First-passage times of non-Markovian processes: The case of a reflecting boundary

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Mean first-passage times (MFPT) of non-Markovian processes driven by Markovian two-state noise of finite correlation time are considered. Absorbing as well as reflecting boundary conditions are constructed, and new results for the first-passage-time density and the MFPT are derived. We extend our study to dichotomic Fokker-Planck processes, i.e., a stochastic dynamics in which the random walker jumps between two different Fokker-Planck processes with a dichotomic noise dynamics. In this general case, too, we derive the boundary conditions explicitly and obtain novel expressions for the MFPT. A number of special cases and limits are considered which elucidate the physics of the more general results. Finally, we consider the problem of bistability driven by dichotomic noise and express the MFPT in terms of the stationary probability density. For the escape rate at weak noise we establish the connection between the MFPT approach and the current overpopulation method.

I. INTRODUCTION

The mean first-passage time (MFPT), i.e., the time at which a stochastic process starting at a given initial value reaches an *a priori* assigned threshold value for the first time, has become a useful concept with many applications in physical, chemical, and engineering sciences.[1,2] Explicit, closed analytical expressions for the MFPT are known for one-dimensional Fokker-Planck processes,[1,2] birth and death processes with nearest-neighbor transitions,[1,3–5] continuous-time random walks with nearest-neighbor jumps,[6,7] and processes involving jumps next to nearest neighbors.[4] The intricate difficulties encountered in obtaining exact first-passage time results for general non-Markovian processes have been illustrated in Ref. 8. Exact results for non-Markovian processes driven by two-state noise have been obtained first in Ref. 9. Recently, those results have been rederived and generalized in a series of papers by various groups.[10] Clearly, for non-Markovian processes the construction of correct boundary conditions is a delicate problem.[8,9] In the case of two-state noise, the construction of absorbing boundaries follows a natural scheme which has already been presented in Ref. 9. In this context, it should be stressed that all of the *recent* work[10] explicitly deals with such *absorbing boundaries only.*

Our objective in this work is to generalize the boundary problem to *reflecting boundaries* as well. As it will turn out, the construction of reflecting boundaries is more subtle because the reflection process can happen on a varying time scale. Moreover, we shall investigate the relationship in bistable situations between the activation rate and the MFPT when the noise strength is small.

The paper is organized as follows. In Sec. II we study free diffusion generated by a two-state Markov process. We review the known results for absorbing boundaries and then study a boundary with immediate reflection. In Sec. III we study generalized diffusion in which a particle jumps back and forth between two different diffusion processes (dichotomic Fokker-Planck processes). MFPT's for reflecting as well as absorbing boundaries are derived for specific situations. Finally, we study in Sec. IV the MFPT for a general, nonlinear flow driven by multiplicative Markovian two-state noise. This process results as a degenerate dichotomic Fokker-Planck process in which the white-noise diffusion is set equal to zero. We present new results for the MFPT, both with absorbing and reflecting exit boundaries. In the Appendix we study the detailed description of the boundary conditions by starting from a discrete random walk on two layers and then performing the continuum limit.

II. FREE DICHOTOMIC FLOW (PERSISTENT DIFFUSION)

Our investigation starts with a free process driven by Markovian two-state noise $\xi(t)=\pm 1$ according to

$$\dot{x}=a\xi(t), \quad a>0 \tag{2.1}$$

with equal jump rates $(1\to -1)=\mu$ and $(-1\to 1)=\mu$, respectively. The symmetric dichotomic noise has zero mean and an exponential correlation

$$\langle \xi(t)\xi(s)\rangle=\exp(-|t-s|/\tau) , \tag{2.2}$$

where $\tau=(2\mu)^{-1}$ is the noise correlation time. Now, let $F_t(y;\pm a)$ denote the probability for a particle that started out at $y\varepsilon[A,B]$, with positive or negative velocity, respectively, to be found at time t still in the safe domain $[A,B]$, $A<B$. This quantity $F_t(y;\pm a)$ obeys the backward equation

$$\dot{F}_t(y;a)=a\frac{\partial F_t(y;+a)}{\partial y}+\mu[F_t(y;-a)-F_t(y;a)] , \tag{2.3a}$$

$$\dot{F}_t(y;-a)=-a\frac{\partial F_t(y;-a)}{\partial y}+\mu[F_t(y;a)-F_t(y;-a)] . \tag{2.3b}$$

The initial condition at time $t=0$ of preparation for $F_t(y;\pm a)$ reads

$$F_{t=0}(y;\pm a)=\Theta(y-A)\Theta(B-y)\ , \tag{2.4}$$

where Θ is the step function. In what follows we also assume that at time $t=0$ we have, for the initial probability W_0 of finding the particle at y with positive and/or negative velocity, $W_0(a\mid y)\equiv w_+$, and $W_0(-a\mid y)=w_-$, respectively. Obviously, one has

$$w_+ + w_- = 1\ . \tag{2.5}$$

The first-passage-time density $\phi_t(y;\pm a)$ itself is given in terms of $F_t(y;\pm a)$ by

$$\phi_t(y;\pm a)=-\frac{\partial}{\partial t}F_t(y;\pm a)\ . \tag{2.6}$$

It obeys with the substitution $-F_t\rightarrow\phi_t$ the same equation (2.3) with initial condition

$$\phi_{t=0}(y;\pm a)=\delta(y-A)\Theta(B-y)+\Theta(y-A)\delta(B-y)\ .$$

In terms of the Laplace transform

$$\begin{aligned}\hat{\phi}_z(y;\pm a)&=\int_0^\infty \phi_t(y;\pm a)e^{-zt}dt\\ &=1-z\hat{F}_z(y;\pm a)\end{aligned} \tag{2.7}$$

(where $\hat{\phi}_z\rightarrow+1$ as $z\rightarrow 0$), the nth moment of the FPT, $T_n(y;\pm a)$ reads

$$\begin{aligned}T_n(y;\pm a)&=\int_0^\infty t^n\phi_t(y;\pm a)dt\\ &=(-1)^n\left.\frac{\partial^n\hat{\phi}_z(y;\pm a)}{\partial z^n}\right|_{z=0}\\ &=(-1)^n\frac{\partial^n}{\partial z^n}[-z\hat{F}_z(y;\pm a)+1]\,|_{z=0}\ ,\\ &\qquad n=0,1,\ldots\ .\end{aligned} \tag{2.8}$$

In the following, the first moment $T_1(y;\pm a)$ is simply denoted by $T_1(y;\pm a)\equiv T(y;\pm a)$.

