Static and dynamic properties of incommensurate smectic-A_{IC} liquid crystals

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We study the elasticity, topological defects, and hydrodynamics of the recently discovered incommensurate smectic (A_{IC}) phase, characterized by two collinear mass density waves of incommensurate spatial frequency. The low-energy long-wavelength excitations of the system can be described by a displacement field $u(\mathbf{x})$ and a "phason" field $w(\mathbf{x})$ associated, respectively, with collective and relative motion of the two constituent density waves. We formulate the elastic free energy in terms of these two variables and find that when $w=0$, its functional dependence on u is identical to that of a conventional smectic liquid crystal, while when $u=0$, its functional dependence on w is the same as that for the angle variable in a slightly anisotropic XY model. An arbitrariness in the definition of u and w allows a choice that eliminates all relevant couplings between them in the long-wavelength elastic energy. The topological defects of the system are dislocations with nonzero u and w components. We introduce a two-dimensional Burgers lattice for these dislocations, and compute the interaction between them. This has two parts: one arising from the u field that is short ranged and identical to the interaction between dislocations in an ordinary smectic liquid crystal, and one arising from the w field that is long ranged and identical to the logarithmic interaction between vortices in an XY model. The hydrodynamic modes of the A_{IC} include first- and second-sound modes whose direction-dependent velocities are identical to those in ordinary smectics. The sound *attenuations* have a different direction dependence, however. The breakdown of hydrodynamics found in conventional smectic liquid crystals, with three of the five viscosities diverging as $1/\omega$ at small frequencies ω, occurs in these systems as well and is identical in all its details. In addition, there is a diffusive phason mode, not found in ordinary smectic liquid crystals, that leads to anomalously slow mechanical response analogous to that predicted in quasicrystals, but on a far more experimentally accessible time scale.

I. INTRODUCTION

Following the theoretical prediction[1] of its existence, a novel smectic liquid crystal[2] whose x-ray diffraction pattern could be generated by *two* collinear vectors $k_1\hat{z}$ and $k_2\hat{z}$ with k_1/k_2 irrational was discovered. This phase is the incommensurate smectic A (A_{IC}), which is quasiperiodically layered in one direction but liquidlike in the other two. We construct here a theory of the energetics and dynamics of long-wavelength distortions and of dislocations in this phase.

In the present work, the A_{IC} is characterized by density waves whose Fourier components are nonzero only on a discrete *incommensurate* set of reciprocal-lattice points. Such a density-wave description[3,4] leads naturally to an elastic theory involving two variables. The first, $u(\mathbf{x})$, is just the usual layer displacement field describing distortions which look locally like uniform translations. The new variable $w(\mathbf{x})$ (the phason) (Ref. 5) is the *relative* displacement of the two fundamental density waves, and arises as a consequence of their incommensurability. Using this description, an elastic theory is constructed. The u dependence of the resulting elastic free energy is identical to that[6] of a commensurate smectic liquid crystal with energies for distortions with wave vector $\mathbf{q}$ parallel to the layers proportional to q^4. The w dependence is similar to that of the phase variable in an XY model[7] with energies proportional to q^2 for all directions of $\mathbf{q}$.

The elastic theory is also used to derive the Debye-Waller factor for this system. It is shown that because phason fluctuations cost energy $\propto q^2$ in all directions, they do not contribute to the destruction of true long-range order. This is effected, as in ordinary smectic liquid crystals,[8] only by the u field. Thus the x-ray scattering intensity around each peak has the same power-law form as in periodic smectic liquid crystals.

Dislocations and a Burgers vector lattice are defined. It is shown that every Burgers vector must have a w component and that hence every dislocation must have a w field. The x-y-like energetics of the w field therefore lead to an energy per unit length of dislocation line that is proportional to the logarithm of the system size, and thus infinite in an infinite system. By contrast, in periodic smectic liquid crystals[9] this energy per unit length is independent of L. Some remarkable consequences of this novel behavior of incommensurate smectic liquid crystals will be explored elsewhere.[10]

We also construct the hydrodynamic equations[11–13] for the A_{IC}. The phason has no inertia since it is a *relative* motion of density waves. It thus gives rise to a *diffusive* mode not present in periodic smectic liquid crystals. For **q** normal to the layers, this mode mixes with the diffusive u mode. First and second sound also exist in A_{IC}. Their velocities are identical to, but their dampings differ slightly from, those of periodic smectic liquid crystals. Thus neither sound velocity vanishes or jumps discontinuously at an A_{IC}-A phase boundary. Both are, however, expected to display the usual[14] $|t|^{1-\alpha}$ singularities at this transition, where $t \equiv (T-T_C)/T_C$ is the reduced temperature of the A_{IC}-A transition, T_C that transition's temperature, and α its specific-heat exponent. Shear diffusion throughout the A_{IC} is unchanged from its form in periodic smectic liquid crystals.

Nonlinearities in the elastic free energy lead to a breakdown of conventional long-wavelength elasticity as in periodic smectic liquid crystals.[15] The same nonlinearities lead to the breakdown of conventional hydrodynamics, in which the three bulk viscosities diverge like $1/\omega$, which has been predicted theoretically[16] and observed experimentally[17] in *periodic* smectic liquid crystals. We argue that our results should, for all practical experimental purposes, be robust against "lock-in" of k_1/k_2 to a rational ratio p/q of relatively prime integers p and q.

The rest of this paper is organized as follows. Section II identifies the broken symmetry modes of the smectic A_{IC}. Section III uses Landau theory to confirm this identification and to provide a framework for the development of the elastic theory, which is discussed in the harmonic approximation in Sec. IV. Section V uses this harmonic elastic theory to calculate the order-parameter correlation functions, thereby demonstrating that in the A_{IC}, as in conventional smectic liquid crystals, long-ranged translational order is destroyed by fluctuations. Section VI discusses the characterization of dislocations, their fields, and their energies. Section VII derives the linearized hydrodynamic equations of motion for the A_{IC}. Section VIII calculates divergent nonlinear corrections to the elasticity theory, correlation functions, and hydrodynamics. The concluding section, Sec. IX, demonstrates the robustness of our theory against commensurate lock-in.

II. BROKEN-SYMMETRY MODES IN INCOMMENSURATE SMECTIC LIQUID CRYSTALS

As discussed in the Introduction, an incommensurate smectic liquid crystal (Sm A_{IC}) is a material that is fluidlike in two dimensions (the xy plane) and quasiperiodic in one dimension (the z axis). This implies that all vectors **G** in the reciprocal lattice L_R of the simplest ideal (i.e., undisturbed) A_{IC} can be expressed as integral linear combinations of two collinear basis vectors $\mathbf{k}_1 = k_1\hat{\mathbf{e}}_2$ and $\mathbf{k}_2 = k_2\hat{\mathbf{e}}_z$, where the ratio $k_1/k_2 < 1$ is irrational; i.e., $\mathbf{G} = G\hat{\mathbf{e}}_z$, where

$$G = pk_1 + qk_2 , \tag{2.1}$$

with p and q integers. Because k_1/k_2 is irrational, there is no nonzero pair of integers (p',q') such that $p'k_1 + q'k_2 = 0$, and the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are integrally linearly independent.[12,13,18] Therefore the vector $\mathbf{G}_0$ with $p = q = 0$ is the unique vector in the reciprocal lattice with zero magnitude. There are, however, vectors **G** with arbitrarily small magnitude because it is possible to find integers p and q such that $|p/q - k_1/k_2|$ is less than any preassigned number. The existence of arbitrarily small G's is a manifestation of the fact that these systems are not spatially periodic. Rank-r reciprocal lattices have bases consisting of r integrally linearly vectors.[12,18] We will consider here only Sm A_{IC} with rank-2 reciprocal lattices.

The mass density of an ideal A_{IC} can be expanded in terms of mass-density waves whose Fourier components are nonzero only at wave vectors in the reciprocal lattice L_R,

$$\rho(\mathbf{x}) = \rho_0 + \sum_{G \in L_R} [\psi_G(\mathbf{x}) e^{iGz} + \text{c.c.}] , \tag{2.2}$$

where

$$\psi_G(\mathbf{x}) \equiv |\psi_G(\mathbf{x})| e^{-i\phi_G(\mathbf{x})} . \tag{2.3}$$

The amplitudes $|\psi_G(\mathbf{x})|$ are assumed to vary very slowly on the scale of G^{-1}, and a minimum free-energy stable (ground) state with $\psi_G(x) = \text{const}$ for all $G \in L_R$ is assumed to exist. If the thermal averages $\langle \psi_G \rangle$ were nonzero, the x-ray scattering intensity from an A_{IC} would have δ-function peaks of intensity $|\langle \psi_G \rangle|^2$ at wave vector **G**. Fluctuations destroy the long-range order implied by a nonvanishing $\langle \psi_G \rangle$ as they do in periodic smectic liquid crystals,[8] and the x-ray scattering intensity will consist of power-law peaks at **G**. As in periodic smectics, in the system studied by Ratna *et al.*,[2] there are observable peaks at only a small number of wave vectors. We will make use of the weakness of these higher harmonics later when we estimate the strength of those terms that tend to lock the smectic liquid crystal into a commensurate structure.

A useful heuristic picture of an A_{IC} can be obtained by considering only the two mass density waves $\psi_1 \equiv \psi_{k_1}$ and $\psi_2 \equiv \psi_{k_2}$ associated with the basis vectors. The densities arising from these two waves are

$$\rho_{1,2} = \text{Re}(\psi_{1,2} e^{ik_{1,2}z}) = |\psi_{1,2}| \cos(k_{1,2}z - \phi_{1,2}) .$$

The zeros of the cosine functions in ρ_1 and ρ_2 define two grids (sequences of planes) perpendicular to the z axis at positions $z_{n,\alpha} = nl_\alpha + u_\alpha$, $\alpha = 1,2$ (n = integer), where $l_\alpha \equiv 2\pi/k_\alpha$ and

$$\phi_1 \equiv k_1 u_1, \quad \phi_2 \equiv k_2 u_2 . \tag{2.4}$$

Thus an incommensurate smectic liquid crystal can be thought of as two interpenetrating sets of equally spaced layers with respective layer spacings l_1 and l_2 with l_1/l_2 irrational (see Fig. 1). It is clear that u_α specifies the origin of the grid with spacing l_α. A uniform translation of

