Effect of a spectral line with Doppler broadening on optical bistability in a ring cavity

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(Received 17 September 1987)

We discuss the effect of a spectral line with Doppler broadening on optical bistability in a ring cavity which contains a saturable absorber of two-level atoms. In this paper, we first derive a generalized equation that is satisfied by the output light intensity of this system from the Langevin equations in H. Haken's *Laser Theory* (Springer, Heidelberg, 1970). Then, we simplify the generalized equation for a single mode. We study the single-mode equation for both nonexisting and existing spontaneous emission and obtain the result that the square of the bistability loop increases with the increase of Doppler broadening of the spectral line.

I. INTRODUCTION

Optical bistability has been one of the important problems in the field of nonlinear optics for several years. As early as 1969, when Szöke *et al.* researched the properties of a Fabry-Pérot étalon which contains a saturable absorber,[1] they found that this system could show a new feature. So they first put forward the concept of the optical bistability and predicted some applications of this system. Unfortunately, at that time they could not observe the bistable output of this system. The bistability was not experimentally demonstrated until 1975.[2] At the same time, many other people studied bistability, some experimentally and some developing only the theoretical prediction of bistability.[3–6]

In fact, there are two physical mechanisms to produce bistability. One is an absorptive operation[5–15] which is due to the saturable absorption of an absorber in the laser cavity and the other is a dispersive operation[16–21] which results from the dispersion of a nonlinear body in the laser cavity. In this paper we will study the first one, considering a two-level-atom system. Many methods have been developed to deal with the various problems in a bistable system which contains the two-level atoms as an absorber. Examples are classical analysis[5] by rate equations, semiclassical analyses by the Bloch equation[6] or the Bloch-Maxwell equation,[7,8] and several quantum treatments by the Von Neumann equation[10,11] or photon statistics.[12,21] But in these papers and in Refs. 13–20, the laser light is treated as monochromatic. In fact, the light exhibits a certain broadening, especially inhomogeneous broadening because of spontaneous emission, movement of atoms in gas, and the lattice distortion caused by impurity atoms in a solid. This inhomogeneous linewidth was included in Ref. 9 when the authors studied a gas-laser system with a saturable absorber. In Ref. 9 the authors mainly discuss the principle of generating the bistability and give some parameters which cause bistability of this system, such as the pumping parameters π, Π, and the detuning $(\Omega-\omega)/k,u$. They did not study the effect of the spectral line with inhomogeneous broadening on the bistability, for their starting point was only to treat the gas-laser system. As the aim of this paper we discuss this effect on the bistability.

The method that we use starts from the Langevin equations in Ref. 22, which gives people a distinct physical image of inhomogeneous broadening. In Sec. II we give the general theory of all procedures and derive the equation of motion of the cavity modes. This gives a multimode case more general than is needed for this paper. In Sec. III we simplify the multimode equation into a single-mode equation using some approximations. Section IV discusses the solution of the single-mode equation and draws the curves of the output light with Doppler broadening. Finally, Sec. V briefly concludes this paper.

II. GENERAL THEORY

Figure 1 shows the principle of the ring cavity which consists of three mirrors. In a ring cavity, an advantage is that the population inversion of atoms is spatially homogeneous because the modes in the ring cavity are running waves and the spatial structure of the modes is not taken into account for running-wave modes.[23,24] In Fig. 1 A represents an amplification cell and B is an absorption cell. In the theoretical model to be used we assume that the amplifying medium and absorbing medium consist of a set of two-level atoms, and that the laser action occurs from the atomic state l to another, k (see Fig. 2). Figure 2 is based on a supposition that the atomic transition in the absorbing cell can be described with a model similar to the amplifying cell, but with different values of the characterizing parameters, and that the atoms and the resonance frequencies in both cells are the same.[9,10] The Doppler broadening is inhomogeneous, thus implying that a given mode is coupled with different strengths to different groups of atoms. This characteristic is illustrated in the subsequent discussion. In Fig. 2, $a^{\dagger}$ and a are the creation and annihilation operators of an atom for the active cell (the subscripts l and k show the high and low levels, respectively); the levels in the amplifying medium decay with the rates γ for the high level and γ_k for low level, and they are populated by a pump-

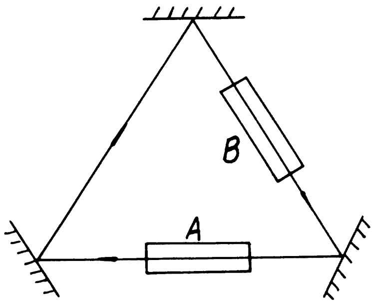

FIG. 1. Principle of the ring cavity.