The first moment

$$T_1(y_1\pm a)\equiv T(y;\pm a)=\int_0^\infty F_t(y;\pm a)dt \tag{2.9}$$

obeys from Eqs. (2.3) and (2.8) the backward equation

$$\begin{aligned}&\mu[T(y;-a)-T(y;a)]+aT'(y;-a)=-1\ ,\\ &\mu[T(y;a)-T(y;-a)]-aT'(y;a)=-1\ ,\end{aligned} \tag{2.10}$$

where $T'\equiv\partial T/\partial y$.

In previous works[9,10] the MFPT problem has been solved for *absorbing boundaries* only. With the exit boundaries A and B both being absorbing, one has[9]

$$\phi_t(A;-a)=0,\quad \phi_t(B;a)=0\ . \tag{2.11a}$$

For the MFPT itself, this implies

$$T(A;-a)=0,\quad T(B;+a)=0\ . \tag{2.11b}$$

The values $T(A;+a)$ and $T(B;-a)$ are determined by (2.10). Generally, both $T(A;+a)$ and $T(B;-a)$ will take on nonvanishing, positive values; this is due to the fact that a particle starting at the boundary with an inward velocity will first jump back into the safe domain $[A,B]$, the average residence time in the corresponding velocity state being of the order of $1/\mu$.

Next we turn to the problem of a *reflecting boundary.* In the following we always identify A to be a reflecting boundary while B remains absorbing. If the particle hits with negative velocity the boundary $x=A$ and if it is reinjected *immediately,* i.e., the corresponding rate constant is infinite (see the Appendix), we have the natural *reflecting boundary condition*

$$\phi_t(A;-a)=\phi_t(A;+a)\ , \tag{2.12a}$$

whence, for the MFPT,

$$T(A;-a)=T(A;+a)\ . \tag{2.12b}$$

For A absorbing and B reflecting, a similar condition is readily derived. If particles are reinjected with a *finite* rate λ, the process of reflection is not unique. For the general derivation of the reflecting boundary condition we refer the reader to the Appendix.

Table I contains a summary of the various results. $T(y)$ and $\phi_t(y)$ are obtained by using

$$T(y)=w_+T(y;a)+w_-T(y;-a) \tag{2.13a}$$

and

$$\phi_t(y)=w_+\phi_t(y;a)+w_-\phi_t(y;-a)\ . \tag{2.13b}$$

III. DICHOTOMIC FOKKER-PLANCK PROCESS

A situation of great practical importance is the case in which a system switches between two (or more) stochastic dynamics.[11] Here, we shall consider the case of a switching between two Fokker-Planck processes, i.e., in terms of the dichotomic noise $\xi(t)=\pm 1$ we write the stochastic differential equation

$$\begin{aligned}\dot{x}=&\tfrac{1}{2}[f_+(x)+g_+(x)\eta(t)+f_-(x)+g_-(x)\eta(t)]\\ &+\tfrac{1}{2}[f_+(x)+g_+(x)\eta(t)-f_-(x)-g_-(x)\eta(t)]\xi(t)\ .\end{aligned} \tag{3.1}$$

Here, $\eta(t)$ is Gaussian white noise of zero mean and correlation $\langle\eta(t)\eta(s)\rangle=2\delta(t-s)$. The product $g(x)\eta(t)$ is interpreted using the Ito rule.[12] For $\xi(t)=1$ one probes the Fokker-Planck dynamics

$$\dot{x}=f_+(x)+g_+(x)\eta(t)\ , \tag{3.2a}$$

whereas for $\xi(t)=-1$ we have

$$\dot{x}(t)=f_-(x)+g_-(x)\eta(t)\ . \tag{3.2b}$$

With $g_+=g_-=0$ and $f_+=-f_-=a$, we recover the free persistent diffusion of Sec. II. The general nonlinear dichotomic flow

$$\dot{x}=f(x)+g(x)\xi(t) \tag{3.3}$$

is recovered from (3.1) if we set $f_+(x)=f(x)+g(x)$; $f_-(x)=f(x)-g(x)$, and $g_+(x)=g_-(x)=0$.

Because the pair (x,ξ) forms a Markov process, the so-

TABLE I. Results for the free dichotomic flow in (2.1). ϕ_z, Laplace transform of the first-passage-time density, see (2.13b); T_1, mean first-passage time, see (2.13a).

