Phase-space method without large-N scaling for the laser and optical bistability

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In the following we present a new set of phase-space equations, based on Haken's laser model, for dealing with atom-cavity situations such as the laser or optical bistability. The most important feature of these equations is that they do not require a scaling in terms of the number of atoms to obtain results, and hence can be used in few-atom situations outside the applicability of other equations. There is a mathematical emphasis in the paper as we develop our equations, but we also use the equations to determine the leading noise terms for the one-atom laser as well as outlining their possible usage in other problems.

I. INTRODUCTION

Most of the theoretical works on lasers and optical bistability are based on Haken's derivation of the laser equations.[1] These have proved very successful for many-atom problems where the derivatives of order greater than 2 can be scaled away in terms of the large parameter N (the number of atoms). However, this scaling assumption means that problems involving a small number of atoms cannot be treated using these equations. In this paper we present a different phase space expansion which leads to a Fokker-Planck equation (FPE) for the laser which no longer contains derivatives of arbitrary order. For these equations in large-N situations we can employ a scaling assumption and arrive at Haken's result for the dominant noise term in the laser operating at threshold. However, using our method we can show that this assumption is not necessary as we have a finite-order generalized FPE, and thus we can always write exact corresponding stochastic differential equations (SDE's) for any N. It is only necessary then to use the assumption of adiabatic elimination of the atoms to arrive at Haken's results.

In Sec. II we develop our laser equations using a characteristic function equivalent to Haken's but, using one more parameter, obtain a new partial differential equation for the characteristic function. In Sec. III we define a transformation of the characteristic function into a quasiprobability function which has a Fokker-Planck equation (FPE) containing only first and second derivatives. This function is not the standard Glauber-Sudarshan-Haken P function of laser theory which can be shown to have a nonpositive diffusion matrix in this situation, but the positive P function developed by Drummond and Gardiner which has been shown to always give rise to positive-definite diffusion. The transformation used then is no longer a Fourier transform but there is now a large body of knowledge on the treatment of such situations.[2] Indeed under an appropriate change of variables the Fokker-Planck equation obtained by this procedure can be shown to be equivalent to an equation obtained by Gordon[3] in the case where N is not a constant.

In Sec. IV we eliminate the extra variable in the characteristic function by fixing the number of atoms as N. We now apply our transformation to obtain the generalized FPE which is the central result of this paper. There are now second- and third-order derivatives but these can be easily interpreted as stochastic differential equations with second- and third-order noise.[4] At this stage (Sec. V) it is possible to employ Haken's large-N scaling argument and obtain agreement for the dominant noise terms obtained by both methods.

In Sec. VI we investigate other scaling arguments which do not rely on N being large. It proves possible to find certain small parameters present in the equations and scale the noise in terms of these. Hence results can be obtained for the one-atom laser and other few-atom problems. These results suggest regimes for which the one-atom laser behaves in very much the same way as the many-atom laser.

Finally we present a short conclusion in which we outline other problems which may be solved, or more easily solved, using the techniques and equations developed in this paper. These areas include optical bistability for both small and large-N situations, Jaynes-Cummings problems involving passive atoms in a cavity,[5] and problems involving squeezed inputs.

II. THE LASER MODEL AND CHARACTERISTIC FUNCTION EQUATION

We begin with the usual model for the laser consisting of an ensemble of N two-level atoms inside an optical cavity. We consider there to be a single cavity mode of frequency ω described by the operators a and $a^\dagger$. The atoms are all assumed to have the same resonant frequency ω and are represented by the pseudospin operators σ_μ^-, σ_μ^+, and $\sigma_{\mu,z}$ which obey the commutation relations

$$[\sigma_i^+, \sigma_j^-]=2\sigma_{z,i}\delta_{ij} \text{ and } [\sigma_{z,i}, \sigma_j^\pm]=\pm\sigma_i^\pm\delta_{ij} .$$