ing mechanism at the rate R_{ua} for the high level and the rate λ_{ua} for the low level. Similarly, in the absorbing medium, $\bar{a}^\dagger$ and $\bar{a}$ represent the creation and annihilation operators of the atom, the decay rates of the two levels are conveniently expressed by γ and γ_k multiplied by some parameter ξ ($\xi<1$),[9,10] and the pumping rates are R_{ub} for the high level and λ_{ub} for the low level. The pumping rates in the amplifying cell must be so intense as to get the population inversion, but on the contrary, in the absorbing cell the pumping rates are weak and no population inversion is induced. In the laser field, $b_j^\dagger$ and b_j are the creation and annihilation operators of photons, and the photon numbers are given by $n_j = b_j^\dagger b_j$ for mode j whose frequency and lifetime are described by Ω_j and $1/k_j$. We assume that the mode j interacts with groups of atoms counted by u, and the strength of the interaction between them is described by g_{uj}. Furthermore we assume that the emission lines of all groups of atoms have the same homogeneous width Γ but different line centers ω_u. These atom lines superimpose upon the Doppler broadening line. Finally we assume that the group of atoms u has a corresponding group in the absorbing cell.

After the above interpretation, we can set up Langevin equations for the field, the dipole, and the atom number. In the field equation, spontaneous emission is taken account of.

The field equation is

$$\dot{b}_j^\dagger = \left[i\Omega_j - \frac{k_j}{2} \right] b_j^\dagger + i \left[1 + \frac{1}{n_j} \right] \sum_u g_{uj}^* (a_l^\dagger a_k)_u - i \left[1 + \frac{1}{n_j} \right] \sum_u g_{uj}^* (\bar{a}_l^\dagger \bar{a}_k)_u + F_j^+ ,$$

the dipole equations are

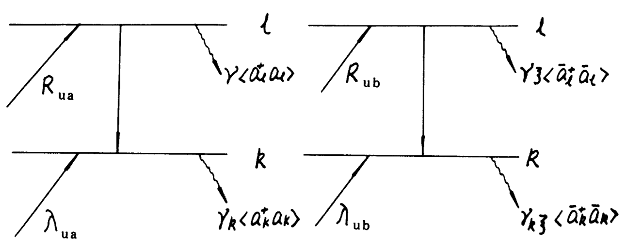

FIG. 2. Transmission of two cells.

$$(a_l^\dagger a_k)'_u = \left[i\omega_u - \frac{\Gamma}{2} \right] (a_l^\dagger a_k)_u - i \sum_j g_{uj} b_j^\dagger (N_l - N_k)_u + \Gamma_{lk,u} ,$$

$$(\bar{a}_l^\dagger \bar{a}_k) = \left[i\omega_u - \frac{\Gamma}{2} \right] (\bar{a}_l^\dagger \bar{a}_k)_u - i \sum_j g_{uj} b_j^\dagger (\bar{N}_l - \bar{N}_k)_u + \bar{\Gamma}_{lk,u} ,$$

and the atom-number equations are

$$\dot{N}_{lu} = R_{ua} + i(a_k^\dagger a_l)_u \sum_j g_{uj} b_j^\dagger - i(a_l^\dagger a_k)_u \sum_j g_{uj}^* b_j - \gamma N_{lu} + \Gamma_{ll,u} ,$$

$$\dot{N}_{ku} = \lambda_{ua} + i(a_l^\dagger a_k) u \sum_j g_{uj}^* b_j - i(a_k^\dagger a_l)_u \sum_j g_{uj} b_j^* - \gamma_k N_{ku} + \Gamma_{kk,u} ,$$

$$\dot{\bar{N}}_{lu} = R_{ub} + i(\bar{a}_k^\dagger \bar{a}_l)_u \sum_j g_{uj} b_j^\dagger - i(\bar{a}_l^\dagger \bar{a}_k)_u \sum_j g_{uj}^* b_j - \xi\gamma N_{lu} + \bar{\Gamma}_{ll,u} ,$$

$$\dot{\bar{N}}_{ku} = \lambda_{ub} + i(\bar{a}_l^\dagger a_k)_u \sum_J g_{uj}^* b_j - i(\bar{a}_k^\dagger \bar{a}_l)_u \sum_j g_{uj} b_j^\dagger - \xi\gamma_k N_{ku} + \bar{\Gamma}_{kk,u} ,$$

where

$$N_{lu} = \langle a_l^\dagger a_l \rangle_u, \quad N_{ku} = \langle a_k^\dagger a_k \rangle_u ,$$
$$\bar{N}_{lu} = \langle \bar{a}_l^\dagger \bar{a}_l \rangle_u, \quad \bar{N}_{ku} = \langle \bar{a}_k^\dagger \bar{a}_k \rangle_u .$$