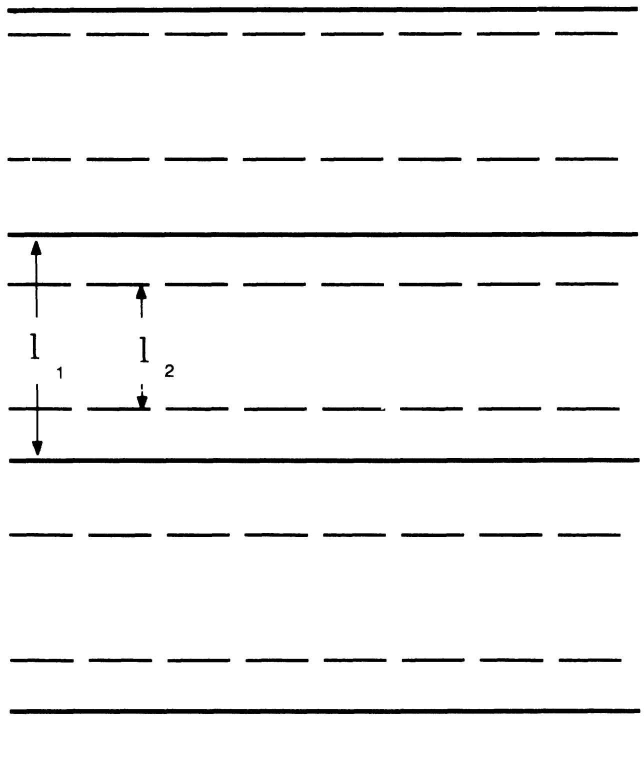

FIG. 1. Schematic representation of an incommensurate smectic liquid crystal. The parallel solid and dashed lines represent the positions of the intensity maxima of the coexisting mass density waves with respective wave numbers $k_1=2\pi/l_1$ and $k_2=2\pi/l_2$ with l_1/l_2 irrational.

grid α is described by a spatially uniform increment of u_α. Such a uniform translation of either grid will change the relative positions of the layers in the two grids. However, because l_1/l_2 is irrational, any sequence of layers in any finite region before translation can be found somewhere in the structure after translation. This implies that the free energy of the A_{IC} does not change under spatially uniform changes in either u_1 or u_2. In the jargon of condensed-matter physics, u_1 and u_2 are "broken-symmetry variables,"[11–13] or elastic variables of the incommensurate smectic. Spatially nonuniform changes do increase the free energy which for slow spatial variations can be expanded in a power series in gradients of u_1 and u_2. The form of the resulting elastic free energy, as will be discussed in Sec. IV, is determined entirely by the symmetry of the SmA_{IC} phase. It is useful, however, to use Landau theory to verify that u_1 and u_2 are indeed elastic variables and to derive the elastic free energy. This will be done in Sec. III.

III. LANDAU THEORY

We will now use the Landau theory to demonstrate[12,13] that u_1 and u_2 are both broken-symmetry variables, and furthermore the *only* broken-symmetry variables of the incommensurate smectic.

Landau theory begins by assuming that the free energy $F(\{\rho(\mathbf{x})\})$ is an analytic, local functional of the density function $\rho(\mathbf{x})$, and hence can be expanded in powers of $\rho(\mathbf{x})$ and its gradients. Carrying out this expansion and using the density-wave expansion (2.2), we obtain the free energy as an expansion in powers of the set of fields $\{\psi_G(\mathbf{x})\}$ and their spatial gradients. To leading (quadratic) order in the gradients of ψ_G and to all orders in the ψ_G's themselves, the terms in this expansion can be separated into two classes: those that involve spatial gradients of the ψ_G's (whose sum we will call F_1), and those that do not (whose sum we will call F_2). The former will determine the form of the elastic energy as a function of spatial gradients of u_1 and u_2. The latter serve to fix the set of amplitudes $|\psi_G|$ and phases ϕ_G of ψ_G that will minimize F; these will be the ground-state configurations. Fluctuations will involve departures from this ground state.

We will not be interested in the determination of the set of amplitudes $|\psi_G|$ since these will in general not be broken-symmetry variables and will therefore have small fluctuations about some energetically preferred value (i.e., they will have a mass). A two component subset of the infinite set of phases ϕ_G will, on the other hand, be broken-symmetry variables: these are precisely u_1 and u_2.

We illustrate this in two steps. First, we show that if, for all $G=pk_1+qk_2$, the set of phases ϕ_G^0 minimizes the free energy, then the set

$$\phi_G=pk_1u_1+qk_2u_2+\phi_G^0 \tag{3.1}$$

also minimizes it, for arbitrary constants u_1 and u_2. This shows that u_1 and u_2 are broken-symmetry variables since the energy is independent of them as long as they are constant in space. Second, we show that no other choice of phases will minimize the free energy. This shows that u_1 and u_2 are the *only* broken-symmetry variables for these systems.

Both results follow from the fact that F_2, when derived as described above, is the sum over all possible sets of $\{G_\alpha\}$ of

$$F_{2m}=C_{2m}\int d^3x\left\{\exp\left[i\left[\sum_{\alpha=1}^{m}G_\alpha\right]z\right]\prod_{\alpha=1}^{m}\psi_{G_\alpha}(\mathbf{x})+\text{c.c.}\right\}, \tag{3.2}$$

where $\{C_{2m}\}$ are phenomenological constants and

$$G_\alpha=p_\alpha k_1+q_\alpha k_2 . \tag{3.3}$$

For $\psi_G(\mathbf{x})=$const (by assumption the ground state) this term will vanish due to the spatial oscillation of the integrand unless

$$\sum_\alpha G_\alpha=\left[\sum_\alpha p_\alpha\right]k_1+\left[\sum_\alpha q_\alpha\right]k_2=0 . \tag{3.4}$$

Because k_1/k_2 is irrational (again by assumption), (3.4) can only be satisfied if

$$\sum_\alpha p_\alpha=0=\sum_\alpha q_\alpha , \tag{3.5}$$

since each p_α and q_α is an integer. Note that this constraint (3.5), which severely limits the number of allowed nonzero terms in F, depends crucially upon the incommensurability of k_1 and k_2.

Given (3.5), F_2's independence of u_1 and u_2 is readily established: each term (3.2) in the sum for F_2 independently remains invariant (i.e., keeps its value at $\phi_G=\phi_G^0$) under the phase change (2.6), so clearly their sum, which is F_2, does. To see this, just insert (3.1) into (3.2), for $\{G_\alpha\}$ obeying (3.4), and find

$$F_{2m}=C_{2m}\int d^3x \prod_{\alpha=1}^{m} |\psi_{G_\alpha}| \exp\left[i\sum_\alpha \phi^0_{G_\alpha}\right] \times\exp\left\{i\left[\left[\sum_\alpha p_\alpha\right]k_1u_1 + \left[\sum_\alpha q_\alpha\right]k_2u_2\right]\right\} = \int d^3x \prod_{\alpha=1}^{m}\psi^0_{G_\alpha}, \tag{3.6}$$

where $\psi^0_{G_\alpha}\equiv|\psi_G|e^{i\phi^0_{G_\alpha}}$ and where we have used (3.5) to set the phase of the second exponential to zero.

To prove our second point, we show that there is at least one particular nonzero term of the form (3.2) for every G_α which *will* change if ϕ_{G_α} departs from (3.1), namely,

$$F'_\alpha=C_\alpha\int d^3x\,(\psi_1^{p_\alpha}\psi_2^{q_\alpha}\psi_{-G_\alpha}+\text{c.c.})\ . \tag{3.7}$$

This term is clearly allowed, since the sum of its G's is $p_\alpha G_1+q_\alpha G_2-G_\alpha=0$. Writing each of the ψ_G's in this expression in terms of an amplitude and a phase, and using the fact that ϕ_1 and ϕ_2 are, by definition, just k_1u_1 and k_2u_2, respectively, we obtain

$$\begin{aligned}F_\alpha&=C_\alpha\int d^3x\,|\psi_1|^{p_\alpha}|\psi_2|^{q_\alpha}|\psi_{G_\alpha}| \times(e^{i(p_\alpha k_1u_1+q_\alpha k_2u_2-\phi_\alpha)}+\text{c.c.})\\&=2C_\alpha\int d^3x\,|\psi_1|^{p_\alpha}|\psi_2|^{q_\alpha}|\psi_{G_\alpha}| \times\cos(p_\alpha k_1u_1+q_\alpha k_2u_2-\phi_\alpha)\ ,\end{aligned} \tag{3.8}$$

where ϕ_α is the phase of ψ_{G_α} and we have used the fact that $\rho(\mathbf{x})$ is real to write $\psi_{-G}=\psi_G^*=|\psi_G|e^{-i\phi_G}$. Now in the assumed ground state $u_1=u_2=0$ and $\phi_{G_\alpha}=\phi^0_{G_\alpha}$. To keep the argument of the cosine in (3.8) and hence the total free energy invariant, one clearly must obey (3.1). Thus u_1 and u_2 embody all the broken-symmetry variable distortions of the system.

We now turn our attention to F_1, the part of the free energy involving spatial gradients of the ψ_G's. As in periodic smectic liquid crystals, F_1 should be invariant with respect to rigid rotations of both the smectic-liquid-crystal layers and the Frank director $\hat{\mathbf{n}}\equiv\hat{\mathbf{z}}+\delta\mathbf{n}$ specifying the average direction of molecular alignment. In the A_{IC} phase, the minimum-energy state by definition has $\hat{\mathbf{n}}$ normal to the layers. This implies that F_1 must be a function only of $\nabla_z\psi_G$ and of the invariant derivatives $(\nabla_\perp-iG\delta\mathbf{n})\psi_G$. The lowest-order (quadratic) term in F_1 is of the form

$$F_1^0=\int\left[\tfrac{1}{2}\sum_G C_G^z|\nabla_z\psi_G|^2+\tfrac{1}{2}\sum_G C_G^\perp|(\nabla_\perp-iG\delta\mathbf{n})\psi_G|^2\right]d^3x\ . \tag{3.9}$$

There are in addition an infinite number of terms second order in gradients involving higher powers of ψ_G. For many purposes, it is sufficient to consider only ψ_1 and ψ_2, in which case

$$\begin{aligned}F_1=\int[&\tfrac{1}{2}C_1^z|\nabla_z\psi_1|^2+\tfrac{1}{2}C_2^z|\nabla_z\psi_2|^2+\tfrac{1}{2}C_1^\perp|(\nabla_\perp-ik_1\delta\mathbf{n})\psi_1|^2+\tfrac{1}{2}C_2^\perp|(\nabla_\perp-ik_2\delta\mathbf{n})\psi_2|^2\\&+\tfrac{1}{2}D_1^z(\psi_1^*\nabla_z\psi_2\psi_2^*\nabla_z\psi_1)+\tfrac{1}{2}D_1^\perp\psi_2^*(\nabla_\perp-ik_2\delta\mathbf{n})\psi_2\psi_1^*(\nabla_\perp-ik_1\delta\mathbf{n})\psi_1+\cdots]d^3x\ ,\end{aligned} \tag{3.10}$$

where we have included the lowest-order terms coupling the gradients of ψ_1 to those of ψ_2. The total free-energy density describing an incommensurate smectic Sm A_{IC} and phase transitions from the Sm A_{IC} is $F=F_1+F_2+F_n$, where F_n is the Frank free energy[19] (a function of $\hat{\mathbf{n}}$) for a nematic liquid crystal.