Hence in the electric-dipole and rotating-wave approximations the Hamiltonian for the laser system may be modeled as[1]

$$H=\sum_{j=1}^{4} H_j ,$$

$$H_1=\hbar\omega a^\dagger a+\hbar\omega\sum_{\mu=1}^{N}\sigma_{\mu,z} ,$$

$$H_2=ig\hbar\sum_{\mu=1}^{N}(a^\dagger\sigma_\mu^- - a\sigma_\mu^+) , \tag{2.1}$$

$$H_3=\sum_{\mu=1}^{N}(\Gamma_a\sigma_\mu^+ + \Gamma_a^\dagger\sigma_\mu^-) ,$$

$$H_4=\Gamma_f a^\dagger + \Gamma_f^\dagger a ,$$

where the constant g represents the dipole coupling strength between the field and atoms. The field reservoir Γ_f describes the loss of energy by dissipation through the cavity mirrors, which we assume to have a decay rate of value κ. The atomic reservoir describes the energy losses of the atoms, and is related to the rate at which the atoms are pumped (pumping rate of ω_{12}), and losses due to spontaneous emission (with decay rate ω_{21}).

Using standard techniques[1,6] one obtains the master equation for the density operator ρ of the atom-field system in the Markov approximation

$$\frac{\partial\rho}{\partial t}=\frac{1}{i\hbar}[H_2,\rho]+\left.\frac{\partial\rho}{\partial t}\right|_A+\left.\frac{\partial\rho}{\partial t}\right|_F , \tag{2.2}$$

$$\left.\frac{\partial\rho}{\partial t}\right|_A=\sum_\mu[\tfrac{1}{2}\omega_{21}(2\sigma_\mu^-\rho\sigma_\mu^+-\sigma_\mu^+\sigma_\mu^-\rho-\rho\sigma_\mu^+\sigma_\mu^-) + \tfrac{1}{2}\omega_{12}(2\sigma_\mu^+\rho\sigma_\mu^- - \sigma_\mu^-\sigma_\mu^+\rho-\rho\sigma_\mu^-\sigma_\mu^+)], \tag{2.3}$$

$$\begin{aligned}\left.\frac{\partial\rho}{\partial t}\right|_F &= \kappa n(2a^\dagger\rho a - aa^\dagger\rho-\rho aa^\dagger)\\ &+\kappa(n+1)(2a\rho a^\dagger - a^\dagger a\rho-\rho a^\dagger a)\\ &-\kappa m(2a^\dagger\rho a^\dagger - a^\dagger a^\dagger\rho-\rho a^\dagger a^\dagger)\\ &-\kappa m^*(2a\rho a - aa\rho-\rho aa) .\end{aligned} \tag{2.4}$$

The situations where m and m^* are nonzero allow for squeezing of the cavity-field reservoir. Now introducing the characteristic function

$$\chi=\mathrm{Tr}[O\rho] , \tag{2.5}$$

where $O=O^A O^F$, then the characteristic function usually used is defined by

$$O^A=\prod_{\mu=1}^{N} e^{i\xi^*\sigma_\mu^+}e^{i\zeta\sigma_{\mu,z}}e^{i\xi\sigma_\mu^-} , \tag{2.6}$$

$$O^F=e^{i\beta^* a^\dagger}e^{i\beta a} . \tag{2.7}$$

Although this characteristic function has an ordering which readily allows the finding of averages when the atomic operators are normally ordered [readers are referred to the recent paper by Carmichael, Satchell, and Sarkar[2] for a full explanation of this, and the general results obtained by the use of (2.6)] it is still quite singular. In fact, O^A can be written explicitly as

$$\begin{aligned}O^A=&\prod_{\mu=1}^{N}(1+i\xi^*\sigma_\mu^+)\\ &\times[\cos(\zeta\sigma_{\mu,z})+i\sin(\zeta\sigma_{\mu,z})](1+i\xi\sigma_\mu^-)\end{aligned} \tag{2.8}$$

whose Fourier transform [to be defined in (3.2)] will contain terms of the form

$$\delta^m(\nu)\delta(D-m')\delta^n(\nu^*) \tag{2.9}$$

for all m,n such that $0\le m,n\le N$ and for all m' such that $-N\le m'\le N$.

The P function conventionally obtained by Haken's truncation procedure amounts to a smoothed version of this quite singular distribution. This is an approximation in the sense that physical averages evaluated with a smooth distribution can be quite close to the exact result obtained with the exact, nonsmooth distribution, although the smoothed distribution is not a good pointwise approximation to the exact distribution.