$F_j^+, \Gamma_{lk,u}, \bar{\Gamma}_{lk,u} \Gamma_{ll,u}, \Gamma_{kk,u}, \bar{\Gamma}_{kk,u}$ are the fluctuating force of the field, dipole, and atom number, which vanish when we average them over the heat bath, i.e.,

$$\langle F_j^+ \rangle = 0, \quad \langle \Gamma_{lk,u} \rangle = \langle \bar{\Gamma}_{lk,u} \rangle = 0 ,$$
$$\langle \Gamma_{ll,u} \rangle = \langle \bar{\Gamma}_{ll,u} \rangle = 0, \quad \langle \Gamma_{kk,u} \rangle = \langle \bar{\Gamma}_{kk,u} \rangle = 0 .$$

The fluctuations result from the microscopic effects, but we know that any macroscopic quantity comes from the statistical average of a lot of microscopic ones corresponding to it. We study the macroscopic quantity so as to compare this theoretical result with the experimental demonstration; hence we average the above equations around the heat bath. The mean-field equations are as follows:

$$b_j^* = \left[i\Omega_j - \frac{K_j}{2} \right] + i \left[1 + \frac{1}{n_j} \right] \sum_u g_{uj}^* \langle a_l^\dagger a_k \rangle - i \left[1 + \frac{1}{n_j} \right] \sum_u g_{uj}^* \langle \bar{a}_l^\dagger \bar{a}_k \rangle , \qquad (1)$$

$$\langle a_l^\dagger a_k\rangle_u' = \left[i\omega_u - \frac{\Gamma}{2}\right]\langle a_l^\dagger a_k\rangle_u - i\sum_j g_{uj} b_j^*(N_l - N_k)_u \ , \tag{2}$$

$$\langle \bar{a}_l^\dagger \bar{a}_k\rangle_u' = \left[i\omega_u - \frac{\Gamma}{2}\right]\langle \bar{a}_l^\dagger \bar{a}_k\rangle_u - i\sum_j g_{uj} b_j^*(N_l - N_k)_u \ , \tag{3}$$

$$\dot{N}_{lu} = R_{ua} + i\langle a_k^\dagger a_l\rangle_u \sum_j g_{uj} b_j^* - i\langle a_l^\dagger a_k\rangle_u \sum_j g_{uj}^* b_j - \gamma N_{lu} \ , \tag{4}$$

$$\dot{N}_{ku} = \lambda_{ua} + i\langle a_l^\dagger a_k\rangle_u \sum_j g_{uj}^* b_j - i\langle a_k^\dagger a_l\rangle_u \sum_j g_{uj} b_j^* - \gamma_k N_{ku} \ , \tag{5}$$

$$\dot{\bar{N}}_{lu} = R_{ub} + i\langle \bar{a}_k^\dagger \bar{a}_l\rangle_u \sum_j g_{uj} b_j^* - i\langle \bar{a}_l^\dagger \bar{a}_k\rangle_u \sum_j g_{uj}^* b_j - \xi\gamma N_{lu} \ , \tag{6}$$

$$\dot{\bar{N}}_{ku} = \lambda_{ub} + i\langle \bar{a}_l^\dagger \bar{a}_k\rangle_u \sum_j g_{uj}^* b_j - i\langle \bar{a}_k^\dagger \bar{a}_l\rangle_u \sum_j g_{uj} b_j - \xi\gamma_k N_{ku} \ . \tag{7}$$

Assume that the atoms rapidly relax to the stable state around Eqs. (2)–(7), i.e.,

$$\Gamma\gamma \gg k_j \ . \tag{8}$$

Thus the atom number and the dipole could be slow time variables. Ignoring the frequency pulling and inserting $b_j^* = B_j^* \exp(i\Omega_j t)$ (B_j is the slow time variable compared with b_j) in Eqs. (2) and (3), we have

$$\langle a_l^\dagger a_k\rangle_u = -i\sum_j g_{uj} b_j^*(N_l - N_k)_u D_j(\Omega_j - \omega_u) \ , \tag{9}$$

$$\langle \bar{a}_l^\dagger \bar{a}_k\rangle_u = -i\sum_j g_{uj} b_j^*(\bar{N}_l - \bar{N}_k)_u D_j(\Omega_j - \omega_u) \ , \tag{10}$$

where

$$D_j(\Omega_j - \omega_u) = \frac{1}{\frac{\Gamma}{2} + i(\Omega_j - \omega_u)} \ . \tag{11}$$

Substituting Eqs. (9) and (10) in Eq. (1), we obtain

$$b_j^* = \left[i\Omega_j - \frac{K_j}{2}\right] b_j^* + \sum_{uj'} g_{uj}^* g_{uj'} b_{j'}^* (N_l - N_k)_u D_j \times (\Omega_j - \omega_u)\left[1 + \frac{1}{n_j}\right] - \sum_{u,j'} g_{uj}^* g_{uj'} b_{j'}^* (\bar{N}_l - \bar{N}_k)_u D_j(\Omega_j - \omega_u)\left[1 + \frac{1}{n_j}\right] \ . \tag{12}$$