IV. ELASTICITY

As we have seen, the free energy of the incommensurate smectic liquid crystal is invariant with respect to spatially uniform translations of u_1 and u_2. It will, therefore, depend only on the spatial derivatives of these variables. The elastic free energy F_{el} of an incommensurate smectic liquid crystal will thus be the sum of two smectic-like free energies for the displacements u_1 and u_2 plus coupling terms between them. The most important coupling term results from the fact that there is a volume energy cost for rotating one of the grids described after Eq. (2.4) relative to the other. To lowest order, the angle (Fig. 2) between the two grids is $\delta\boldsymbol{\Omega}=\nabla_\perp u_1-\nabla_\perp u_2$, and there will be a term in F_α proportional to $|\delta\boldsymbol{\Omega}|^2$. In addition there will be cross terms proportional to $\partial_z u_1\partial_z u_2$ and $\nabla_\perp^2u_1\nabla_\perp^2u_2$. The incommensurate elastic free-energy density is, therefore,

$$\begin{aligned}f=&\tfrac{1}{2}B_1(\partial_zu_1)^2+\tfrac{1}{2}K_{11}(\nabla_\perp^2u_1)^2\\&+\tfrac{1}{2}B_2(\partial_zu_2)^2+\tfrac{1}{2}K_{22}(\nabla_\perp^2u_2)^2\\&+\tfrac{1}{2}D|\nabla_\perp u_1-\nabla_\perp u_2|^2+B_{12}\partial_zu_1\partial_zu_2\\&+K_{12}(\nabla_\perp^2u_1)(\nabla_\perp^2u_2)\ .\end{aligned} \tag{4.1}$$

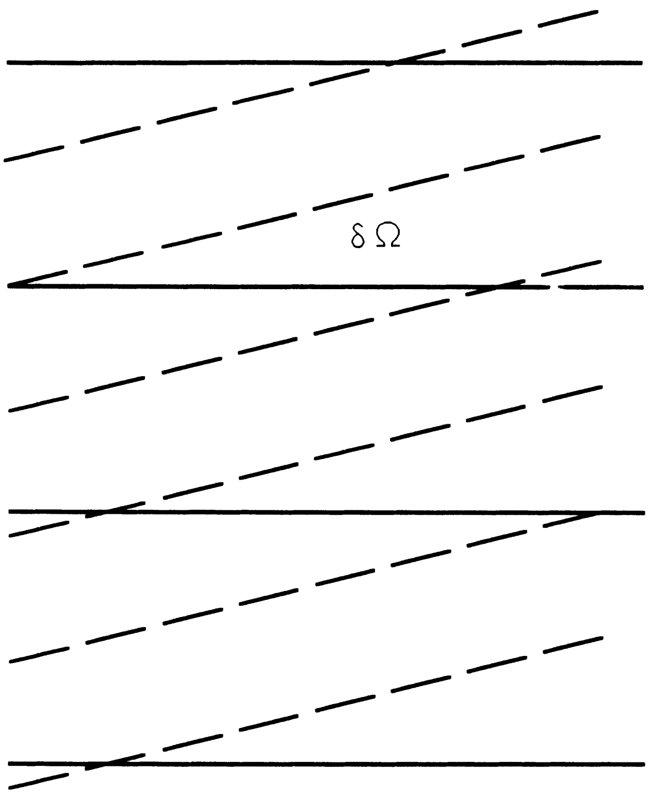

FIG. 2. Schematic representation of an incommensurate smectic liquid crystal in which the directions of the periodic modulation of the two constituent mass-density waves are at a relative angle $\delta\Omega$. There is an energy cost proportional to the volume times $(\delta\Omega)^2$ for such a distortion from the ideal configuration shown in Fig. 1.

This free energy could have been derived from the Landau theory just presented by taking $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$ to have constant amplitude (since fluctuations in these amplitudes are not broken-symmetry variables) and allowing only their phases $k_1u_1(\mathbf{x})$ and $k_2u_2(\mathbf{x})$ to vary with $\mathbf{x}$. Upon inserting this into the Landau theory just presented and minimizing over $\delta\mathbf{n}$, we would obtain (4.1) with B_α proportional to $|\psi_\alpha|^2$, $\alpha=1,2$, and B_{12} to $|\psi_1|^2|\psi_2|^2$. The other coefficients are more complicated functions of the parameters of the Landau theory (including the Frank elastic constants) and of $|\psi_1|$ and $|\psi_2|$.

As we have seen, u_1 and u_2 describe translations of the component sublattices of the incommensurate smectic liquid crystal. The physics of displacements in which the layers move together ($u_1=u_2$) differs from that in which they do not ($u_1\neq u_2$). It is therefore useful to make a change of variables[12] to an overall displacement variable u and a relative displacement or phason variable w via

$$u_1=u-\alpha s w,\quad u_2=u+\alpha(1-s)w \ , \tag{4.2}$$

so that

$$u=(1-s)u_1+su_2,\quad w=\alpha^{-1}(u_2-u_1) \ . \tag{4.3}$$

The parameters α and s are arbitrary and can be chosen to simplify calculations. u is the usual displacement variable describing common motion of the two sublattices, and w is the phason variable describing relative motion of the two sublattices. The phases ϕ_G [Eq. (3.1)] can be reexpressed in terms of u and w via

$$\phi_G=Gu+\alpha G^{\perp}w+\phi_G^0 \ , \tag{4.4}$$

where

$$G^{\perp}=-spk_1+(1-s)qk_2 \tag{4.5}$$

is a complementary vector associated with G and indexed by the same integers p and q as G.

The elastic free energy expressed in terms of u and w is

$$\begin{aligned} f=&\tfrac{1}{2}B_u(\partial_z u)^2+\tfrac{1}{2}K_u(\nabla_\perp^2 u)^2\\ &+B_{uw}(\partial_z u)(\partial_z w)+\tfrac{1}{2}C_w|\nabla_\perp w|^2+\tfrac{1}{2}B_w(\partial_z w)^2\\ &+\tfrac{1}{2}K_w(\nabla_\perp^2 w)^2+K_{uw}(\nabla_\perp^2 u)(\nabla_\perp^2 w) \ , \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} &B_u=B_1+B_2+2B_{12} \ ,\\ &B_w=\alpha^2[s^2B_1+(1-s)^2B_2-2s(1-s)B_{12}] \ ,\\ &B_{uw}=\alpha[-sB_1+(1-s)B_2+(1-2s)B_{12}] \ ,\\ &C_w=\alpha^2 D \ ,\\ &K_u=K_1+K_2+2K_{12} \ ,\\ &K_w=\alpha^2[s^2K_1+(1-s)^2K_2-2s(1-s)K_{12}] \ ,\\ &K_{uw}=\alpha[-sK_1+(1-s)K_2+(1-2s)K_{12}] \ . \end{aligned} \tag{4.7}$$

The choice $s=(B_2+B_{12})/B_u$ eliminates B_{uw} and leads to $B_w=\alpha^2(B_1B_2-B_{12}^2)/B_u$. Stability requires that B_u and B_w be positive, of course.

Using the Landau theory expressions for B_1, B_2, and B_{12} derived earlier, we can show that

$$\begin{aligned} &B_u\propto|\psi_1|^2+\sigma|\psi_2|^2 \ ,\\ &B_w\propto\frac{|\psi_1|^2|\psi_2|^2}{|\psi_1|^2+\sigma|\psi_2|^2} \ , \end{aligned} \tag{4.8}$$

where σ is a parameter depending on α and s and the constants in $f_{\rm el}$. This equation enables us to predict that B_w will vanish while B_u will remain finite at a *continuous* transition between the A_{IC} and a periodic smectic liquid crystal in which *one* of the components (say, ψ_1) of the density wave vanishes. (This is to be expected, of course, since there is no w mode in periodic smectic liquid crystals). In Landau theory, $|\psi_1|^2\propto|T_c-T|$ near the transition, and hence $B_w\propto|T_c-T|$. A more correct argument, including fluctuations, shows that B_w vanishes like the superfluid density at the superfluid-normal transition, which by the Josephson[20] argument implies,

$$B_w\propto|T_c-T|^{\nu_{XY}} \ , \tag{4.9}$$

where $\nu_{XY}\approx\frac{2}{3}$ is the correlation length exponent of the $d=3XY$ transition, to which universality class the $A_{IC}\rightarrow A$ transition can readily be shown to belong. The terms in Eq. (4.6) involving $\nabla_\perp^2 w$ are subdominant so that u and w are effectively uncoupled in f

$$\begin{aligned} f &= f_u + f_w , \\ f_u &= \tfrac{1}{2}B_u(\partial_z u)^2 + \tfrac{1}{2}K_u(\nabla_\perp^2 u)^2 , \\ f_w &= \tfrac{1}{2}B_w(\partial_z w)^2 + \tfrac{1}{2}C_w \,|\nabla_\perp w|^2 . \end{aligned} \tag{4.10}$$

f_u is identical in form to the elastic free energy of a periodic smectic liquid crystal,[6] whereas f_w is the elastic free energy[7] of an anisotropic xy model. The presence of both a smectic- and an xy-like piece in f is responsible for many of the unusual properties of incommensurate smectic liquid crystals to be discussed below.

V. CORRELATION FUNCTIONS

The smectic form of f_u immediately implies that there is no long-range order in incommensurate smectic liquid crystals as in periodic smectic liquid crystals.[8] This can be seen from

$$\langle \psi_G \rangle \cong |\psi_G|\, e^{i\phi_G^0} e^{-(1/2)[\alpha^2(G^\perp)^2\langle w^2(\mathbf{x})\rangle + G^2\langle u^2(\mathbf{x})\rangle]} , \tag{5.1}$$

an expression which is valid in the harmonic approximation, where the statistics of u and w are Gaussian. $\langle w^2(\mathbf{x})\rangle$ is finite since $\langle w_q^2\rangle \sim q^{-2}$, while $\langle u^2(\mathbf{x})\rangle$ as in a periodic smectic liquid crystal[8] is proportional (in the harmonic approximation) to $\ln L$ in a sample of size L. Thus $\langle u^2(\mathbf{x})\rangle$ determines $\langle \psi_G\rangle$, which is zero for $L\to\infty$. The density-density correlation function is then

$$\langle \rho(\mathbf{x})\rho(\mathbf{0})\rangle = \sum_G |\psi_G|^2 e^{iGz} e^{-G^2 g_{uu}(\mathbf{x}) - \alpha^2(G^\perp)^2 g_{ww}(\mathbf{x})} , \tag{5.2}$$

where $g_{uu}(\mathbf{x}) = \langle [u(\mathbf{x}) - u(\mathbf{0})]^2\rangle/2$ and $g_{ww}(\mathbf{x}) = \langle [w(\mathbf{x}) - w(\mathbf{0})]^2\rangle/2$. As in a periodic smectic liquid crystal,[8]

$$g_{uu}(\mathbf{x}) = \begin{cases} \dfrac{1}{8\pi}\dfrac{k_BT}{\sqrt{B_uK_u}}\ln x_\perp^2, & \lambda z \ll x_\perp^2 \\[2ex] \dfrac{1}{8\pi}\dfrac{k_BT}{\sqrt{B_uK_u}}\ln z, & \lambda z \gg x_\perp^2 \end{cases} \tag{5.3}$$

with $\lambda \equiv \sqrt{K_u/B_u}$, the splay coherence length, and

$$g_{ww}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1-e^{i\mathbf{k}\cdot\mathbf{x}}}{B_w k_z^2 + C_w k_\perp^2} \to \text{const as } |\mathbf{x}|\to\infty . \tag{5.4}$$

Thus the components of the correlation function at wave number $G\neq 0$ decay algebraically at large $|\mathbf{x}|$:

$$\langle \rho(\mathbf{x})\rho(\mathbf{0})\rangle = \sum_G A_G e^{iGz} f_G(\mathbf{x}) ,$$

where

$$f_G(\mathbf{x}) = \begin{cases} x_\perp^{-2\eta G^2}, & \lambda z \ll x_\perp^2 \\ z^{-\eta G^2}, & \lambda z \gg x_\perp^2 \end{cases} \tag{5.5}$$

and where

$$\eta \equiv \frac{k_BT}{8\pi\sqrt{B_uK_u}} .$$

The x-ray scattering intensity is the Fourier transform of this correlation function, and at wave vector $\mathbf{q} = (q_z, \mathbf{q}_\perp)$ near $\mathbf{G} = (G,0)$ is proportional to $|q_z - G|^{-2+G^2\eta}$ for $\mathbf{q}_\perp = 0$ and $|\mathbf{q}_\perp|^{-4+2\eta G^2}$ for $q_z = G$. This is exactly the same form as the scattering intensity of a periodic smectic liquid crystal.[8] The phason fluctuations alter the coefficients in front of the preceding power laws but are not responsible for the destruction of long-range order, nor do they change the exponent η.