It is clear, however, that O^A is, in fact, simply a linear combination of the Pauli spin matrices and the identity matrix, so a simpler way of looking at the characteristic function is to change to a new set of variables and define

$$\begin{aligned}O^A&=\prod_{\mu=1}^{N}(b+c^+\sigma_\mu^+ + c^-\sigma_\mu^- + c\sigma_{\mu,z})\\ &=\prod_{\mu=1}^{N}(b+\mathbf{c}\cdot\boldsymbol{\sigma}_\mu) .\end{aligned} \tag{2.10}$$

Notice that we have a characteristic function defined in six variables rather than five as previously. The characteristic function obtained by the use of (2.10) is not bounded—indeed it is a polynomial of order N, which could be very large. Only in the sense of a distribution will it have a Fourier transform, but we have already seen that even the exact P function for the better-behaved characteristic function (2.6) is a singular distribution so this need not be regarded as a disadvantage. What is important is the simplicity of the resulting equations of motion—they are now much simpler than those obtained by the use of (2.6).

We can of course use our new definition of the characteristic function with (2.10) to calculate operator averages. In general the results for normally ordered operators are not as straightforward as those obtained via the use of (2.6), however in the case of simple averages we still have

$$\left\langle\sum_\mu\sigma_\mu\right\rangle=\left.\frac{\partial\chi}{\partial\mathbf{c}}\right|_{b=1,\ \mathbf{c}=0} \tag{2.11}$$

and

$$N=\left.\frac{\partial\chi}{\partial b}\right|_{b=1,\ \mathbf{c}=0} . \tag{2.12}$$

More extensive results on how to calculate correlation

functions and moments of the atomic operators are given in Appendix B. In Appendix C we similarly show how general results for multitime averages may be obtained using (2.10). These demonstrate clearly the correspondence which exists between the quantum-mechanical operator averages, and the averages obtained for the classical variables of the Fokker-Planck equation.

Using much the same methods as usual, we can derive an equation of motion for the characteristic function (2.5) with the operators O^A and O^F as defined in (2.7) and (2.10). Then as is shown in Appendix A we obtain the equation

$$\frac{\partial\chi}{\partial t}=\left[\kappa\left[-i\beta\frac{\partial}{\partial i\beta}-i\beta^*\frac{\partial}{\partial i\beta^*}+2ni\beta^*i\beta+(i\beta)^2m+(i\beta^*)^2m^*\right]-\frac{(\omega_{12}+\omega_{21})}{2}\left[c^+\frac{\partial}{\partial c^+}+c^-\frac{\partial}{\partial c^-}+2c\frac{\partial}{\partial c}\right]\right.$$
$$+\frac{(\omega_{12}-\omega_{21})}{2}c\frac{\partial}{\partial b}+g\left\{-c\frac{\partial^2}{\partial c^+\partial i\beta}+2c^-\frac{\partial^2}{\partial c\,\partial i\beta}+i\beta\left[(b-\tfrac{1}{2}c)\frac{\partial}{\partial c^-}+c^+\left[\frac{1}{2}\frac{\partial}{\partial b}+\frac{\partial}{\partial c}\right]\right]\right.$$
$$\left.\left.-c\frac{\partial^2}{\partial c^-\partial i\beta^*}+2c^+\frac{\partial^2}{\partial c\,\partial i\beta^*}+i\beta^*\left[(b-\frac{1}{2}c)\frac{\partial}{\partial c^+}+c^-\left[\frac{1}{2}\frac{\partial}{\partial b}+\frac{\partial}{\partial c}\right]\right]\right\}\right]\chi\ . \tag{2.13}$$

III. THE TRANSFORMATION TO THE FOKKER-PLANCK EQUATION

For simplicity we make the change of variables

$$b\rightarrow 1+b'\ ,\qquad \frac{\partial}{\partial b}\rightarrow\frac{\partial}{\partial b'}\ , \tag{3.1}$$

which means that the expectation values (2.9) and (2.10) are evaluated at $b'=0$.