Multiplying Eq. (12) by b_j and adding the complex-conjugate expression to the result, we obtain

$$\dot{n}_j = -k_j n_j + \left[1 + \frac{1}{n_j}\right]\sum_{u,j'} [g_{uj}^* g_{uj'} b_{j'}^* b_j D_j(\Omega_j - \omega_u) + g_{uj} g_{uj'}^* b_{j'} b_j^* D_j(\Omega_j - \omega_u)](N_l - N_k)_u - \left[1 + \frac{1}{n_j}\right]\sum_{u,j'} [g_{uj}^* g_{uj'} b_{j'}^* b_j D_j(\Omega_j - \omega_u) + g_{uj} g_{uj'}^* b_{j'} b_j^* D_j(\Omega_j - \omega_u)](\bar{N}_l - \bar{N}_k)_u \ . \tag{13}$$

In order to go on we must neglect all terms containing $b_{j'}^* b_j$ ($j \neq j'$). This can be justified as follows: As long as no phase locking occurs, the phase fluctuations of the different modes are uncorrelated. Thus the mixed terms ($j \neq j'$) on the right-hand side of Eq. (13) vanish if an average is taken over the phases.

The rate equation for the photon number thus reads

$$\dot{n}_j = -K_j n_j + \frac{\sum_u 4|g_{uj}|^2 L(\Omega_j - \omega_u)(N_l - N_k)_u}{\Gamma} - \frac{\sum_u 4|g_{uj}|^2 L(\Omega_j - \omega_u)(\bar{N}_l - \bar{N}_k)_u}{\Gamma} \ , \tag{14}$$

where

$$n_j = b_j^* b_j \ ,$$
$$L(\Omega_j - \omega_u) = \frac{\Gamma^2}{4(\Omega_j - \omega_u)^2 + \Gamma^2} \ . \tag{15}$$

When we insert Eq. (9) into Eqs. (4) and (5), insert Eq. (10) into Eqs. (6) and (7), and apply the same reasoning, we find

$$\dot{N}_{lu} = R_{ua} - \gamma N_{lu} - \frac{\sum_j 4|g_{uj}|^2 n_j L(\Omega_j - \omega_u)(N_l - N_k)_u}{\Gamma} \ , \tag{16}$$

$$\dot{N}_{ku}=\lambda_{ua}-\gamma_k N_{ku}+\frac{\sum_j 4|g_{uj}|^2 n_j L(\Omega_j-\omega_u)(N_l-N_k)_u}{\Gamma}\ , \tag{17}$$

$$\dot{\overline{N}}_{lu}=R_{ub}-\xi\gamma\overline{N}_{lu}-\frac{\sum_j 4|g_{uj}|^2 n_j L(\Omega_j-\omega_u)(\overline{N}_l-\overline{N}_k)_u}{\Gamma}\ , \tag{18}$$

$$\dot{\overline{N}}_{ku}=\lambda_{ub}-\xi\gamma_k\overline{N}_{ku}+\frac{\sum_j 4|g_{uj}|^2 n_j L(\Omega_j-\omega_u)(\overline{N}_l-\overline{N}_k)_u}{\Gamma}\ . \tag{19}$$

We define the stationary state by the condition that all time derivatives vanish in Eqs. (14), (16)–(19), i.e.,

$$\dot{n}_j=\dot{N}_{lu}=\dot{N}_{ku}=\dot{\overline{N}}_{lu}=\dot{\overline{N}}_{ku}=0\ .$$

Then from the set of equations (16),(17) and (18),(19), we have

$$N_{lu}-N_{ku}=\frac{(R_{ua}/\gamma-\lambda_{ju}/\gamma_k)}{1+\sum_j I_j L(\Omega_j-\omega_u)/I_{sa}}\ , \tag{20}$$

$$\overline{N}_{lu}-\overline{N}_{ku}=\frac{(R_{ub}/\gamma-\lambda_{ub}/\gamma_k)}{1+\sum_j I_j L(\Omega_j-\omega_u)/I_{sb}}\ , \tag{21}$$

where

$$I_j=4|g_{uj}|^2 n_j/\gamma\ , \tag{22}$$

$$I_{sa}=(1-\gamma/\gamma_k)^{-1}\Gamma\ ,\quad I_{sb}=(1-\gamma/\gamma_k)^{-1}\Gamma\ . \tag{23}$$

