theta(z)$ and $\phi(z)$ may be arbitrary, but slowly varying over the optical wavelength. Then, the evolution of the polarization of the light in traversing the medium is described by the Hamiltonian

$$H_{\text{opt}}(-l_z,\psi)=\tfrac{1}{2}[\bar{n}(\theta)-n_o]\{(I/c) +(\omega/c)[(I/\omega)^2-l_z^2]^{1/2} \times\cos[2(\psi-\phi)]\} , \quad (20)$$

where $-l_z$ and ψ are the conjugate momentum and coordinate, respectively. The Hamiltonian H_{opt} is obtained from Eq. (17) by dropping out the elastic term H_0. The term In_o/c already appearing in Eq. (8) has been omitted, because it is a constant. One can easily verify that the Hamilton equations associated with H_{opt} are Eqs. (12) [or, equivalently, Eqs. (15) and (16)]. The polar angles $\theta(z)$ and $\phi(z)$ appear in the Hamiltonian H_{opt} as slowly varying parameters. It is natural, therefore, to search for the adiabatic invariant A of H_{opt}, given by

$$A=\oint l_z d\psi =\text{const} =\pi(I/\omega)\{1-(1-e^2)^{1/2}\cos[2(\psi-\phi)]\} \quad (21)$$

(the quantity A is the area enclosed by the curve $H_{\text{opt}}=\text{const}$ in the phase space). If θ and ϕ were constant, then $A=\text{const}$ would be equivalent to $H_{\text{opt}}=\text{const}$. If θ and ϕ are slowly varying functions of z, the Hamiltonian H_{opt} is no longer constant, but the invariant A remains still constant. For transparent media ($I/\omega=\text{const}$), Eqs. (20) and (21) imply the existence of the two adiabatic invariants

$$I_1=H_{\text{opt}}/\{\omega[\bar{n}(\theta)-n_o]\}=\text{const} , \qquad I_2=(1-e^2)^{1/2}\cos[2(\psi-\phi)]=\text{const} . \quad (22)$$

One can easily verify from Eqs. (12) and (13) that $I_2=\mathbf{\Omega}\cdot\mathbf{s}$. Then, $I_2=\text{const}$ implies $\mathbf{\Omega}\cdot\mathbf{s}=\text{const}$ and the component of the Stokes vector along $\mathbf{\Omega}$ is conserved adiabatically. If, in particular, $\mathbf{s}$ is parallel or antiparallel to $\mathbf{\Omega}$ (i.e., $\mathbf{\Omega}\cdot\mathbf{s}=\pm1$), it will remain so along the whole medium, as stated in Mauguin's theorem.

Equations (22) generalize Mauguin's theorem to the case of elliptically polarized light and of slowly varying angles θ and ϕ. To our knowledge, this is the first time that the adiabatic invariants I_1 and I_2 are derived explicitly in the framework of crystal optics.

VI. CASE OF PURE TWIST

The nonlinear equations (11) and (15) can be solved analytically in the important case of purely twisted structures [θ=const, $\phi=\phi(z)$]. This is the case, for example, of twisted nematic or cholesteric liquid crystals with large pitch (in these cases $\theta=\pi/2$), but the solution applies also to smectic-C liquid crystals having the smectic layers perpendicular to the direction of the light beam. In the last case, the angle θ is to be considered as a given parameter of the smectic material, since very strong fields are required to vary it appreciably.

Setting θ=const, the conservation of the Hamiltonian (17) yields

$$k/2(d\phi/dz)^2+(I/c)(n_o+(\Delta n/2)\{1+(1-e^2)^{1/2}\cos[2(\psi-\phi)]\})=E=\text{const} , \quad (23)$$

where $k=(k_{22}\sin^2\theta+k_{33}\cos^2\theta)\sin^2\theta$ and $\Delta n=\bar{n}(\theta)-n_o$ are given constants, characteristic of the material.

The conservation of the total angular momentum yields

$$k(d\phi/dz)+(I/\omega)e=M=\text{const} . \quad (24)$$

Eliminating $d\phi/dz$ between Eqs. (23) and (24) yields

$$E=(I/c)(n_o+\Delta n/2)+(1/2k)(M-Ie/\omega)^2 +(I\Delta n/2c)(1-e^2)^{1/2}\cos[2(\psi-\phi)] . \quad (25)$$

Squaring this equation and using the second of Hamilton's equations (15), we finally obtain

$$(de/dz)^2=(\omega\Delta n/c)^2(1-e^2) -[E-(I/c)(n_o+\Delta n/2) -(1/2k)(M-Ie/\omega)^2]^2(2\omega/I)^2 . \quad (26)$$

This equation has the form $(de/dz)^2=P_4(e)$, where $P_4(e)$ is a fourth-order polynomial and therefore it can be solved analytically in terms of elliptic integrals. The arbitrary constants E and M must be determined from the boundary conditions. Although the general solution may be interesting in its own right, a complete discussion of it is very long and will be presented elsewhere. In this paper, we shall limit ourselves to a discussion of the occurrence of the optical Fréedericksz transition in twisted NLC film.