VI. DISLOCATIONS

We now turn to dislocations in incommensurate smectic liquid crystals. Dislocations in three dimensions are the topologically stable line defects[21] of any translationally ordered state. In a complete circuit around any path enclosing a dislocation line, the broken-symmetry variables change in such a way that the phase of every mass-density wave amplitude ψ_G changes by an integral multiple of 2π. This ensures that ψ_G itself has no abrupt discontinuities in space—a desirable requirement since such discontinuities would cost a large gradient energy. In incommensurate smectic liquid crystals ϕ_G satisfies Eq. (3.1), and changes in u_1 and u_2 by integral multiples of $l_1 = 2\pi/k_1$ and $l_2 = 2\pi/k_2$ produce the desired changes in ϕ_G. Thus the broken-symmetry variables u_1 and u_2 in incommensurate smectic liquid crystals satisfy

$$\int_P \nabla u_\alpha \cdot dl = b_\alpha = m_\alpha l_\alpha, \quad \alpha = 1,2 \tag{6.1}$$

where m_α is an integer and P is any closed path enclosing

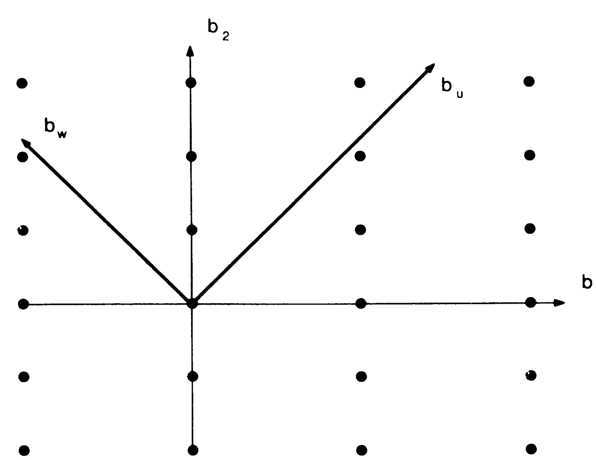

FIG. 3. The two-dimensional Burgers vector lattice of an incommensurate smectic. The vectors in the lattice are indexed by two integers n and m and are of the form $\mathbf{b} = (nl_1, ml_2)$. The components b_u and b_w of a Burgers vector along the displacement and phason directions are its projections onto the b_u and b_w axes shown in the figure. The tangent of the angle between the b_u and b_1 axes is irrational so that no Burgers vector lies precisely on the b_u axis, i.e., every Burgers vector must have a nonzero b_w component.

the dislocation line. The two-component vector (b_1,b_2) is the Burgers vector of the dislocation. The set of Burgers vectors define a two-dimensional rectangular Burgers vector lattice depicted in Fig. 3.

The integers m_1 and m_2 are independent. This then implies that a dislocation with Burgers vector $(m_1 l_1, m_2 l_2)$ corresponds to the insertion of m_1 layers in the primary density wave ψ_1 and m_2 layers in ψ_2. The dislocation line is the path of the ends of these extra layers. An edge dislocation with $\mathbf{b}=(l_1,0)$ is depicted in Fig. 4.

Since the conditions of Eq. (6.1) are linear, they can equally well be expressed with the aid of Eq. (4.3) in terms of u and w,

$$\begin{aligned} \int_P \nabla u \cdot dl &= b_u \ , \\ \int_P \nabla w \cdot dl &= b_w \ , \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} b_u &= (1-s)m_1 l_1 + s m_2 l_2 = (1-s)b_1 + s b_2 \ , \\ b_w &= \frac{m_2 l_2 - m_1 l_1}{\alpha} = (b_2 - b_1)\frac{1}{\alpha} \ . \end{aligned} \tag{6.3}$$

These equations describe the transformation of the Burgers vector lattice to the new coordinates b_u and b_w as shown in Fig. 3.

An important point should be noted about the parametrization of the Burgers vector in terms of b_u and b_w. Because of the incommensurability of l_1 and l_2, b_w *never* vanishes for any point on the Burgers lattice. This is a general property of dislocations in incommensurate structures[12] including quasicrystals.[22] This is important physically because, as discussed earlier, the w field has no "soft" directions in which distortions are easy. As a result, forcing a dislocation to carry a w component (as a nonzero b_w does) makes the energy of a dislocation much higher, and also (as we shall see) makes the interaction between dislocations long ranged. This property distinguishes incommensurate from ordinary commensurate smectic liquid crystals; dislocations in the latter have only a short-ranged interaction. This should have a number of experimentally observable consequences, for example, for grain boundaries, which will be discussed in a future publication.[10]

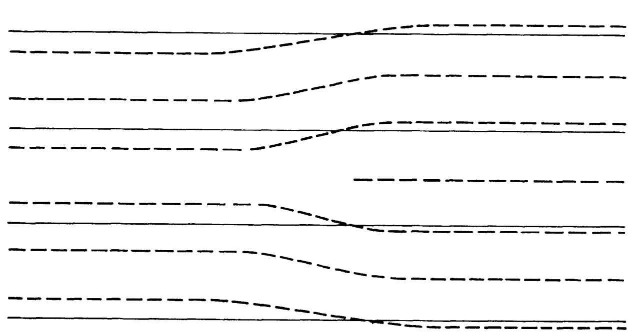

FIG. 4. Schematic representation of an edge dislocation with Burgers vector $\mathbf{b}=(l,0)$ in which an extra layer of the grid with spacing l has been inserted from the right.

The conditions of Eq. (6.2) imply[9,23] that the curls of the vector fields $\mathbf{\Omega}_u = \nabla u$ and $\mathbf{\Omega}_w = \nabla w$ satisfy

$$\nabla \times \mathbf{\Omega}_u(\mathbf{x}) = \mathbf{m}_u(\mathbf{x}) \ , \tag{6.4a}$$

$$\nabla \times \mathbf{\Omega}_w(\mathbf{x}) = \mathbf{m}_w(\mathbf{x}) \ , \tag{6.4b}$$

where

$$\mathbf{m}_u(\mathbf{x}) \equiv \sum_\alpha b_u^{(\alpha)} \int \delta(\mathbf{x}-\mathbf{x}^\alpha(s)) \hat{\mathbf{t}}_\alpha(s) ds^{(\alpha)} \ , \tag{6.5a}$$

$$\mathbf{m}_w(\mathbf{x}) \equiv \sum_\alpha b_w^{(\alpha)} \int \delta(\mathbf{x}-\mathbf{x}^\alpha(s)) \hat{\mathbf{t}}_\alpha(s) ds^{(\alpha)} \ , \tag{6.5b}$$

are the u and w components of the dislocation density. In Eq. (6.5), $\mathbf{x}^\alpha(s)$ is the position and $\hat{\mathbf{t}}^{(\alpha)}(s)$ the unit tangent vector of the αth dislocation line as a function of its arc length s.

The characterization of dislocations in terms of b_u and b_w is more useful than that in terms of b_1 and b_2 because u and w are decoupled in the elastic free energy [Eq. (4.10)]. The u and w components of the elastic free energy can be expressed in terms of $\mathbf{\Omega}_u$ and $\mathbf{\Omega}_w$ as

$$F_u = \int \tfrac{1}{2}(B_u \Omega_{uz}^2 + K_u \,|\, \nabla_\perp \cdot \mathbf{\Omega}_{u\perp} |^2) d^3x \ , \tag{6.6}$$

$$F_w = \int \tfrac{1}{2}(B_w \Omega_{wz}^2 + C_w \,|\, \mathbf{\Omega}_{w\perp} |^2) d^3x \ . \tag{6.7}$$

$\mathbf{\Omega}_u$ and $\mathbf{\Omega}_w$ are determined in the usual way[9,24] by solving

$$\left.\frac{\delta F}{\delta u}\right|_{\nabla u = \Omega_u} = 0, \quad \left.\frac{\delta F}{\delta w}\right|_{\nabla w = \Omega_w} = 0 \tag{6.8}$$

simultaneously with Eqs. (6.4).

It is clear that the solutions for $\mathbf{\Omega}_u$ and u in terms of $\mathbf{m}_u$ are identical to the solutions for the corresponding quantities in periodic smectic liquid crystals,[9,24] since the conditions (6.4a) and (6.8) and the u-dependent part of the free energy (6.6) are all precisely the same. Thus we can just copy the known solution[24] to this problem, and obtain the desired solution for $\mathbf{\Omega}_u(\mathbf{m}_u)$. The general result is most conveniently written in terms of the spatial Fourier transforms $\mathbf{\Omega}_u(\mathbf{q})$ and $\mathbf{m}_u(\mathbf{q})$ of $\mathbf{\Omega}_u(\mathbf{x})$ and $\mathbf{m}_u(\mathbf{x})$,

$$\mathbf{\Omega}_u(\mathbf{q}) = \frac{i\mathbf{q}\times\mathbf{m}_u(\mathbf{q})}{q^2} - \frac{i\mathbf{q}\hat{\mathbf{z}}\cdot[\mathbf{q}\times\mathbf{m}_u(\mathbf{q})]q_z}{q^2(q_z^2+\lambda^2 q_\perp^4)} \ . \tag{6.9}$$

In two simple special cases, this result simplifies to $u = -[\tan^{-1}(x/y)]b_u/2\pi$ for a straight screw dislocation and

$$u = \tfrac{1}{4} b_u [\mathrm{erf}(x/\sqrt{4\lambda\,|\,z\,|}\,)+1]\mathrm{sgn}(z)$$

for a straight edge dislocation. Edge and screw dislocations lie entirely within and orthogonal to the layers, respectively.