$\dot{x}=a\xi$	$\hat{\phi}_z(y)$	$T_1(y)$	$\langle T_1(y)\rangle = \frac{1}{(B-A)}\int_A^B T_1(y)dy$
$\mu^+=\mu^-=\mu$	$\lambda=[z(z+2\mu)]^{1/2}/a$		
A, absorbing B, absorbing	$\{[w^+(\lambda a+z)+\mu]e^{\lambda(y-A)}$ $+[w^+(\lambda a-z)-\mu]e^{-\lambda(y-A)}$ $+[w^-(\lambda a+z)+\mu]e^{\lambda(B-y)}$ $+[w^-(\lambda a-z)-\mu]e^{-\lambda(B-y)}\}$ $\times[(\lambda a+z+\mu)e^{\lambda(B-A)}+(\lambda a-z-\mu)e^{-\lambda(B-A)}]^{-1}$	$\frac{(B-A)}{2a}-\frac{\mu}{a^2}(y-A)(y-B)$ $-\frac{[y-\frac{1}{2}(A+B)]}{a}(w^+-w^-)$[a]	$\frac{(B-A)}{2a}\left[1+\frac{1}{3}\frac{\mu}{a}(B-A)\right]$[b]
A, reflecting B, absorbing	$\{e^{\lambda(y-A)}[\lambda a+z(w^+-w^-)]$ $+e^{-\lambda(y-A)}[\lambda a-z(w^+-w^-)]\}$ $\times[e^{\lambda(B-A)}(\lambda a+z)+e^{-\lambda(B-A)}(\lambda a-z)]^{-1}$	$\frac{-\mu}{a^2}(y+B-2A)(y-B)$ $+\frac{(B-A)}{a}-\frac{(y-A)}{a}(w^+-w^-)$	$\frac{(B-A)}{a}\left[1-\frac{(w^+-w^-)}{2}+\frac{2\mu}{3a}(B-A)\right]$
$A=-\infty$, reflecting B, absorbing	$\frac{[\lambda a+z(w^+-w^-)]e^{-\lambda(B-y)}}{\lambda a+z}$	∞	∞

[a]Reference 9.
[b]Reference 10(b).

journ probabilities $F_t(y;+1)$ and $F_t(y;-1)$, respectively, obey the backward equations

$$\dot{F}_t(y;+1)=\left[f_+(y)\frac{\partial}{\partial y}+g_+^2(y)\frac{\partial^2}{\partial y^2}\right]F_t(y;+1)-\mu[F_t(y;+1)-F_t(y;-1)]\ , \tag{3.4a}$$

$$\dot{F}_t(y;-1)=\left[f_-(y)\frac{\partial}{\partial y}+g_-^2(y)\frac{\partial^2}{\partial y^2}\right]F_t(y;-1)-\mu[F_t(y;-1)-F_t(y;+1)]\ . \tag{3.4b}$$

Again, these equations must be supplemented with boundary conditions. A detailed derivation of these boundary conditions is given in the Appendix. Here we merely state the results and elucidate their meaning.

(a) *Absorbing boundaries at A and B.* This implies for the MFPT density, assuming $g_+(x)\neq 0, g_-(x)\neq 0$

$$\phi_t(A;-1)=\phi_t(A;+1)=0\ , \tag{3.5a}$$

$$\phi_t(B;-1)=\phi_t(B;+1)=0\ , \tag{3.5b}$$

and for the first moment

$$T(A;-1)=T(A;+1)=0\ , \tag{3.5c}$$

$$T(B;-1)=T(B;+1)=0\ . \tag{3.5d}$$

(b) *A (reflecting) and B (absorbing).*

$$[g_+(y)]^2\frac{\partial\phi_t(y;1)}{\partial y}\bigg|_{y=A}=0\ , \tag{3.6a}$$

$$\phi_t(A;-1)=\phi_t(A;+1)\ , \tag{3.6b}$$

while at B we have, as before,

$$\phi_t(B;-1)=\phi_t(B;+1)=0\ , \tag{3.6c}$$

with corresponding conditions for the first moment, i.e., $T(A;-1)=T(A;+1)$, etc.

(c) *B (reflecting) and A (absorbing).*

$$[g_-(y)]^2\frac{\partial\phi_t(y,-1)}{\partial y}\bigg|_{y=B}=0\ , \tag{3.7a}$$

$$\phi_t(B;-1)=\phi_t(B;+1)\ , \tag{3.7b}$$

and at A

$$\phi_t(A;-1)=\phi_t(A;+1)=0\ . \tag{3.7c}$$

We note first that each boundary implies *two* boundary conditions, yielding a total of *four* boundary conditions. This is due to the presence of the Fokker-Planck dynamics (a second-order differential operator) both for the $(+1)$ dynamics and the (-1) dynamics. Moreover, at a "reflecting" boundary, one has no *unique* reflection mechanism. The process of reflection at the boundary must be specified in detail. The reflection condition in Eqs. (3.6a) and (3.6b) corresponds to an *immediate* reinjection of particles that have hit the boundary $y=A$ from the "minus" state $[\xi(t)=-1]$ into the "plus" state $[\xi(t)=+1]$. Note that (3.6a) corresponds to the well-known reflecting boundary condition for Fokker-Planck dynamics in the plus state $[\xi(t)=1$ at all times], while the condition (3.6b) amounts to an instantaneous reinjection from the minus into the plus state as already encountered in Sec. II. Other, even more complex, types of boundary conditions are, of course, also possible. For example, one can construct situations that resemble radiation boundary conditions[2(b)] by incorporating a suitable mechanism by which particles are reflected from the $(\pm)$ state to the $(\mp)$ state at the boundaries.