VII. OPTICAL FRÉEDERICKSZ TRANSITION

Consider a twisted nematic liquid-crystal film of thickness d. The sample walls are rubbed for planar alignment along their respective easy directions and twisted by an angle α. We assume strong anchoring. Then, without loss of generality we can pose $\phi(0)=0$ and $\phi(d)=\alpha$ at the planes $z=0$ and $z=d$, respectively. In the absence of the light beam, the sample is uniformly twisted:

$$\phi(z)=az \quad (0\le z\le d) , \quad (27)$$

with $a=\alpha/d$. With the light beam present, the uniform distortion (27) is perturbed. The perturbation is in general small but finite, even if the light intensity I is weak. No threshold occurs in these conditions. The occurrence of the OFT requires, in fact, that the distortion (27) remain unchanged, even in the presence of a finite light intensity I. For the uniform twist (27) we have $p=1/2ka^2$=const. Then the angular momentum conservation Eq. (19) yields l_z=const and Eqs. (11a) and (14) yield $\sin[2(\psi-\phi)]=0$ and e=const. The condition $\sin[2(\psi-\phi)]=0$ implies $\psi-\phi=0$ or $\pi/2$. Finally, Eqs. (15) yield

$$d\psi/dz=-(\omega/2c)\Delta n[e/(1-e^2)^{1/2}]=\text{const} . \quad (28)$$

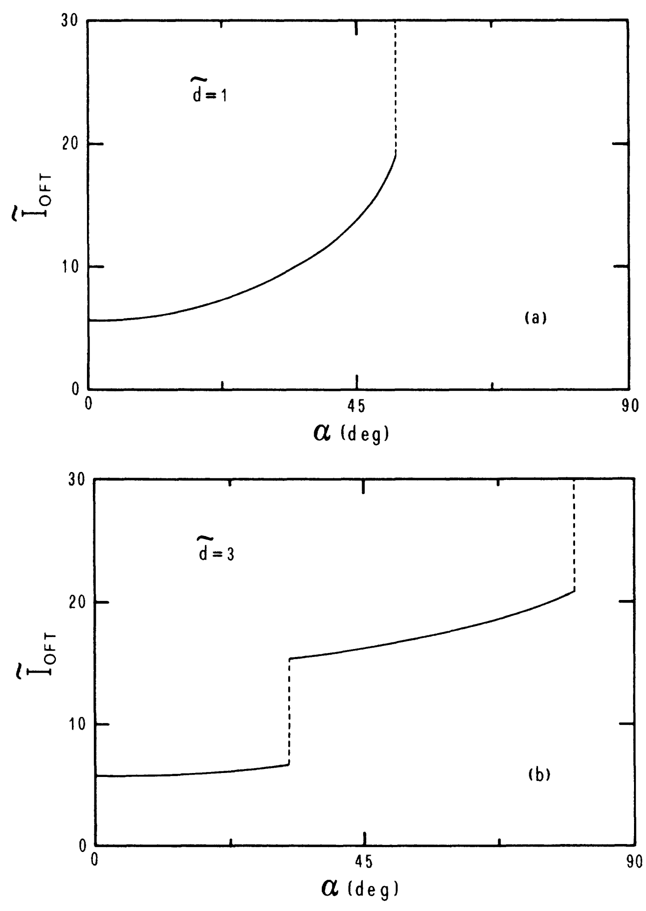

FIG. 2. Lowest intensity threshold $\tilde{I}_{OFT}$ as a function of the twist angle α for two sample thicknesses: (a) $\tilde{d}=1$; (b) $\tilde{d}=3$.

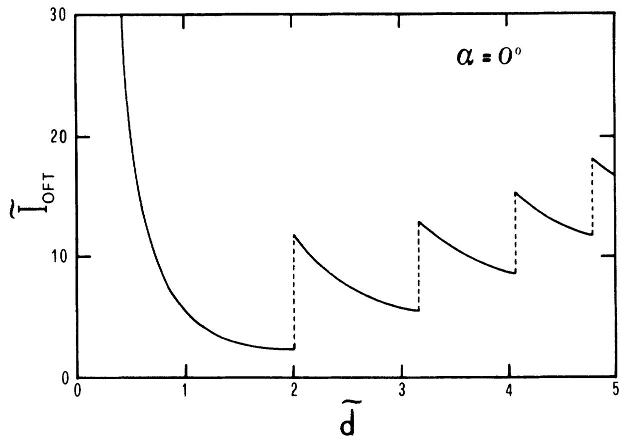

FIG. 3. Lowest threshold $\tilde{I}_{OFT}$ as a function of the reduced thickness $\tilde{d}$ for a planarly aligned ($\alpha=0$) nematic sample.