If B_w were equal to C_w, the free energy for w would be identical with that for the phase variable in an XY model.

In that case we would have $w=-\theta b_w$ for both screw and edge dislocations and, indeed, for any straight dislocation regardless of its orientation. (Here θ is the angle the radius vector from the dislocation core makes with some fixed axis orthogonal to the core.) The anisotropy implied by $C_w \neq B_w$ leads to differences between edge and screw dislocations. We, therefore, present the general solution for $\mathbf{\Omega}_w$ for an arbitrary complexion of dislocations, and then use it to obtain the specific solutions for the special cases of straight screw and edge dislocations as well.

Like any three-dimensional vector, $\mathbf{\Omega}_w$ can be written as the sum of a curl and gradient,

$$\mathbf{\Omega}_w = \nabla \times \mathbf{A} + \nabla \phi \ . \tag{6.10}$$

This decomposition is somewhat arbitrary. This arbitrariness is a gauge freedom, and may be eliminated by choosing the gauge $\nabla \cdot \mathbf{A}=0$. ϕ drops out when (6.10) is inserted into (6.4b), leaving

$$\nabla \times (\nabla \times \mathbf{A}) = \mathbf{m}_w \ . \tag{6.11}$$

Using the vector calculus identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ and our earlier gauge choice $\nabla \cdot \mathbf{A}=0$, we obtain

$$\mathbf{A}(\mathbf{q}) = \frac{\mathbf{m}_w(\mathbf{q})}{q^2} \tag{6.12}$$

after a Fourier transform in space.

Now we can determine ϕ from the Euler-Lagrange equation

$$\left.\frac{\delta H}{\delta w}\right|_{\nabla w = \Omega w} = B_w \partial_z (\Omega_w)_z + C_w \nabla_\perp \cdot (\mathbf{\Omega}_w)_\perp = 0 \ . \tag{6.13}$$

Fourier transforming this and our decomposition (6.10) of $\mathbf{\Omega}_w$, inserting the latter into the former and using our solution (6.12) for $\mathbf{A}(\mathbf{q})$, gives us a simple linear algebraic equation for $\phi(\mathbf{q})$ in terms of $\mathbf{m}(\mathbf{q})$, whose solution is

$$\phi(\mathbf{q}) = \frac{(C_w - B_w) q_z \hat{\mathbf{z}} \cdot (\mathbf{q} \times \mathbf{m}_w)}{q^2 (B_w q_z^2 + C_w q_\perp^2)} \ , \tag{6.14}$$

which, when reinserted back into the Fourier transform of (6.10) along with our solution (6.12) for $\mathbf{A}(\mathbf{q})$, gives us our final expression for $\mathbf{\Omega}_w(\mathbf{q})$,

$$\mathbf{\Omega}_w(\mathbf{q}) = \frac{i}{q^2} \left[\mathbf{q} \times \mathbf{m}_w(\mathbf{q}) + \frac{(C_w - B_w) \mathbf{q} q_z}{B_w q_z^2 + C_w q_\perp^2} \hat{\mathbf{z}} \cdot [\mathbf{q} \times \mathbf{m}_w(\mathbf{q})] \right] . \tag{6.15}$$

The dislocation line of a single straight screw dislocation is parallel to the z axis and

$$\mathbf{m}_w = \delta^2(\mathbf{x}_\perp) b_u \hat{\mathbf{z}} \quad \text{(screw)} \ . \tag{6.16}$$

The path of single straight edge dislocation is perpendicular to $\hat{\mathbf{z}}$. This leads to

$$\mathbf{m}_w = b_w \delta(y) \delta(z) \hat{\mathbf{x}} \quad \text{(edge)} \ , \tag{6.17}$$

where $\mathbf{x}_\perp$ is the projection of the position vector $\mathbf{x}$ perpendicular to $\hat{\mathbf{z}}$ and, where we have arbitrarily chosen the edge dislocation to run along the x axis. Using Eq. (6.15), it is straightforward to show that

$$w(\mathbf{x}_\perp) = -\tan^{-1}(x/y) bw/2\pi \tag{6.18}$$

for a screw dislocation and

$$w(\mathbf{x}) = \frac{b_w}{2\pi} [2 \tan^{-1}(z/y) - \tan^{-1}(\sqrt{\gamma} z/y)] \tag{6.19}$$

for an edge dislocation, where $\gamma = C_w/B_w$. In screw dislocations, $w(\mathbf{x})$ and $u(\mathbf{x})$ have the same linear dependence on $\theta = \tan^{-1}(x/y)$. Thus

$$u_\alpha = \frac{\theta b_\alpha}{2\pi}, \quad \alpha = 1,2 \tag{6.20}$$

for screw dislocations.

The elastic energy arising from dislocations follows from the free energy [Eqs. (6.6) and (6.7)] and the solutions for $\mathbf{\Omega}_u$ and $\mathbf{\Omega}_w$ in terms of $\mathbf{m}_u$ and $\mathbf{m}_w$. The resulting dislocation energy is

$$H(\mathbf{m}_u(\mathbf{q}), \mathbf{m}_w(\mathbf{q})) = \tfrac{1}{2} \int_q [m_{ui}(\mathbf{q}) U_{ij}^u(\mathbf{q}) m_{uj}(-\mathbf{q}) + m_{wi}(\mathbf{q}) U_{ij}^w(\mathbf{q}) m_{wj}(-\mathbf{q})] \ , \tag{6.21}$$

with the potentials $U_{ij}^{u,w}(\mathbf{q})$ given by

$$U_{ij}^{(u)}(\mathbf{q}) = \frac{P_{ij}^\perp K_u q_\perp^2}{q_z^2 + \lambda^2 q_\perp^4} + \delta_{ij} E_{\text{core}}^u \tag{6.22}$$

and

$$U_{ij}^{(w)}(\mathbf{q}) = (\delta_{ij}^\perp B_w + \delta_{ij}^z C_w) C_w / (B_w q_z^2 + C_w q_\perp^2) + \delta_{ij} E_{\text{core}}^w \ , \tag{6.23}$$

where $\delta_{ij}^\perp \equiv \delta_{ij} - \delta_{ij}^z$; $\delta_{ij}^z = 1$, $i=j=z$ and 0 otherwise; and $P_{ij}^\perp \equiv \delta_{ij}^\perp - q_i^\perp q_j^\perp / q_\perp^2$. In deriving these results, we made use of the fact that $\nabla \cdot \mathbf{m}(\mathbf{x}) = 0 = \mathbf{q} \cdot \mathbf{m}(\mathbf{q})$. The most important thing to note about these expressions is that for most directions of $\mathbf{q}$ ($q_z=0$ being the only exception), the u-u interaction is short ranged (in the sense that U_{ij}^u remains finite as $|\mathbf{q}| \to 0$), while U_{ij}^w is long ranged for *all* directions (diverging as $1/q^2$ as $|\mathbf{q}| \to 0$). Thus the energy per unit length of a single straight screw dislocation is $E_{\text{screw}} = E_{\text{core}}^u b_u^2 + C_w b_w^2 \ln L$ and that of a straight edge dislocation is $E_{\text{edge}} = E_u^{\text{core}} b_u^2 + b_w^2 \sqrt{C_w B_w} \ln L$. Similarly, the interaction between the w part of dislocations is long range ($\propto \ln|\mathbf{x}|$), whereas that between the u part is short range as in periodic smectic liquid crystals. In light of this fact, it is extremely significant that b_w can never be zero for any dislocation, since this then implies that all dislocations must have some long-range interaction. This is in direct contrast to conventional commensurate smectic liquid crystals, where only the U_{ij}^u term in the interaction is present, and hence interactions between dislocations are short ranged.

VII. HYDRODYNAMICS

The long-wavelength, low-frequency excitations of incommensurate smectic liquid crystals are naturally described by hydrodynamic equations. The hydrodynamic variables are the momentum density $\mathbf{g}$, the mass density ρ, the energy density ε, and the broken-symmetry variables u and w. In order to keep the analysis simple, we will ignore the energy density in what follows. A detailed analysis[25] shows that the only effects of including it are the usual ones, namely, the following.

(1) It introduces a new diffusive mode, which is, of course, thermal diffusion,

(2) It slightly modifies the sound speeds and dampings.

The dynamical equations are derived most conveniently using Poisson brackets.[26] For a general set $\{A_i\}$ of variables, governed by an effective free energy F, the hydrodynamic equations are

$$\partial_t A_i + \{A_i, A_j\}\frac{\partial F}{\partial A_j} + \Gamma_{ij}\frac{\partial F}{\partial A_j} = 0 . \tag{7.1}$$

Here $\{\ \}$ is a classical Poisson bracket, and Γ_{ij} is a matrix of kinetic coefficients. Since [16,27]

$$\{\mathbf{g}(\mathbf{x}), u_\alpha(\mathbf{x}')\} = [\hat{\mathbf{z}} - \nabla u_\alpha(\mathbf{x})]\delta(\mathbf{x}-\mathbf{x}') \tag{7.2}$$

for $\alpha = 1,2$ we see, using the definition (4.3) of u and w, that

$$\begin{aligned}\{\mathbf{g}(\mathbf{x}), u(\mathbf{x}')\} &= (1-s)[\hat{\mathbf{z}} - \nabla u_1(\mathbf{x})]\delta(\mathbf{x}-\mathbf{x}') \\ &\quad + s[\hat{\mathbf{z}} - \nabla u_2(\mathbf{x}')]\delta(\mathbf{x}-\mathbf{x}') \\ &= [\hat{\mathbf{z}} - \nabla u(\mathbf{x})]\delta(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{7.3}$$

and

$$\begin{aligned}\{\mathbf{g}(\mathbf{x}), w(\mathbf{x}')\} &= \alpha^{-1}[\hat{\mathbf{z}} - \nabla u_2(\mathbf{x})]\delta(\mathbf{x}-\mathbf{x}') \\ &\quad - \alpha^{-1}[\hat{\mathbf{z}} - \nabla u_1(\mathbf{x})]\delta(\mathbf{x}-\mathbf{x}') \\ &= -\alpha^{-1}\nabla(u_2 - u_1)\delta(\mathbf{x}-\mathbf{x}') \\ &= -\nabla w(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}') .\end{aligned} \tag{7.4}$$