In terms of the Laplace transform

$$\hat{\phi}_z(y;\pm 1)=\int_0^\infty \phi_t(y;\pm 1)e^{-zt}dt \tag{3.8}$$

$$=-z\hat{F}_z(y;\pm 1)+1\ , \tag{3.9}$$

one finds from (3.4) the coupled first-order equations

$$-1=-z\hat{F}_z(y;+1)+\left[f_+\frac{\partial}{\partial y}+g_+^2\frac{\partial^2}{\partial y^2}\right]\hat{F}_z(y;+1)-\mu[\hat{F}_z(y;+1)-\hat{F}_z(y;-1)]\ , \tag{3.10a}$$

$$-1=-z\hat{F}_z(y;-1)+\left[f_-\frac{\partial}{\partial y}+g_-^2\frac{\partial^2}{\partial y^2}\right]\hat{F}_z(y;-1)-\mu[\hat{F}_z(y;-1)-\hat{F}_z(y;+1)]\ , \tag{3.10b}$$

obeying the boundary conditions mentioned above with $\phi_t(y;\pm 1)$ replaced by $F_t(y;\pm 1)$. As $\lim_{z\to 0}\hat{F}_z(y;\pm 1)=T(y;\pm 1)$ we obtain from (3.10) for the MFPT's,

$$-1=\left[f_+\frac{\partial}{\partial y}+g_+^2\frac{\partial^2}{\partial y^2}\right]T(y;+1)-\mu[T(y;+1)-T(y;-1)]\ , \tag{3.11a}$$

$$-1=\left[f_-\frac{\partial}{\partial y}+g_-^2\frac{\partial^2}{\partial y}\right]T(y;-1)-\mu[T(y;-1)-T(y;+1)]\ . \tag{3.11b}$$

The set of Eqs. (3.10), or, via Eq. (2.8) those for the corresponding moments $T_n(y;\pm 1)$, cannot be solved explicitly, in general. In the following we shall consider only the first moment, given by Eqs. (3.11), which we study for some special choices of the set of functions $\{f_+(x),f_-(x),g_+(x),g_-(x)\}$.

A. Dichotomic diffusion

This process is defined by $f_+=f_-=0$ and $g_+=D_+^{1/2}, g_-=D_-^{1/2}$. Then $x(t)$ obeys the stochastic equation

$$\dot{x}(t)=\tfrac{1}{2}(D_+^{1/2}+D_-^{1/2})\eta(t)+\tfrac{1}{2}(D_+^{1/2}-D_-^{1/2})\eta(t)\xi(t)\ . \tag{3.12}$$

This flow corresponds to a Brownian motion which jumps between a diffusion process of strength D_+ and a diffusion process of strength D_-. From (3.11) we obtain

$$D_-T''(y;-1)+D_+T''(y;+1)=-2\ ,$$

$$D_+T''(y;+1)-\mu[T(y;+1)-T(y;-1)]=-1\ ,$$

or upon eliminating $T(y;-1)$ the fourth-order equation

$$(D_-+D_+)T''(y;+1)-\frac{D_+D_-}{\mu}T''''(y;+1)=-2\ . \tag{3.13}$$

When both A and B are absorbing boundaries, we obtain from (3.5c) and (3.5d) the following explicit result:

$$\begin{aligned}T^{\rm abs}(y)&=w_+T(y;1)+w_-T(y;-1)\\&=\frac{(y-A)(B-y)}{D_++D_-}+2(w_+D_--w_-D_+)\frac{(D_+-D_-)}{\mu(D_++D_-)^2}\sinh[\tfrac{1}{2}\lambda(y-A)]\sinh[\tfrac{1}{2}\lambda(y-B)]/\cosh[\tfrac{1}{2}\lambda(B-A)]\ ,\end{aligned} \tag{3.14}$$

where

$$\lambda^2=\mu(D_+^{-1}+D_-^{-1})\ . \tag{3.15}$$

For the average $\langle T^{\rm abs}(y)\rangle$ over y uniformly distributed in $[A,B]$, one has

$$\langle T^{\rm abs}(y)\rangle=\frac{(B-A)^2}{6(D_++D_-)}+(w_+D_--w_-D_+)\frac{(D_+-D_-)}{\mu(D_++D_-)^2}\left[\frac{2\tanh[\frac{1}{2}\lambda(B-A)]}{\lambda(B-A)}-1\right]\ . \tag{3.16}$$

When A is an instantaneously reflecting boundary one obtains, using (3.4), (3.6b), and (3.6c), the result

$$\begin{aligned}T^{\rm refl}(y)=\frac{(B-y)(B-2A+y)}{D_++D_-}&+(w_+D_--w_-D_+)\frac{(D_+-D_-)}{\mu(D_++D_-)^2}\{2\sinh[\tfrac{1}{2}\lambda(y-A)]\sinh[\tfrac{1}{2}\lambda(y-B)]\\&+\lambda(y-B)\sinh[\tfrac{1}{2}\lambda(B-A)]\}/\cosh[\tfrac{1}{2}\lambda(B-A)]+\frac{(D_+-D_-)}{\mu(D_++D_-)}w_-\lambda(y-B)\tanh[\tfrac{1}{2}\lambda(B-A)]\ ,\end{aligned} \tag{3.17}$$

with λ defined in (3.15). Its average $\langle T^{\rm refl}\rangle=(B-A)^{-1}\int_A^B T^{\rm refl}(y)dy$ can readily be evaluated.

A few specific cases are of special interest. These also serve as useful checks of the results derived above.

$$\text{(i) } D_+=D_-=D\ . \tag{3.18}$$

In this case we have

$$T^{\rm abs}(y)=\frac{(y-A)(B-y)}{2D}\ , \tag{3.19}$$

$$T^{\rm refl}(y)=\frac{(B-y)(B-2A+y)}{2D}\ . \tag{3.20}$$

These are the well known MFPT's for a diffusion process with diffusion constant D. This is evident if one notes that with $D_+=D_-$, a switching mechanism between the plus and minus state does not affect the duffision processes.