In the amplifying cell, because of Doppler broadening the pumping rates usually distribute inhomogeneously for all modes. Assume that the high-level pumping of the amplifying cell is

$$R_{ua}=\frac{\varepsilon^2 R_0}{\varepsilon^2+4(\omega_u-\Omega_0)^2}\ , \tag{24}$$

where ε is the inhomogeneous width of the mode j, R_0 is the pumping rate corresponding to the central frequency Ω_0 of the mode, and other pumpings (R_{ub}, λ_{ua}, and λ_{ub}) are assumed as constants over the range of ω_u $(0<\omega_u<\infty)$.

$$I_j=\sum_u\left[\frac{4|g_{uj}|^2}{\gamma}+I_j\right]\left[\frac{[\varepsilon^2R_0/\varepsilon^2+4(\omega_u-\Omega_0)^2]-\gamma\lambda_{ua}/\gamma_k}{1+\sum_i I_iL(\Omega_i-\omega_u)/I_{sa}}-\frac{[R_{ub}-(\gamma/\gamma_k)\lambda_{ub}]}{\xi+\sum_i I_iL(\Omega_i-\omega_u)/I_{sb}}\right]\frac{4|g_{uj}|^2}{K_j\gamma\Gamma}L(\Omega_j-\omega_u)\ . \tag{25}$$

In the above derivation, we have used Eq. (22) and $I_{sb}=I_{sa}\xi$. Equation (25) is just the equation which is satisfied by the multimode output light of a ring cavity.

III. SINGLE-MODE APPROXIMATION

In Sec. II Eq. (25) gives an equation about any mode j, in which there are two summations over u and j, and the summation over j is in the denominator. It is clear that Eq. (25) cannot be calculated without an appropriate approximation. When there is only one mode oscillating in the cavity the summation over j vanishes.

In Eq. (25), g_{uj} describes the interaction between the atom u and the mode j. If we admit that the atoms in two cells have a spatial homogeneous distribution for a ring cavity, g_{uj} can be considered constant. Thus in Eq. (25) we could delete the subscript j and replace $|g_{uj}|^2$ by $|g|^2$ so the single-mode equation is

$$I=\left[\frac{4|g|^2}{\gamma}+I\right]\times\left[\frac{(\varepsilon^2R_0/\varepsilon^2+4(\omega_u-\Omega_0)-\gamma\lambda_{ua}/\gamma_k}{\Gamma^2+I\Gamma^2/I_{sa}+4(\omega_u-\Omega)^2}-\frac{R_{ub}-(\gamma/\gamma_k)\lambda_{ub}}{\xi\Gamma^2+I\Gamma^2/I_{sa}+4\xi(\omega_u-\Omega)}\right]\frac{4|g|^2}{\gamma k}\Gamma\ . \tag{26}$$

In our procedure we have used Eq. (15).

As mentioned in Secs. I and II, there are many small modes in the inhomogeneous width ξ because the atom groups emit an individual line with the same homogeneous width Γ but a different central frequency ω_u. If $\Gamma\ll\varepsilon$, the following approximation can be supported:

$$\sum_u A_u\to\frac{2}{\pi\Gamma}\int^{\infty}A_\omega d\omega\ . \tag{27}$$

Using the replacement $\omega_u-\Omega_0\to\omega_0-\Omega$ in Eq. (26) and calculating it, this leads us to

$$I=\left[\frac{4|g|^2}{\gamma}+I\right]\times\left[\frac{[\varepsilon^2R_0/\varepsilon^2+4(\omega_0-\Omega)^2-(\gamma/\gamma_k)\lambda_{ub}]}{\sqrt{1+I/I_{sa}}}-\frac{\frac{1}{3}[R_{ub}-(\gamma/\gamma_k)\lambda_{ub}]}{\sqrt{1+I/I_{sa}\cdot\xi}}\right]\frac{4|g|^2}{k\Gamma\gamma}\ , \tag{28}$$

where ω_0 is the atom group frequency corresponding to the central frequency Ω_0 and Ω is the frequency of the mode.

In the vicinity of the stimulation threshold, I is much smaller than I_{sa}, i.e., $I/I_{sa} \ll 1$, so we could take the following expansion:

$$\sqrt{1+I/I_{sa}}=1+I/2I_{sa} \ ,$$
$$\sqrt{1+I/I_{sa}\xi}=1+\frac{I}{2\xi I_{sa}} \ . \qquad (29)$$

After developing the above derivation, we understand that the term $4|g|^2/\gamma$ in the factor $(4|g|^2/\gamma+I)$ in Eq. (28) results from spontaneous emission, so it is much smaller than the output intensity I. In Sec. IV we shall discuss the effect of the Doppler broadening in two cases: spontaneous emission does exist or spontaneous emission does not exist.