The remaining Poisson brackets are the same as those of an ordinary fluid.[16,27] The effective free-energy governing the variables $\mathbf{g}$, ρ, u, and w is $F_{\text{eff}} = \int g^2/2\rho + U(\rho,u,w)$, with

$$U(\rho,u,w) = \int \left[\frac{A}{2}\left(\frac{\delta\rho}{\rho_0}\right)^2 + C\frac{\delta\rho}{\rho_0}\frac{\partial u}{\partial z} + D\frac{\delta\rho}{\rho_0}\frac{\partial w}{\partial z} + f_0(u,w)\right] d^3x , \tag{7.5}$$

where ρ_0 is the average density and A, C, and D are phenomenological constants. The first term in F_{eff} is the kinetic energy. The first term in U describes, in the harmonic approximation, the energy cost of long-wavelength changes in the density, and the second term describes the coupling between layer compressions and overall changes in the density. These two terms of course also arise in *periodic* smectic liquid crystals. The third term in U is a coupling between the w field and the density. That it must in general be present can be seen as follows. The individual sublattice displacement fields u_1 and u_2 must couple to the density through terms of the form $\delta\rho\,\partial_z u_1$ and $\delta\rho\,\partial_z u_2$. The coefficients of these terms will in general be unequal, since no symmetry dictates that they be equal. When these terms are combined, therefore, both the C and the D terms in [Eq. (7.5)] will result. The fourth term in U is just the elastic free energy from Eq. (4.10) except that now the elastic constants are appropriate derivatives of the free energy evaluated at constant density ρ rather than at constant pressure P, as in (4.10). In general, the choice of s in the definition (4.3) of u and w that we made earlier, which makes the $\partial_z u\,\partial_z w$ cross term vanish in a constant pressure ensemble, will not make it vanish in a constant density ensemble. Therefore, to simplify our analysis of the dynamics, we will make a different choice of s that makes this cross term vanish in the constant density ensemble, rather than our earlier choice. This amounts to a different definition of u and w from that which we used earlier, and also leads to slightly different values of the elastic constants. To avoid confusion, we will denote the new u and w fields described above as u_d and w_d, and will use a superscript d to denote the new elastic constants. This replacement is necessary to ensure that when $\delta\rho$ is integrated out of the partition function $Z \equiv \int D\,\delta\rho\,DuDwe^{-\beta F}$ the resultant effective free energy for u and w along be that given in Eq. (4.10).

The hydrodynamic equations which follow from the preceding analysis are

$$\begin{aligned}&\partial_t\rho + \nabla\cdot\mathbf{g} = 0 , \\ &\partial_t u - v_z + \mathbf{v}\cdot\nabla u_d + \Gamma_u\frac{\delta F}{\delta u_d} + \Gamma_{uw}\frac{\delta F}{\delta w_d} = 0 , \\ &\partial_t w + \mathbf{v}\cdot\nabla w_d + \Gamma_{uw}\frac{\delta F}{\delta u_d} + \Gamma_w\frac{\delta F}{\delta w_d} = 0 , \\ &\partial_t g_i + \nabla\cdot(g_i\mathbf{v}) - \nabla_j(\eta_{ijkl}\nabla_k g_l) + \rho\nabla_i\frac{\delta F}{\delta\rho} \\ &\qquad + (\delta_{iz} - \nabla_i u_d)\frac{\delta F}{\delta u_d} - \nabla_i w_d\frac{\delta F}{\delta w_d} = 0 ,\end{aligned} \tag{7.6}$$

where $\mathbf{v} = \mathbf{g}/\rho$, and Γ_u, Γ_{uw}, and Γ_w are kinetic coefficients governing permeation and phason relaxation. They are expected to be of approximately the same order of magnitude. Since the SmA_{IC} has uniaxial symmetry, the viscosity tensor η_{ijkl} has the same form as it does in a periodic smectic liquid crystal,[11]

$$\begin{aligned}\eta_{ijkl} = {}& \eta_2(\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}) + (\eta_3 - \eta_2)[\delta_{jz}(\delta_{il}\delta_{kz} + \delta_{lz}\delta_{ik}) + \delta_{iz}(\delta_{jk}\delta_{lz} + \delta_{kz}\delta_{lj})] \\ &+ (\eta_4 - \eta_2)\delta_{ij}\delta_{kl} + (\eta_1 + \eta_2 - 4\eta_3 + \eta_4 - 2\eta_5)\delta^z_{ijkl} + (\eta_5 - \eta_4 + \eta_2)(\delta_{ij}\delta^z_{kl} + \delta_{kl}\delta^z_{ij}) ,\end{aligned} \tag{7.7}$$

where the symbols δ^z_{ij} and δ^z_{ijkl} are equal to 1 when all of their indices $= z$, and zero otherwise.

Linearization of the equations of motion, and Fourier transformation in space and time gives

$$
\begin{aligned}
&[-i\omega+\Gamma_u(B_u^d q_z^2+K_u^d q_\perp^4)]u_d+\Gamma_{uw}(B_w^d q_z^2+C_w^d q_\perp^2)w_d-\frac{i}{\rho_0}(\Gamma_u C+\Gamma_{uw}D)q_z\delta\rho-\frac{g_z}{\rho_0}=0\ ,\\
&[-i\omega+\Gamma_w(B_w^d q_z^2+C_w^d q_\perp^2)]w_d+\Gamma_{uw}(B_u^d q_z^2+K_u^d q_\perp^4)u_d-\frac{i}{\rho_0}(\Gamma_w D+\Gamma_{uw}C)q_z\delta\rho=0\ ,\\
&[-i\omega+\nu_{zz}(\mathbf{q})]g_z+\nu_{z\perp}(\mathbf{q})g_\perp+[(B_u^d-C)q_z^2+K_u^d q_\perp^4]u_d-Dq_z^2 w_d+\frac{i(A-C)}{\rho_0}q_z\delta\rho=0\ ,\\
&[-i\omega+\nu_{\perp\perp}(\mathbf{q})]g_\perp+\nu_{z\perp}(\mathbf{q})g_z-Cq_\perp q_z u_d-Dq_\perp q_z w_d+\frac{iA}{\rho_0}q_\perp\delta\rho=0\ ,\\
&[-i\omega+\nu_t(\mathbf{q})]g_t=0\ ,\\
&-i\omega\delta\rho+i(q_\perp g_\perp+q_z g_z)=0\ ,
\end{aligned}
\tag{7.8}
$$

where $g_\perp$ is the component of $\mathbf{g}$ orthogonal to $\hat{\mathbf{z}}$ in the $\mathbf{q}$-$\hat{\mathbf{z}}$ plane, g_t is the component of $\mathbf{g}$ orthogonal to both $\mathbf{q}$ and $\hat{\mathbf{z}}$, and $\nu_{zz}(\mathbf{q})\equiv\eta_3 q_\perp^2+\eta_1 q_z^2$, $\nu_{z\perp}(\mathbf{q})\equiv(\eta_3+\eta_5)q_\perp q_z$, $\eta_{\perp\perp}(\mathbf{q})\equiv(\eta_2+\eta_4)q_\perp^2+\eta_3 q_z^2$, and $\nu_t(\mathbf{q})\equiv\eta_2 q_\perp^2+\eta_3 q_z^2$. Note that all of the ν's are $O(q^2)$.

These equations show that for most directions of $\mathbf{q}$, incommensurate smectic liquid crystals have the usual first- and second-sound modes and transverse shear mode associated with g_t that are found in periodic smectic liquid crystals.[11] In addition, they have a new diffusive mode, namely, that of phason relaxation.[5,12,13,27] We will first consider the mode structure for these generic $\mathbf{q}$'s, and then we will consider the two exceptional cases $q_z=0$ and $q_\perp=0$.

Turning first to the generic case, we seek the pole structure of the equations of motion (7.8). We shall do this in two steps. First, we will seek eigenfrequencies ω of the equations of motion with $\omega=cq$ to leading order. This will determine the speeds of the propagating sound modes. Corrections to these frequencies at next order in q give the damping of sound. Next, we look for frequencies of order q^2; one of these is the aforementioned transverse shear mode, and the other gives the relaxation rate for phasons.

First we seek frequencies $\omega=cq$. For such modes the w_d equation of motion (7.8) immediately implies that

$$
w_d=-i\frac{\Gamma_{uw}B_u^d}{cq}q_z^2 u_d-\frac{\bar{D}_2}{\rho_0}\frac{q_z}{cq}\delta\rho\ ,
\tag{7.9}
$$

where $\bar{D}_2\equiv\Gamma_w D+\Gamma_{uw}C$. This expression for w_d implies that w_d is negligible to leading order in q in all other equations of motion. The remaining leading order in q terms in those equations are precisely the same as those for a periodic smectic liquid crystal;[11] thus we find, as there, that there are two pairs of propagating sound modes with speeds $c_1(\theta)$ and $c_2(\theta)$ given by

$$
\begin{aligned}
&\rho_0(c_1^2+c_2^2)=A+(B_u^d-2C)\cos^2\theta\ ,\\
&\rho_0^2c_1^2c_2^2=(AB_u^d-C^2)\sin^2\theta\cos^2\theta\ ,
\end{aligned}
\tag{7.10}
$$

where θ is the angle between the direction of propagation ($\mathbf{q}$) and the z axis.

Using these results and Eq. (7.9), we can now compute the eigenvalue ω to next order in q. The result is that the complex frequencies ω_j $(j=1,2)$ of first and second sound are given by

$$
\omega_j=\pm c_j(\theta)q-iq^2c_j^3(\theta)\varepsilon_j(\theta)\ .
\tag{7.11a}
$$

The sound attenuation coefficient $\varepsilon_j(\theta)$ is given by

$$
\begin{aligned}
\varepsilon_j(\theta)=\frac{(-1)^j}{2\rho_0c_j^3(\theta)[c_1^2(\theta)-c_2^2(\theta)]}\Bigg[&\sin^2\theta\cos^2\theta[A\eta_1+(A+B_u^d-2C)(\eta_2+\eta_4)+2(C-2A)\eta_3+2(C-A)\eta_5]\\
&+\eta_3[A+(B_u^d-2C)\cos^4\theta]-\rho_0c_j^2(\theta)[\eta_3+\eta_1\cos^2\theta+(\eta_2+\eta_4)\sin^2\theta]\\
&+\Gamma_uB_u^d\left[(A+D-C)\cos^4\theta+\frac{D(A-C)}{\rho_0c_j^2(\theta)}\sin^2\theta\cos^2\theta-\rho_0c_j^2(\theta)\right]\\
&+\bar{D}_1\left[B_u^d\cos^2\theta-C+\frac{A(B_u^d-C)}{\rho_0c_j^2(\theta)}\cos^2\theta\sin^2\theta\right]\cos^2\theta+D\bar{D}_2\left[\frac{B_u^d\cos^2\theta}{\rho_0c_j^2(\theta)}-1\right]\sin^2\theta\cos^2\theta\Bigg]
\end{aligned}
\tag{7.11b}
$$

and we have defined $\bar{D}_1\equiv\Gamma_uC+\Gamma_{uw}D$ and $\bar{D}_2\equiv\Gamma_wD+\Gamma_{uw}C$. In ultrasonic attenuation experiments, the quantity of interest is $q(\omega)\equiv|\mathbf{q}(\omega)|$, rather than $\omega(\mathbf{q})$. From Eq. (7.11a) we find

$$
q_j(\omega,\theta)=\frac{\omega}{c_j(\theta)}+i\alpha_j(\omega,\theta)\ ,
\tag{7.12}
$$

where $\alpha_j(\omega)=\varepsilon_j(\theta)\omega^2$.