(ii) Limit of infinite switching rate, $\mu\to\infty$. This implies a standard diffusion process and thus yields again the above results (3.19) and (3.20), respectively, with an effective overall diffusion constant D given by

$$D=\tfrac{1}{2}(D_++D_-)\ . \tag{3.21}$$

(iii) The limit $\mu\to 0$.

$$T^{\rm abs}(y)\overset{\mu\to 0}{\to}\frac{(y-A)(B-y)}{2D_+}w_++\frac{(y-A)(B-y)}{2D_-}w_-\ , \tag{3.22}$$

$$\begin{aligned}T^{\rm refl}(y)\overset{\mu\to 0}{\to}&\frac{(B-2A+y)(B-y)}{D_++D_-}+\frac{D_+-D_-}{2D_+D_-}w_-(y-B)(B-A)\\&+\left[\frac{w_+}{D_+}-\frac{w_-}{D_-}\right]\frac{D_+-D_-}{2(D_++D_-)}(B-2A+y)(y-B)\ .\end{aligned} \tag{3.23}$$

The result in (3.22) follows because with $\mu=0$ (and A and B both absorbing), the two diffusion layers are not in contact with each other: Thus Eq. (3.19) applies, with corresponding initial preparation weights w_+ and w_-. With the plus and minus states behaving differently at the reflecting boundary A [see (3.6)], the result in (3.23) is naturally more complex.

(iv) $D_-=0$. Now the minus state is not mobile. This implies, from Eqs. (3.14) and (3.17), respectively,

$$\lim_{D_-\to 0} T^{\rm abs}(y)=\frac{(y-A)(B-y)}{D_+}+\frac{w_-}{\mu} \tag{3.24}$$

and

$$\lim_{D_-\to 0} T^{\rm refl}(y)=\frac{(B-2A+y)(B-y)}{D_+}+\frac{w_-}{\mu} \ . \tag{3.25}$$

In other words, with $w_-=0$, one obtains twice the answers in (3.19) and (3.20), respectively. When $w_->0$, one has to add also the residence time in this state, namely, w_-/μ.

(v) $D_+\to 0$. For absorbing boundaries one has simply (3.24) with w_- replaced by w_+ and D_+ by D_-. The situation for $T^{\rm refl}(y)$ when the plus state has a mobility approaching zero is more tricky. One finds from (3.17)

$$T^{\rm refl}(y)\to\frac{(B-y)}{(\mu D_+)^{1/2}} \quad \text{as } D_+\to 0 \ , \tag{3.26}$$

i.e., the MFPT is (in leading order) independent of D_-, and it diverges like $D_+^{-1/2}$. The result in (3.26) can be understood intuitively as follows: The boundary A is hit by the particle while it is in the mobile minus state. It is then transferred to the plus state, in which it resides for an average time $1/\mu$ and diffuses during this time away from the boundary A a distance of the order of $(D_+/\mu)^{1/2}$. It then switches back to the minus state. The process repeats over and over again, the number of times A is hit being given by $(B-y)/(D_+/\mu)^{1/2}$, yielding a total residence time (or MFPT) of $\mu^{-1}(B-y)/(D_+/\mu)^{1/2}$, as in (3.26).

B. Forced dichotomic diffusion

Next we consider dichotomic diffusion driven by the two-state process $\xi(t)$ defined as in Eq. (2.1). In the following we therefore set $f_+=a$, $f_-=-a$, and use a "symmetric" diffusion $g_+=g_-=D^{1/2}$. The diffusive process reads

$$\dot{x}=a\xi(t)+D^{1/2}\eta(t) \ . \tag{3.27}$$

Using (3.11), we obtain on eliminating $T(y;-1)$ the equation obeyed by $T(y;+1)$, i.e.,

$$\left[2D+\frac{a^2}{\mu}\right]T''(y;+1)-\frac{D^2}{\mu}T''''(y;+1)=-2 \ . \tag{3.28}$$

Due to the symmetry we find for absorbing boundaries at A and B

$$T^{\rm abs}(y;+1)=T^{\rm abs}(A+B-y;-1) \ . \tag{3.29}$$

With

$$\rho^2=(a^2+2\mu D)/D^2 \ , \tag{3.30}$$

one finds for the explicit solution of (3.28) with A, B both absorbing [see Eqs. (3.5)]

$$\begin{aligned} T^{\rm abs}(y;+1)=&\Bigg\{4a^2(B-A)\left[(D\rho^2-2\mu)\sinh\left[\frac{B-A}{2}\rho\right]+(B-A)\rho\mu\cosh\left[\frac{B-A}{2}\rho\right]\right]\\ &\times\left[\cosh\{\rho[y-\tfrac{1}{2}(A+B)]\}-\cosh\left[\frac{B-A}{2}\rho\right]\right]\\ &-4a\rho\left[a^2\sinh\left[\frac{B-A}{2}\rho\right]+(B-A)\rho\mu D\cosh\left[\frac{B-A}{2}\rho\right]\right]\\ &\times\left[(B-A)\sinh\{\rho[y-\tfrac{1}{2}(A+B)]\}-2[y-\tfrac{1}{2}(A+B)]\sinh\left[\frac{B-A}{2}\rho\right]\right]\\ &+\Bigg[2a^2\mu\rho^2(B-A)\sinh[(B-A)\rho]-2\mu D^2\rho^4(B-A)\sinh[(B-A)\rho]\\ &-8\mu\rho a^2\sinh^2\left[\frac{B-A}{2}\rho\right]\Bigg](B-y)(y-A)\Bigg\}[(4(a^2+2\mu D)\sinh[\tfrac{1}{2}\rho(B-A)]\\ &\times\{(B-A)\rho^2(a^2-\rho^2D^2)\cosh[\tfrac{1}{2}\rho(B-A)]-2\rho a^2\sinh[\tfrac{1}{2}\rho(B-A)]\})\ . \end{aligned} \tag{3.31}$$

In the limit $a\to 0$, Eq. (3.31) reduces to the known result in (3.19). The limit $D\to 0$ is more elucidative and interesting: In the open interval (A,B) the result (3.31) for the MFPT reduces to the result in Table I with $w^+=1$. In agreement with the boundary condition in (3.5c) we find

$$\lim_{D\to 0}\lim_{y\uparrow A} T^{\rm abs}(y;+1)=0 \ . \tag{3.32}$$

Upon an interchange of limits, however, we find

$$\lim_{y\uparrow A}\lim_{D\to 0} T^{\rm abs}(y;+1)=(B-A)/a \ . \tag{3.33}$$

Indeed, the latter result follows for a particle located close to $y=A$ which starts with a positive velocity, $+a$, and moves deterministically $(D\to 0)$ towards the exit boundary at $y=B$.