At the end of this section, we expand Eq. (28) in the following form, in which we use Eq. (29):

$$I^3+2\Bigg\{I_{sa}(\xi+1)-\frac{4|g|^2 I_{sa}}{k\Gamma\gamma}(\{[\varepsilon^2R_0/\varepsilon^2+4(\omega_0-\Omega)^2]-\gamma\lambda_{ua}/\gamma\}-(R_{ub}-\gamma\lambda_{jb}/\gamma_k))I^2$$
$$+4\Bigg[I_{sa}\xi-\frac{8|g|^4 I_{sa}}{k\Gamma\gamma^2}[(\{[\varepsilon^2R_0/\varepsilon^2+4(\omega_0-\Omega)^2]-\gamma\lambda_{ua}/\gamma_k\})-(R_{ub}-\gamma\lambda_{ub}/\gamma_k)]\Bigg]$$
$$-\frac{4|g|^2\xi}{k\Gamma\gamma}\Bigg[[\varepsilon^2R_0/\varepsilon^2+4(\omega_0-\Omega)^2-\gamma\lambda_{ua}/\gamma_k]-\frac{1}{3}\Bigg[R_{ub}-\frac{\gamma}{\gamma_k}\lambda_{ub}\Bigg]\Bigg]\Bigg\}I$$
$$+\frac{64|g|^4 I_{sa}^2\xi}{k\Gamma\gamma^2}\Bigg[[\varepsilon^2R_0/\varepsilon^2+4(\omega_0-\Omega)^2-\gamma\lambda_{ua}/\gamma_k]-\frac{1}{3}\Bigg[R_{ub}-\frac{\gamma}{\gamma_k}\lambda_{ub}\Bigg]\Bigg]=0 \ . \qquad (30)$$

This is a cubic nonhomogeneous equation for the intensity I. We can see from it that spontaneous emission leads to the terms proportional to $|g|^2/\gamma$ in Eq. (30). As the constant term in Eq. (30) comes from spontaneous emission, it is a small quantity. Equation (30) can be solved by the method of iteration; thus

$$\frac{I}{2I_{sa}}=\frac{I'}{2I_{sa}}+\frac{\Delta I}{2I_{sa}} \ ,$$

where $I/2I_{sa}$ is a solution of the homogeneous form of Eq. (30). These results will be given in Sec. IV.

IV. DISCUSSION OF THE SOLUTION

In order to see the effect of the width ε on the bistability, we shall discuss Eq. (28) in two cases.

A. Absence of spontaneous emission

If there is no spontaneous emission, Eq. (28) could be written as

$$\frac{I}{2I_{sa}}\left[1-\frac{A}{1+I/2I_{sa}}+\frac{B}{1+aI/2I_{sa}}\right]=0 \ , \qquad (31)$$

where

$$\xi a=1 \ ,$$
$$A=\left[\varepsilon^2R_0/\varepsilon^2+4(\omega_0-\Omega)^2-\frac{\gamma}{\gamma_k}\lambda_{ua}\right]\frac{4|g|^2}{k\Gamma\gamma} \ , \qquad (32)$$
$$B=(R_{ub}-\gamma/\gamma_k\lambda_{ub})\frac{4|g|^2}{k\Gamma\gamma}a \ ,$$

and A and B are nondimensional quantities.

Obviously, the variable A is a function of the width ε, and $\partial A/\partial\varepsilon>0$ $(0<\varepsilon<\infty)$; thus A increases monotonically with ε. In further discussion we can replace ε by A to describe the variance of the bistability with ε because A has a one-to-one correspondence with ε.

A physically means the population inversion of the atoms in the amplifying cell and B is the same in the absorbing cell. On account of the arrangement of the symbols in Eq. (1), A and B are larger than zero, i.e., $A>0$, $B>0$.

The solutions of Eq. (31) are

$$\frac{I_0}{2I_{sa}}=0 \ ,$$
$$\frac{I_\pm}{2I_{sa}}=\frac{1}{2a}\{a(A-1)-(B+1)$$
$$\pm[a(A-1)-(B+1)^2$$
$$-4a(B+1-A)]^{1/2}\} \ , \qquad (33)$$

where $I_0/2I_{sa}$ and $I\pm/2I_{sa}$ are nondimensional quantities. We know that the admissible solutions in physics must be non-negative and real quantities. According to this statement the physical solutions of Eq. (33) are shown in two regions.

(i) For $A<a/(a-1)$,

$$\frac{I}{2I_{sa}}=\begin{cases} I_0/2I_{sa} & \text{for } B>A-1 \\ I_0/2I_{sa},\ I_+/2I_{sa} & \text{for } B<A-1 \ . \end{cases} \qquad (34)$$

In this case the function $I/2I_{sa}=(1/2I_{sa})(B)$ is concave towards the B axis (see Fig. 3).