Now we seek the frequencies of order q^2. From the equation of motion for u, we find that for such modes $g_z\sim O(q^2u,q^2w,q\delta\rho)$. Inserting this into the equation of motion for g_z shows that g_z is itself negligible to leading order in q in its own equation of motion. Thus we can solve that equation for $\delta\rho$, finding

$$\frac{\delta\rho}{\rho_0}=\frac{iq_z}{A-C}[(B_u^d-C)u_d-Dw_d]\ . \tag{7.13}$$

Inserting this into the equation of motion for $\delta\rho$ shows that for these modes $g_\perp$ is $O(q^2u_d,q^2w_d)$, which in turn implies that $g_\perp$ is also negligible for small q in its own equation of motion. We can thus solve *that* equation for $\delta\rho$, finding

$$\frac{\delta\rho}{\rho_0}=-\frac{iq_z}{A}(Cu_d+Dw_d)\ . \tag{7.14}$$

Requiring that this be consistent with (7.13) enables us to solve for u_d in terms of w_d,

$$u_d=\frac{DC}{AB_u^d-C^2}w_d\ , \tag{7.15}$$

which can be combined with Eq. (7.14) to give

$$\frac{\delta\rho}{\rho_0}=-\frac{iB_u^dDq_z}{AB_u^d-C^2}w_d\ . \tag{7.16}$$

Inserting (7.15) and (7.16) into the equation of motion (7.8) for w_d gives an equation for w_d alone,

$$(-i\omega+\bar{D}_\perp q_\perp^2+\bar{D}_z q_z^2)w_d=0\ ,$$

with $\bar{D}_\perp\equiv\Gamma_w C_w$ and

$$\bar{D}_z\equiv\Gamma_w\left[B_w^d-\frac{B_u^dD^2}{AB_u^d-C^2}\right]\ .$$

This can be trivially solved for the eigenfrequency of the phason mode $\omega=-i(\bar{D}_\perp q_\perp^2+\bar{D}_z q_z^2)$, which as can be seen is diffusive but anisotropic. Had we included the energy density in our hydrodynamic variables, we would have obtained two coupled (phason and thermal) diffusive modes here, instead of just this one phason mode. The diffusive character of this mode remains, however.

We now turn to the two special cases $q_z\lesssim\lambda q_\perp^2$, and $q_\perp=0$. When $q_z\lesssim\lambda q_\perp^2$, the propagating second-sound mode splits into a pair of diffusive modes. The other features of the mode structure remain unchanged. The analysis of this case proceeds as follows: First, we note that in the limit $q_z\ll q_\perp$, and under the assumption $\omega\ll cq$ for any c (as will be verified *a posteriori*), the $g_\perp$ and g_z terms in the $g_\perp$ equation of motion are negligible for $q\to 0$, $\omega\ll cq$ compared to the $\delta\rho$, u_d, and w_d terms therein. This can be shown for g_z by solving the u_d equation of motion for g_z and using the result in terms of $\delta\rho$, u_d, and w_d in the $g_\perp$ equation of motion; every term thereby generated is manifestly higher order in q than at least one of those already present. That $g_\perp$ is likewise negligible follows from inserting this solution for g_z into the equation of motion for $\delta\rho$; then solving the resultant equation for $g_\perp$ in terms of $\delta\rho$, u_d, and w_d, and finally substituting this solution into the $g_\perp$ equation of motion. Again, each term thereby generated is higher order in q than one of those already present.

Having justified neglecting $g_\perp$ and g_z in the $g_\perp$ equation of motion, we can solve this equation for $\delta\rho$ in the limit $q_z\ll q_\perp$; doing so shows that Eq. (7.14) still holds in this direction of $\mathbf{q}$. Inserting this into our earlier solution for $g_\perp$ enables us to establish its irrelevance in the g_z equation of motion. Using (7.14) in the g_z, u, and w equations of motion then leads to a closed system of equations for these fields, which, neglecting irrelevant terms, reads

$$\begin{gathered}-i\omega u_d+\Gamma_{uw}C_w^dq_\perp^2w_d-g_z=0\ ,\\ O(q_\perp^4)u_d+(-i\omega+\bar{D}_\perp q_\perp^2)w_d+O(q_\perp^4)g_z=0\ ,\\ (B_u'q_z^2+K_u^dq_\perp^4)u_d+O(q_\perp^4)w_d+(-i\omega+\eta_3q_\perp^2)g_z=0\ ,\end{gathered} \tag{7.17}$$

where $B_u'=B_u^d-C/A$. Solving these eigenvalue equations shows that w_d decouples completely from u_d and g_z to leading order in q. The modes involving u_d, as a result, are determined by the system of equations

$$\begin{gathered}-i\omega u_d-\frac{g_z}{\rho_0}=0\ ,\\ (B_u'q_z^2+K_u^dq_\perp^4)u_d+(-i\omega+\eta_3q_\perp^2)g_z=0\ ,\end{gathered} \tag{7.18}$$

which are precisely the equations for the u mode in this limit in periodic smectic liquid crystals. The resulting eigenfrequencies are

$$\omega=\frac{-i\eta_3q_\perp^2}{2}\pm\left[\frac{B_u'}{\rho_0}q_z^2+\left[\frac{K_u^d}{\rho_0}-\frac{\eta_3^2}{4}\right]q_\perp^4\right]^{1/2}\ . \tag{7.19}$$

When $q_\perp=0$, the $g_\perp$ equation cannot be used to determine $\delta\rho$ in terms of u_d and w_d, and Eq. (7.14) no longer applies. In this case there are two coupled u-w diffusive modes and $g_\perp$ and g_t diffusive modes.

The analysis of this case proceeds as follows: First we note that Eq. (7.13) still holds for modes with $q_\perp=0$, since its derivation did not depend on any assumptions about the relative magnitudes of q_z and $q_\perp$. Now when $q_\perp=0$, we can immediately solve the continuity equation for g_z; obtaining $g_z=\omega/q_z\delta\rho$. This, together with (7.13), allows us to write a closed system of equations of motion for u and w, for this special case $q_\perp=0$, which read

$$(-i\omega+\bar{D}_3q_z^2)u_d+\left[i\omega\left[\frac{D}{\rho_0c_1^2(0)}\right]+\bar{D}_4q_z^2\right]w_d=0\ , \tag{7.20}$$

$$(-i\omega+\bar{D}_5q_z^2)w_d+\bar{D}_6q_z^2u_d=0\ , \tag{7.21}$$

where we have defined

$$\bar{D}_3\equiv\frac{\Gamma_uB_u^d(A-C)+\bar{D}_1(B_u^d-C)}{\rho_0c_1^2(0)}\ ,$$

$$\bar{D}_4\equiv\frac{\Gamma_{uw}B_w^d(A-C)-\bar{D}_1D}{\rho_0c_1^2(0)}\ ,$$

$$\bar{D}_5\equiv\Gamma_wB_w^d-\frac{\bar{D}_2D}{A-C}\ ,$$

$$\bar{D}_6\equiv\Gamma_{uw}B_u^d+\bar{D}_2\left[\frac{B_u^d-C}{A-C}\right]\ ,$$

and

$$c_1(0)=[(A+B_u^d-2C)/\rho_0]^{1/2}$$

is the first-sound speed for $\theta=0$, as obtained from Eq. (7.10). Solving Eqs. (7.20) and (7.21) for the eigenfrequencies gives two modes with $\omega \sim O(q^2)$,

$$\omega_\pm=\frac{-iq^2}{2}\{\bar{D}_7\pm[\bar{D}_7^2+4(\bar{D}_4\bar{D}_6-\bar{D}_5\bar{D}_3)]^{1/2}\} \tag{7.22}$$

where $\bar{D}_7\equiv\bar{D}_3+\bar{D}_5+\bar{D}_6[D/\rho_0 c_1^2(0)]$. These eigenfrequencies can be seen to be purely imaginary (which physically implies that the modes are purely diffusive) provided $\bar{D}_3$, $\bar{D}_4$, $\bar{D}_5$, and $\bar{D}_6$ are all greater than 0, as required by stability. In particular, the argument of the square root in (7.22) is positive definite, ruling out the possibility of any propagating character to these modes. Had we included the energy density in our formulation, we would have gotten three coupled diffusive modes here (thermal diffusion again mixing in with the phason and phonon modes). Again, the diffusive ($\omega \sim iq^2$) character of these modes would remain, however.

VIII. NONLINEARITIES

All of our static and dynamic results presented thus far have been derived using the purely harmonic elastic theory developed in Sec. IV. It is known[15] that in periodic smectic liquid crystals anharmonicities in ∇u couple thermally excited undulations of the layers to the longitudinal modes and lead to a qualitatively different form for the elasticity, fluctuations, and hydrodynamics[16] of the u field at long wavelengths. Precisely the same anharmonicities are present here, and they lead to precisely the same quantitative effects on the elasticity, fluctuations, and hydrodynamics of the A_{IC}.

We will consider first the anharmonic effects on the elasticity. These can be summarized as follows: at finite wave vector q, the u field behaves as if the quantities B_u and K_u in Eq. (4.10) were wave-vector dependent, with

$$\begin{aligned} B_u(q)&=B_u(\Lambda)\left[1+\frac{5\gamma}{64\pi}\ln\left[\frac{\Lambda}{q}\right]\right]^{-4/5},\\ K_u(q)&=K_u(\Lambda)\left[1+\frac{5\gamma}{64\pi}\ln\left[\frac{\Lambda}{q}\right]\right]^{2/5}, \end{aligned} \tag{8.1}$$

where Λ is an ultraviolet cutoff of order k_1 or k_2. Here also

$$\gamma\equiv k_B T\left[\frac{B_u(\Lambda)}{K_u^3(\Lambda)}\right]^{1/2}$$

is a dimensionless parameter which is of order 1 in conventional smectics, and most probably of that order in incommensurate smectic liquid crystals as well. For many applications, this weak logarithmic dependence can be ignored, and B_u and K_u can be treated as constants since the length scale

$$l_{NL}\equiv\frac{2\pi}{q_{NL}}=\frac{2\pi}{\Lambda}\exp\left[\frac{64\pi}{5\gamma}\right]$$

on which $B_u(q_{NL})$ and $K_u(q_{NL})$ begin to differ appreciably from their values at $q=\Lambda$ can be very long for typical values of γ (e.g., for $\gamma=1$ and $2\pi/\Lambda=20$ Å, $l_{NL}=582\times10^6$ km, which is roughly the distance from here to Jupiter). In addition, for our choice of s, there are *no* relevant anharmonicities involving w (i.e., none that qualitatively change the long-wavelength behavior), so the quadratic terms displayed explicitly in (4.10) accurately describe the elasticity and fluctuations of w at long wavelengths.