IV. NONLINEAR DICHOTOMIC FLOW

The study of activation rates in nonlinear flows driven by non-Gaussian noise has recently figured in several theoretical discussions.[13–17] Following Refs. 13 and 14, we shall investigate the nonlinear flow

$$\dot{x}=f(x)+ag(x)\xi(t), \quad a>0, \quad g(x)>0 \tag{4.1}$$

where $\xi(t)$ is a dichotomic noise with unit jumps ± 1, as in Eqs. (2.1) and (2.2). This flow is a special case of the dichotomic diffusion considered in Sec. III if both diffusive components $g_+(x)$ and $g_-(x)$ are set equal to zero. We shall assume further that $f(x)$ exhibits bistability, with $[ag(x)\pm f(x)]$ being positive within the bistable region (see Ref. 14). In other words (with $x_1<x_u<x_2$) $f(x_1)=f(x_2)=0$ specifies the metastable states and $f(x_u)=0$ the unstable state. Equation (4.1) follows from (3.1) if we set

$$\begin{aligned} &f_+(x)=f(x)+ag(x)\,,\\ &f_-(x)=f(x)-ag(x), \quad g_+(x)=g_-(x)=0\,. \end{aligned} \tag{4.2}$$

Setting $T(y;+1)\equiv T_+(y)$ and $T(y;-1)\equiv T_-(y)$, respectively, we obtain from (3.11)

$$\begin{aligned} &(f+ag)T'_+ -\mu T_+ +\mu T_- = -1\,,\\ &(f-ag)T'_- -\mu T_- +\mu T_+ = -1\,. \end{aligned} \tag{4.3}$$

Upon eliminating T_+ or T_- we get a first-order differential equation in T'_+ or T'_-, respectively. The boundary conditions are as follows:

$$\begin{aligned} &T_-(A)=0, \quad A \text{ absorbing}\\ &T_+(B)=0, \quad B \text{ absorbing} \end{aligned} \tag{4.4}$$

$$\begin{aligned} &T_+(A)=T_-(A), \quad A \text{ reflecting}\\ &T_+(B)=0, \quad B \text{ absorbing}\,. \end{aligned} \tag{4.5}$$

The stationary probability distribution corresponding to the stochastic flow (4.1), $p(x)$, is explicitly given by[13,14]

$$p(x)=\frac{g}{(g+f/a)(g-f/a)} \times \exp\int_0^x \frac{f}{D(g-f/a)(g+f/a)}dy\,, \tag{4.6}$$

where $D=a^2\tau=a^2/2\mu$ is the noise intensity. The differential equation obeyed by $T_+(y)$ reads, from Eqs. (4.3),

$$D\left[\left(\frac{f}{a}\right)^2-g^2\right]T''_+ + \left[-f+D\left(\frac{f}{a}-g\right)\left(\frac{f'}{a}+g'\right)\right]T'_+ = +1\,, \tag{4.7a}$$

while $T_-(y)$ satisfies

$$D\left[\left(\frac{f}{a}\right)^2-g^2\right]T''_+ + \left[-f+D\left(\frac{f}{a}+g\right)\left(\frac{f'}{a}-g'\right)\right]T'_- = +1\,. \tag{4.7b}$$

These equations can be solved explicitly with the boundary conditions in (4.4) or (4.5). In terms of the auxiliary functions

$$y_0^+(z)=g\left[\left(g+\frac{f}{a}\right)^2\left(g-\frac{f}{a}\right)p\right]^{-1}, \tag{4.8a}$$

$$y_1^+(z)=\left[\int_z^0 dy\frac{(g+f/a)p}{Dg}\right]y_0^+\,, \tag{4.8b}$$

$$y_0^-(z)=g\left[\left(g-\frac{f}{a}\right)^2\left(g+\frac{f}{a}\right)p\right]^{-1}, \tag{4.8c}$$

and

$$y_1^-(z)=\left[\int_z^0 dy\frac{(g-f/a)p}{Dg}\right]y_0^-\,, \tag{4.8d}$$

the resulting MFPT's have the following explicit forms.

(a) *A and B both absorbing:*

$$T_+^{\text{abs}}(y)=\int_B^y y_0^+(z)dz\left[\int_z^0\frac{(g+f/a)p}{Dg}du\right] + C_+^{\text{abs}}\int_B^y y_0^+(u)du\,, \tag{4.9a}$$

where

$$C_+^{\text{abs}}=\frac{-y_1^+(A)(f/a+g)(A)-\dfrac{1}{a}+\dfrac{a}{2D}\displaystyle\int_B^A y_1^+(z)dz}{\left[y_0^+(A)(f/a+g)(A)-\dfrac{a}{2D}\displaystyle\int_B^A y_0^+(z)dz\right]}\,, \tag{4.9b}$$

with $(a^{-1}f+g)(A)$ denoting $a^{-1}f(A)+g(A)$, etc.