(ii) For $A>a/(a-1)$,

$$\frac{I}{2I_{sa}}=\begin{cases} I_0/2I_{sa} & \text{for } B>A-1 \\ I_0/2I_{sa},\ I_+/2I_{sa},\ I_-/2I_{sa} & \text{for } A-1<B<X_- \\ I_0/2I_{sa},\ I_+/2I_{sa} & \text{for } B<A-1\ . \end{cases} \tag{35}$$

In this case the admissible solution condition—nonnegative and a real number—is fulfilled in the region of $\underline{A-1<B<X}$ by all solutions, where $X_-=a(A+1)\pm 2\sqrt{aA(a-1)}$. Thus a new feature will appear. Figure 4 shows the behavior of I in the $(I/2I_{sa}, B)$ plane. In the domain of $A-1<B<X_-$ there are three values for a given B. Figure 5 describes a hysteresis cycle because analysis[10] shows that $I_-/2I_{sa}$ in the case of $A>a/(a-1)$ and $A-1<B<X_-$ is unstable. So the domain of $A-1<B<X_-$ shows a bistability $I_+/2I_{sa}$ and $I_0/2I_{sa}$. We define the bistability loop as the area surrounded by the hysteresis cycle, whose representation is

$$S=\frac{1}{2a}\left[(a-1)^2A^2-3a(A-1)A-2a^2 +4a\sqrt{aA(a-1)}+\frac{a(a-1)A}{4}\ln\frac{a-1}{a}A\right]. \tag{36}$$

This shows that S increases monotonically with A. Taking into account $\partial A/\partial\varepsilon>0$, we could find that the bistable loop area increases with the increase of the Doppler broadening after some analysis. This feature has been shown in Fig. 4.

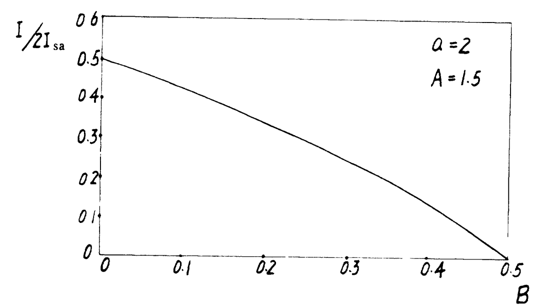

FIG. 3. Behavior of the output intensity at $A<a/(a-1)$.

B. Presence of spontaneous emission

In this case we can rewrite Eq. (28) as follows:

$$C=\left[C+\frac{I}{2I_{sa}}\right]\left[1-\frac{A}{1+I/2I_{sa}}+\frac{B}{1+Ia/2I_{sa}}\right], \tag{37}$$

where A and B are the same as in Eq. (32) and $C=4|g|^2/2\gamma I_{sa}$ is the nondimensional intensity of spontaneous emission, i.e., $C>0$.

Using the iteration method explained in Sec. III, Eq. (37) can be solved as follows:

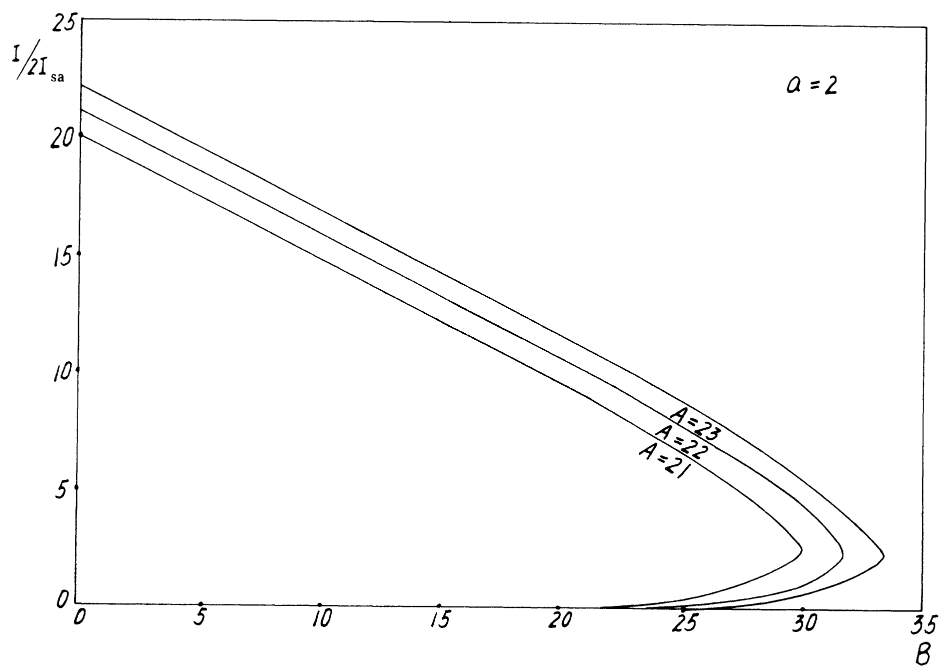

FIG. 4. Behavior of the output intensity when $A>a/(a-1)$. In the domain $A-1<B<X_-$, there are three solutions, $I_0/2I_{sa}$, $I_+/2I_{sa}$, and $I_-/2I_{sa}$.