This may all be seen as follows: the nonlinearities in question arise by making the free energy invariant under uniform rotations to all orders in ∇u, rather than just to quadratic order, as in Eq. (4.1). This can be accomplished[28] simply by replacing $\partial_z u_{1,2}$ everywhere they appear in (4.1) by the fully rotation invariant objects $E_{1,2}\equiv\partial_z u_{1,2}-\frac{1}{2}|\nabla u_{1,2}|^2$. [The same substitution must be made into the free energy (7.5) coupling u and w to $\delta\rho$; the only change that affects the long distance properties is the replacement of $\partial u/\partial z$ in the second term of that equation by $E\equiv\partial_z u-\frac{1}{2}|\nabla u|^2$]. Making the necessary changes in (4.1), and changing variables to u and w, we find

$$\begin{aligned} f=f_{\text{quad}}&-B_u\partial_z u\,|\nabla u|^2\\ &+\frac{B_u}{4}|\nabla u|^2-B_{uw}\partial_z w\,|\nabla u|^2\\ &+O(|\nabla w|^3,\partial_z u\,|\nabla w|^2,\partial_z u\nabla w\cdot\nabla u), \end{aligned} \tag{8.2}$$

where f_{quad} is the quadratic free energy (4.10) and $B_u(s)$ and $B_{uw}(s)$ are as defined in Sec. IV.

The terms cubic and quadratic in u are precisely those found in periodic smectic liquid crystals; since the quadratic part of the u Hamiltonian is also the same, these will lead to precisely the effects found in periodic smectic liquid crystals.[15,16] The

$$O(|\nabla w|^3,\partial_z u\,|\nabla w|^2,\partial_z u\nabla u\cdot\nabla w)$$

terms can be shown by power counting in graphical perturbation theory to be irrelevant at long wavelengths. The same power-counting argument shows that the $\partial_z w\,|\nabla u|^2$ term *would* lead to significant long-wavelength effects; however, for our choice of s this term vanishes since B_{uw} does.

We now turn to the dynamical effects of these anharmonicities. In periodic smectic liquid crystals, the nonlinear coupling of the momentum density to the undulation mode (that is, to displacements with the wave vector parallel to the layers) gives rise to a breakdown of conventional hydrodynamics.[16] The viscosities η_1, η_4, and η_5 are predicted[16] and observed[17] to diverge as $1/\omega$ at low frequencies ω. It is straightforward to see that this also happens in the incommensurate smectic liquid crystal. This is because the calculations in Ref. 16 of the fluctuation corrections to the viscosities depended on only the following three features of that problem.

(1) The form of the static u-u correlation function,

(2) The form of the nonlinearities in u in the equation of motion for **g**,

(3) The decay rates in the linearized equations of motion of the eigenmodes associated with u for wave vec-

tors $q_z \ll q_\perp$ (the region where the static u-u correlations are largest).

All three of these features are exactly reproduced in incommensurate smectic liquid crystals. We have already demonstrated (1) in our discussion of the static correlation functions. (2) follows from the fact that the anharmonic in u terms in f are precisely the same as in periodic smectic liquid crystals, coupled with the fact that the terms involving u in the $\mathbf{g}$ equation of motion all arise from precisely the same functional derivatives (i.e., $\delta F/\delta u$ and $\delta F/\delta\rho$) as in periodic smectics, and with precisely the same dependence thereon. Furthermore, the additional nonlinearities in the equations of motion (e.g., those involving w) are readily shown by the power-counting arguments of the type used in Ref. 16 to be irrelevant at long wavelengths (i.e., they do not change the asymptotic hydrodynamic behavior as q and $\omega \rightarrow 0$). (3) was verified in Sec. VII.

This completes our demonstration that the divergent fluctuation contributions to the viscosities in these systems are *exactly* the same as in periodic smectic liquid crystals. In particular, it implies that the direction dependence of the resulting anomaly in the sound attenuation is identical to that in periodic smectic liquid crystals, and that the relation $\Delta\eta_1\Delta\eta_4=(\Delta\eta_5)^2$ for the divergent parts $\Delta\eta_i$ of the viscosities still holds. We have also checked that the inclusion of the energy density as a hydrodynamic variable does not modify this result, just as it does not in periodic smectic liquid crystals.

The viscosities appearing in the expressions (7.11) for the sound damping must therefore be interpreted as singular frequency-dependent quantities, not constants. No such complication arises for the phason mode or for the transverse shear mode, since fluctuation corrections to η_2, η_3, and Γ_w are regular for $q \rightarrow 0$, $w \rightarrow 0$.

IX. PHASONS IN NEARLY INCOMMENSURATE SMECTIC LIQUID CRYSTALS

Our argument that the phason field w was a broken-symmetry mode depended crucially on the incommensurability of the ratio k_2/k_1. Since arbitrarily close to any irrational number we can find a rational number, it clearly behooves us to consider what happens when the ratio k_1/k_2 locks in to a rational value p/q, where p and q are mutually prime integers.[29] In particular, we wish to focus on "nearly incommensurate" structures where $q \gg 1$. We will argue that for sufficiently large q, u_1 and u_2 will still behave like broken-symmetry modes of the system out to length scales L that grow exponentially with q. In practical terms, using realistic numerical estimates for the values of various Landau parameters, one only need worry in actual experiments about lock-in to a relatively small ($\lesssim 10$) number of commensurate ratios k_2/k_1. If one can establish experimentally that k_2/k_1 is not equal to any of these commensurate ratios, then one will have established that for length scales of order a centimeter or less, the system will behave like a truly incommensurate smectic liquid crystal.

The argument[30] goes as follows: if $k_1/k_2=p/q$ for relatively prime integers p and q, then the nonvanishing term of lowest order in ψ_1 and ψ_2 coupling spatially uniform u_1 to u_2 in the Landau expansion (and hence the largest term since, as mentioned earlier, ψ_1 and ψ_2 are small in real smectic liquid crystals) is

$$F_{p,q}=\tfrac{1}{2}C_{p,q}\int d^3x\,[\psi_1^p(\psi_2^*)^q+\text{c.c.}]$$
$$=C_{p,q}\,|\psi_1|^p\,|\psi_2|^q\int d^3x\,\cos(qk_2w)\;, \tag{9.1}$$

where we have used $pk_1=qk_2$ and $w=u_1-u_2$. This term clearly gives an energy cost to spatially uniform w, thereby eliminating w as a broken-symmetry mode, and leaving only uniform translation u as a broken-symmetry variable.

Since ψ_1 and ψ_2 are both "small," we see that for large q (i.e., lock-in to high commensurability structures), this term will be exponentially small as $q \rightarrow \infty$. To estimate quantitatively the length scale at which, for a given q, this lock-in term becomes important, we need to look at the terms it competes with, namely the gradient energy for w. Clearly, for sufficiently long wavelengths, (9.1) will dominate any gradient terms. An estimate of the critical wavelength l_c beyond which (9.1) dominates can be obtained by estimating, from Landau theory, the magnitude of these gradient terms.

As shown in Sec. IV, the lowest order (in powers of $\psi_{1,2}$ and the gradient operator) gradient terms in w are (focusing just on the $\nabla_z\psi$ terms)

$$f_{\text{Grad}}(w)\cong C_1^\perp\psi_1^2\,|\nabla_z w\,|^2\;. \tag{9.2}$$

Fourier transforming in space, we obtain

$$f_{\text{Grad}}\cong C_1^\perp\psi_1^2k_z^2w^2\;. \tag{9.3}$$

The *incommensurate* hydrodynamics we develop here will also describe the *commensurate* system for k_z's sufficiently large so that (9.3) is much larger than the w-dependent part of (9.1). For small w, Eq. (9.1) can be expanded,

$$f_{p,q}=\text{const}-\tfrac{1}{2}q^2k_2^2\,|\psi_1|^p\,|\psi_2|^qC_{p,q}w^2\;. \tag{9.4}$$

This "mass" term will be negligible relative to the incommensurate smectic elastic term (9.2) for wave vectors k_z that satisfy

$$k_z \gg k_c=\left[\frac{C_{p,q}}{C_1^\perp}\right]^{1/2}|\psi_1|^{p/2-1}\,|\psi_2|^{q/2}qk_2\;. \tag{9.5}$$

To estimate quantitatively the value of k_c, we need to estimate the ratio $C_{p,q}/C_1^\perp$. We can do this by dimensional analysis: $C_1^\perp$ has the dimensions of E/ρ^2V, where E, ρ, and V are energy, mass density, and volume, respectively, while $C_{p,q}$ has the dimensions $E/\rho^{p+q}V$. Their ratio thus has the dimensions of $1/\rho^{p+q-2}$.

Now we can estimate the numerical value of this ratio as $(1/\rho_{\text{char}})^{p+q-2}$. Our choice of the characteristic density ρ_{char} cannot be either ψ_1 or ψ_2, since, by the rules of the Landau-theory game, parameters of the Landau theory must be taken to be smooth through any phase transitions, while ψ_1 and ψ_2 vanish singularly there instead, and hence are unsuitable choices for ρ_{char}. The only other quantity in the theory with the dimensions of density is the background mass density ρ_0, which is smooth through all transitions. Taking this to be $\rho_{\rm char}$, (9.5) becomes

$$k_c = \left[\frac{|\psi_1|}{\rho_0} \right]^{p/2-1} \left[\frac{|\psi_2|}{\rho_0} \right]^{q/2} q k_2 \ . \tag{9.6}$$

We can estimate the numerical values of the ratios in (9.6) fairly easily. ψ_1/ρ_0 and ψ_2/ρ_0 are just the ratios of the changes in the mass density as we move through the density wave to the background density. Typically, in smectic liquid crystals, these ratios are small ($\lesssim 0.1$). Thus, for $p+q>20$, and $k_2 \sim 2\pi/(20\ \text{Å})$, we get $k_c \lesssim 2\pi/(20\ \text{cm})$, or $l_c \equiv 2\pi/k_c > 20$ cm, which is a macroscopic distance. Thus, with x-ray measurements of peak positions sufficiently precise to show that $k_1/k_2 \neq p/q$ for $p+q<20$, which would require measuring k_1/k_2 to an accuracy $\sim 1\%$ (which should be practical) and eliminating ~ 30 pairs of (p,q), experimentalists can show that their system behaves according to incommensurate smectic hydrodynamics as developed here out to macroscopic length scales. More generally, the limitation (9.6) on the validity of incommensurate hydrodynamics can be applied, using the experimentally determined values of ψ_1/ρ_0 and ψ_2/ρ_0, these just being the square roots of the ratios of the integrated intensities of the peaks at k_1 and k_2 to the peak at $G=0$, all of which can be measured experimentally.

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[29]S. R. thanks Jiang Wang (private communication) for bringing

up the question of lock-in many years ago and apologizes for not taking it seriously at the time.

[30]The argument we present here is very similar to (and was, in fact, inspired by) that used by O. Biham, D. Mukamel, and S. Shtrikman, Phys. Rev. Lett. **56**, 2191 (1986), and to be published in *Quasi Periodic Structures,* Vol. 1 of *Aperiodic Crystals,* edited by Marco Jaric (Academic, Boston, 1988), to demonstrate the local stability of quasicrystals against commensurate lock-in.