The last equation is compatible with the requirement $\sin[2(\psi-\phi)]=0$ only if $d\psi/dz=d\phi/dz=a$. This finally leads to a relation between the ellipticity of the polarization of the light beam and the twist constant a:

$$e=\frac{a}{[(\omega\,\Delta n/2c)^2+a^2]^{1/2}} \; . \tag{29}$$

The condition $\psi-\phi=0$ or $\pi/2$ is satisfied in all points in the sample so that the polarization ellipse rotates rigidly as the beam propagates in the medium, following adiabatically the local optical axis, as stated in Mauguin's theorem. It is worth noting, however, that this particular solution obeys Mauguin's theorem for any value of the light intensity I. For this reason, we will refer to the particular solution of Eqs. (11a) and (15) having θ=const, $\psi-\phi=0$ or $\pi/2$, and e given by Eq. (29) as the *Mauguin's solutions.* For a homogeneously aligned sample ($a=0$), the Mauguin solutions are linearly polarized.

The OFT in twisted nematics can be observed only for Mauguin's solutions. Mathematically, the OFT corresponds to a Hopf's bifurcation of Mauguin's solutions. The threshold intensity I_{th} where the bifurcation occurs can be found by investigating the stability of Eqs. (11a) and (15), after having linearized them around the Mauguin solution. The standard Lyapunov stability criterion leads to the following transcendental equation for the threshold intensity I_{th}:

$$\tilde{d}^{\,2}+4\alpha^2=(\tilde{d}^{\,2}+4\alpha^2-x^2)(\sin x)/x \; , \tag{30}$$

where

$$x=\{\tilde{d}^{\,2}+4\alpha^2\mp[2\tilde{I}_{th}\tilde{d}^{\,3}/(\tilde{d}^{\,2}+4\alpha^2)^{1/2}]\}^{1/2}$$

and $\tilde{I}_{th}$ and $\tilde{d}$ are the dimensionless threshold and thickness given by

$$\tilde{I}_{th}=(c/\omega)^2(I_{th}/ck\,\Delta n) \; ,$$

$$\tilde{d}=(\omega/c)\Delta n\, d \; .$$

Equation (30) has no positive root when the upper sign is taken. This means that the Mauguin solution with $\psi-\phi=0$ is always stable and no reorientation can be induced in the sample for any intensity I of the incident beam. This result may be useful when one wants to prevent reorientational effects in the sample, e.g., to study the contribution of thermal effects only.

When the lower sign is taken ($\psi-\phi=\pi/2$), Eq. (30) has multiple roots for $\tilde{I}_{th}$, for fixed thickness $\tilde{d}$, and twist angle α. The threshold $\tilde{I}_{OFT}$ for the optical Fréedericksz transition is given by the lowest of the roots of Eq. (30). In Figs. 2(a) and 2(b), $\tilde{I}_{OFT}$ is plotted as a function of the twist angle α between the rubbing directions on the sample walls for $\tilde{d}=1$ and $\tilde{d}=3$. We note that $\tilde{I}_{OFT}$ is an increasing function of α, having discontinuous jumps at critical values of α. In any case, the lowest threshold is obtained for a homogeneously aligned ($\alpha=0$) sample. The threshold $\tilde{I}_{OFT}$ for a nontwisted sample is reported in Fig. 3 as a function of the reduced thickness $\tilde{d}$. As previously noted,[12] I_{OFT} scales as $\tilde{d}^{\,-2}$ as $\tilde{d}\to 0$. The jumps in Fig. 3 are due to the successive excitation of higher-order modes. The threshold increases, since higher modes have higher elastic free energy. On the average, the threshold $\tilde{I}_{OFT}$ increases with the sample thickness.

VIII. CONCLUSIONS

We have presented a Hamiltonian approach to the problem of the propagation of a light beam in an arbitrarily oriented liquid-crystal sample, for normal incidence. The Hamiltonian equations for the molecular director and for the evolution of the beam polarization in the medium were derived from the same Hamiltonian. The conservation of the total (elastic plus optical) angular momentum [Eq. (19)] and total free energy [Eq. (17)] was derived from Noether's theorems. The use of the theory of the adiabatic invariants led us to give an alternative rigorous proof of Mauguin's theorem of crystal optics. We presented also a general analytical solution of the propagation problem in the important case of pure twist. The solution is expressed in terms of elliptic integrals. Finally, the occurrence of the optical Fréedericksz transition for twisted structures was investigated. An analytical formula [Eq. (31)] for the optical threshold was also presented.

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