For $T_-^{\text{abs}}(y)$ one finds

$$T_-^{\text{abs}}(y)=\int_A^y y_0^-(z)dz\left[\int_z^0\frac{(g-f/a)p}{Dg}du\right] + C_-^{\text{abs}}\int_A^y y_0^-(u)du\,, \tag{4.10a}$$

where

$$C_-^{\text{abs}}=\frac{-y_1^-(B)(g-f/a)(B)+\dfrac{1}{a}+\dfrac{a}{2D}\displaystyle\int_B^A y_1^-(z)dz}{\left[y_0^-(B)(g-f/a)(B)-\dfrac{a}{2D}\displaystyle\int_B^A y_0^-(z)dz\right]}\,. \tag{4.10b}$$

In the Gaussian white-noise limit, i.e., $a\to\infty$, $\mu\to\infty$, such that $D=a^2/2\mu$ finite, $T_+(y)$ equals $T_-(y)$ and coincides with the well-known expression[1,2] for an (Ito)-Fokker-Planck process, i.e., $\dot{x}=f(x)+g(x)\xi(t)$, where the noise ξ is δ-correlated according to $\langle\xi(t)\xi(s)\rangle=2D\delta(t-s)$.

Next we turn to the problem of escape and its connection with the concept of the MFPT. In order to compare mean first-passage times with activation rates we must evaluate the MFPT with A taken to be a *reflecting boundary*. We obtain the following results.

(b) *A reflecting and B absorbing:*

$$T_+^{\text{refl}}(y)=\int_B^y y_0^+(z)dz\left[\int_z^B \frac{(g+f/a)p}{Dg}du\right]+C_+^{\text{ref}}\int_B^y y_0^+(z)dz\;, \tag{4.11a}$$

with

$$C_+^{\text{refl}}=\left[\int_B^A \frac{(g+f/a)p}{Dg}du\right]-\frac{[g^2-(f/a)^2](A)p(A)}{ag(A)}\;. \tag{4.11b}$$

Equations (4.11a) and (4.11b) can be combined to yield the useful expression

$$T_+^{\text{refl}}(y)=\int_y^B \frac{g\,dz}{D(g+f/a)^2(g-f/a)p}\int_A^z \frac{p(g+f/a)du}{g}+\frac{D}{a}\frac{p(A)[g^2-(f/a)^2](A)}{g(A)}\times\left[\int_y^B \frac{g\,du}{D(g+f/a)^2(g-f/a)p}\right]. \tag{4.12}$$

Likewise, we obtain for $T_-^{\text{refl}}(y)$,

$$T_-^{\text{refl}}(y)=\int_y^B \frac{g\,dz}{D(g-f/a)^2(g+f/a)p}\int_A^z \frac{p(g-f/a)du}{g}-\frac{D}{a}\frac{p(A)[g^2-(f/a)^2](A)}{g(A)}\times\int_y^B \frac{g\,du}{D(g-f/a)^2(g+f/a)p}\Bigg]+T_-^{\text{refl}}(B)\;. \tag{4.13}$$

Here $T_-^{\text{refl}}(B)$ is obtained by setting $y=A$ in Eq. (4.13), using the identity $T_-^{\text{refl}}(A)=T_+^{\text{refl}}(A)$, and involving Eq. (4.12) with $y=A$ for the latter quantity. In the limit of Gaussian white noise, both expressions (4.12) and (4.13) yield the well-known expression[1,2]

$$\lim_{a\to\infty} T_+^{\text{refl}}(y)=\lim_{a\to\infty} T_-^{\text{refl}}(y)=\int_y^B \frac{dx}{D(x)\bar{p}(x)}\int_A^x \bar{p}(y)dy\;, \tag{4.14}$$

where $D(x)\cong Dg^2(x)$ is the white-noise diffusion coefficient, and $\bar{p}(x)$ denotes the corresponding stationary probability.

For the activation rate in the case of weak noise (i.e., $D\ll 1$), we have for the forward rate Γ^+ with $A<x_1$, $B=x_2$,

$$\Gamma^+=\frac{1}{T_+^{\text{refl}}}=\frac{1}{T_-^{\text{refl}}}\quad \text{as } D\to 0\;. \tag{4.15}$$

From both (4.12) and (4.13) we obtain, by the use of a steepest-descent approximation ($D\ll 1$),

$$\Gamma^+=\frac{(\lambda_1|\lambda_u|)^{1/2}}{2\pi}\exp\left[-\frac{\Delta\Phi}{D}\right], \tag{4.16a}$$

where

$$\lambda_1=-f'(x_1)>0,\quad \lambda_u=-f'(x_u)<0\;, \tag{4.16b}$$

and

$$\Delta\Phi=-\int_{x_1}^{x_u}\frac{f}{(g-f/a)(g+f/a)}dy\;. \tag{4.16c}$$

This result coincides in leading order [i.e., to order $O(D^0)$] with the result derived in Refs. 13 and 14 by means of the "flux-over-population" approach.

V. CONCLUSIONS

Following the recent activity (see Refs. 9 and 10) aimed at obtaining exact results for the MFPT of non-Markovian processes that are driven by two-state noise, we have considered in this work the case of a reflecting boundary. Moreover, we have extended our previous studies[9,13,14] to the dichotomic Fokker-Planck process, i.e., a process in which a random walker hops between two different one-dimensional Fokker-Planck dynamics. The main new results obtained in the present work are (i) exact results for the MFPT and the first-passage-time density for a free dichotomic flow (Sec. II) with both absorbing and reflecting boundaries. (ii) Exact results for the MFPT for free and forced dichotomic diffusion [Eqs. (3.14), (3.17), and (3.31)] which in some specific limits reduce to rather simple expressions that help elucidate the underlying physics. (iii) Closed-form expressions (in terms of the exact stationary probability density) for the MFPT of nonlinear, bistable dichotomic flows (Sec. V). These exact results are closely related to similar, well-known expressions for the MFPT of one-dimensional Fokker-Planck processes.[1,2]

The results of Sec. V also clarify the connection between the rate approach[13,14] and the MFPT concept. Using appropriate boundary conditions, the well-known relationship to the effect that the rate equals the inverse MFPT for leaving the metastable state x_1 (with a reflecting boundary to the left of x_1) and being absorbed in the neighboring metastable state x_2 is shown to hold well also for one-dimensional non-Markov processes of the kinds considered.