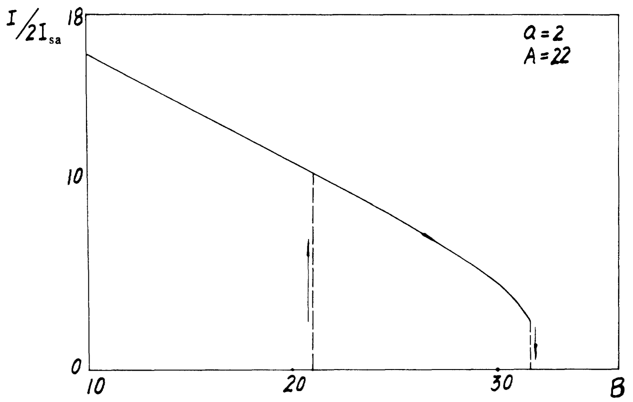

FIG. 5. Hysteresis cycle.

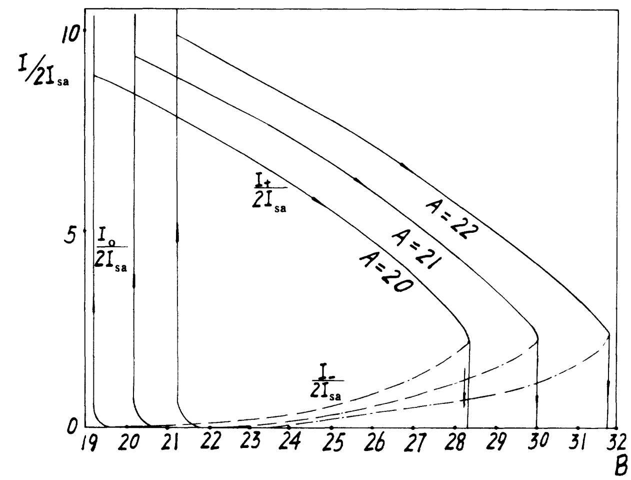

FIG. 6. Hysteresis cycle in the presence of spontaneous emission. $C=0.01$, $A=20,21,22$.

$$\frac{I_0}{2I_{sa}}=\frac{(A-B)C}{B-A+BC-ACa-1}\,,$$
$$\frac{I_\pm}{2I_{sa}}=\frac{1}{2a}(a(A-1)-B-1 \pm\{[a(A-1)-B-1]^2 -4a(B-A+BC-ACa+1)\}^{1/2})\,. \tag{38}$$

In order to make the problem clear, we only pay attention to the bistability. The turning points of the bistable domain are just the two ends of $I_0/2I_{sa}$. By some analysis[10] the two ends of $I/2I_{sa}$ are

$$\frac{A(1-Ca)-1}{1+C}<B<X_-\,, \tag{39}$$

where

$$X_-=a(A+1)-1+2aC -2\sqrt{aA(a-1)-aC(1-a-aC)}\,.$$

For the solution of Eq. (38), it is necessary to make the restriction as follows:

$$\frac{I_0}{2I_{sa}}=\frac{(A-B)C}{B-A+BC-ACa+1}\quad \text{for } B\le A\,,$$
$$\frac{I_0}{2I_{sa}}=0 \quad \text{for } B>A\,,$$

to keep the physical significance of the solution.

If we properly select $C<1/A(a-1)$, we could find $A>[A(1+Ca)-1/(1+C)]$. Because C is the intensity of the spontaneous emission, it is much smaller than the intensity of the output light, and this selection for C is suitable.

Computer testing tells us that if $C=0.01$, $a=2$, and $A=20,21,22$, Eqs. (38) and (39) are fulfilled. Figure 6 shows the bistable cycles in the presence of spontaneous emission. It shows that $I_0/2I_{sa}$ has a nonzero value in the bistable domain, but when $B\to A$, the curve of $I_0/2I_{sa}\to\infty$. This is not a good behavior. On the other hand, if the $(\Delta I/2I_{sa})^2$ term is kept in the iteration, we can overcome this indefiniteness, but the solution is so complicated that we must analyze the existence of $\Delta I/2I_{sa}$.

V. CONCLUSION

We have introduced a simple mode to describe a ring cavity with an absorbing cell, and we have studied it by quantum-mechanical Langevin equations. The mathematical analyses in this paper are simpler than any others used previously.

In Sec. IV the bistable-loop area formula where spontaneous emission exists is not derived because the derivation method is so tedious and hardly explains any new feature. The increase of the bistable-loop area with the increase of Doppler broadening can be seen in Fig. 6 clearly. The result of our analysis is that the bistable-loop area increases with increasing Doppler broadening.

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