APPENDIX

Consider a random walk on two sets $(n,+)$ and $(n,-)$, $n=0,1,\ldots$, of discrete states (see Fig. 1). Exchanges between adjacent cells take place with birth and death rates $\Gamma^+_{n;n\pm1}$ to go from $(n,+)$ to $(n\pm1,+)$, and $\Gamma^-_{n;n\pm1}$ to go from $(n,-)$ to $(n\pm1,-)$, respectively. Transitions between the two layers occur for $n>0$ with a rate μ. At the boundary itself the transition rate from $(0,+)\to(0,-)$ will be denoted by ν^-, and that for $(0,-)\to(0,+)$ by ν^+. Our goal is to take the limit in which the random walk on the layers approaches a continuous-time Fokker-Planck process, while the boundary states $(0,+)$ and $(0,-)$ will be kept separately to facilitate the precise implementation of various boundary conditions. A variety of cases corresponding to different types of boundary conditions can thus be investigated. The procedure can be applied to both the forward and the backward Kolmogorov equations. The detailed limiting procedure to go from a discrete random walk to a diffusion process is already known[18] and will

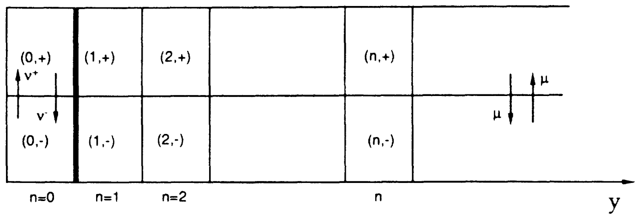

FIG. 1. Discrete dichotomic Fokker-Planck process. A random walker hops between two sets of states $(n,+),(n,-)$ with rates $\mu^-=\mu^+=\mu$; at the boundary, the corresponding rates are ν^+ and ν^-.

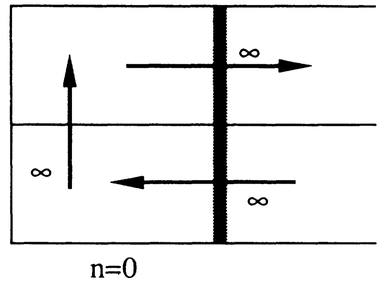

FIG. 2. Discrete dichotomic Fokker-Planck process. The case of immediate reflection at the boundary.

not be repeated here explicitly. For the sojourn probability $F_t(y;\pm)$ introduced in Sec. II, one finds the backward equations (3.4a) and (3.4b), respectively. The most general boundary conditions emerge as

$$[g_+(y)]^2 \left.\frac{\partial F_t(y,-)}{\partial y}\right|_{y=0} = \Gamma_{\text{abs}}^{-}[-F_t(b,+)+F_t(y=0,+)] \quad \text{(A1)}$$

and

$$[g_-(y)]^2 \left.\frac{\partial F_t(y,-)}{\partial y}\right|_{y=0} = \Gamma_{\text{abs}}^{-}[-F_t(b,-)+F_t(y=0,-)] \ . \quad \text{(A2)}$$

These equations connect the flux at $y=0$ to the value of the function $F_t(y,\pm)$ at the boundary of the continuum states (Fig. 1); the latter quantity is denoted by $F_t(b,\pm)$. These boundary states obey

$$\dot{F}_t(b,+) = +\Gamma_{\text{des}}^{+}[F_t(y=0,+)-F_t(b,+)] + \nu^-[F_t(b,-)-F_t(b,+)] \quad \text{(A3)}$$

and

$$\dot{F}_t(b,-) = +\Gamma_{\text{des}}^{-}[F_t(y=0,-)-F_t(b,-)] + \nu^+[F_t(b,+)-F_t(b,-)] \ . \quad \text{(A4)}$$

Here $\Gamma_{\text{des}}^{+}=\Gamma_{0;1}^{-}, \Gamma_{\text{des}}^{-}=\Gamma_{0;1}^{-}$ and $\Gamma_{\text{abs}}^{\pm}=\lim_{\lambda\to 0}\Gamma_{1;0}^{\pm}\lambda$ (λ denotes the length of a cell) is the rate constant in a given layer describing the in-flow and outflow of probability from the continuum to the boundary (Fig. 2).

The general boundary conditions in (A1) and (A2) can be simplified if we consider special situations. The correct choice for the boundary condition obviously depends on the physical absorption and desorption mechanisms at the boundary itself. Here, we shall consider only the case of "*immediate reinjection*" from the $(-)$ state to the $(+)$ state; i.e., particles that touch the boundary at $y=0$ in the $(-)$ state are immediately reinserted at $y=0$ in the $(+)$ state. As is physically clear, this can be achieved as follows. We set $\Gamma_{\text{abs}}^{+}=0$, i.e., no particles enter the boundary from $(y=0,+)$. From (A1) one then finds the usual condition for a reflecting boundary for the $(+)$ layer, i.e.,

$$[g_+(y)]^2 \left.\frac{\partial F_t(y,+)}{\partial y}\right|_{y=0} = 0 \ . \quad \text{(A5)}$$

Immediate reinjection can be realized by letting Γ_{abs}^{-}, ν^-, and Γ_{des}^{+} go to infinity (see Fig. 2). Upon elimination of the boundary states, one finds from (A2)–(A4) the result

$$F_t(y=0,+) = F_t(y=0,-) \ . \quad \text{(A6)}$$

This condition is also intuitively clear on inspecting Fig. 2 with $\Gamma_{\text{abs}}^{-}=\nu^-=\Gamma_{\text{des}}^{+}=\infty$.

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