The standard approach following Haken's method is to define a quasiprobability function $P(\mathbf{x})=P(\alpha^*,\alpha,\nu^*,D,\nu)$ via

$$\chi(\beta,\beta^*,\xi^*,\zeta,\xi)=\int d\alpha^*\int d\alpha\int d\nu^*\int dD\int d\nu\, e^{i\beta^*\alpha^*}e^{i\beta\alpha}e^{i\xi^*\nu^*}e^{1/2i\zeta D}e^{i\xi\nu}P(\mathbf{x})\ . \tag{3.2}$$

In this case we have two independent complex variables and one real variable in both χ and P, so that (3.2) defines $P(\mathbf{x})$ to be the five-dimensional Fourier transform of the characteristic function. We may then define $P(\mathbf{x})$ via the inverse Fourier transform

$$P(\mathbf{x})=\int d\beta^*\int d\beta\int d\xi^*\int d\zeta\int d\xi\, e^{-i\beta^*\alpha^*}e^{-i\beta\alpha}e^{-i\xi^*\nu^*}e^{-1/2i\zeta D}e^{-i\xi\nu}\chi(\beta^*,\beta,\xi^*,\zeta,\xi)\ . \tag{3.3}$$

However, a major problem arises with the Glauber-Sudarshan-Haken P representation defined in this way for the laser theory, in that the diffusion matrix of the Fokker-Planck equation obtained is not positive definite in the five-dimensional physical space. To overcome this problem we follow the work of Drummond and Gardiner[7] and consider all the variables in both the characteristic function and the P function to be independent complex variables. Thus we no longer have $\alpha^*=(\alpha)^*$ but must treat α and α^* as independent complex variables. To make this clear in the following we shall adopt the notation $\alpha^*=\alpha^+$ to explicitly display the lack of conjugacy. So making all the notational changes to the variables, we define the characteristic function χ in a ten dimensional phase space by

$$\chi=\int d^2\alpha^+\int d^2\alpha\int d^2\nu^+\int d^2D\int d^2\nu\exp[i(\beta^+,\beta,\xi^+,\tfrac{1}{2}\zeta,\xi)\cdot\mathbf{x}]P(\mathbf{x})\ , \tag{3.4}$$

where $\mathbf{x}=(\alpha^+,\alpha,\nu^+,D,\nu)$.

It is now no longer possible to uniquely define the quasiprobability function $P(\mathbf{x})$ by the inverse Fourier transform, as was done between (3.2) and (3.3). However it is this nonuniqueness which makes the guarantee of positive-definite diffusion possible. The flexibility that arises in choosing a positive quasiprobability function with which to expand χ, allows us to find one such function which satisfies a Fokker-Planck equation. We therefore make the particular choice for the P function of

$$P(\mathbf{x})=\int d^2\beta^+\int d^2\beta\int d^2\xi^+\int d^2\zeta\int d^2\xi\exp[-i(\beta^+,\beta,\xi^+,\tfrac{1}{2}\zeta,\xi)\cdot\mathbf{x}]\chi(\beta^+,\beta,\xi^+,\zeta,\xi)\ . \tag{3.5}$$

With this definition of the positive P function it was shown by Drummond and Gardiner that the Fokker-Planck equation obtained in the double-dimensioned space always has positive-definite diffusion, and yet leads to the same stochastic differential equations as if the normal P function had been naively employed.

So for the characteristic function given by (2.7) and (2.10) we then use the positive P representation and make the particular choice for the quasiprobability function $P(\mathbf{x})$ in twelve-dimensional space as

$$P(\mathbf{x})=\int d^2\beta^+\int d^2\beta\int d^2c^+\int d^2c\int d^2c^-\int d^2b'\exp[-i(\beta^+,\beta,c^+,\tfrac{1}{2}c,c^-,b')\cdot\mathbf{x}]\chi(\beta^+,\beta,c^+,c,c^-,b')\ , \tag{3.6}$$

where $\mathbf{x}=(\alpha^+,\alpha,\nu^+,D,\nu,B)$.

Now applying this transformation to the differential equation for χ we arrive at a generalized Fokker-Planck equation, which is equivalent to making the following transformation in variables:

$$\begin{aligned}
&i\beta\rightarrow-\frac{\partial}{\partial\alpha},\quad\frac{\partial}{\partial i\beta}\rightarrow\alpha\ ,\\
&i\beta^+\rightarrow-\frac{\partial}{\partial\alpha^+},\quad\frac{\partial}{\partial i\beta^+}\rightarrow\alpha^+\ ,\\
&c^-\rightarrow-\frac{\partial}{\partial\nu},\quad\frac{\partial}{\partial c^-}\rightarrow\nu\ ,\\
&c^+\rightarrow-\frac{\partial}{\partial\nu^+},\quad\frac{\partial}{\partial c^+}\rightarrow\nu^+\ ,\\
&c\rightarrow-2\frac{\partial}{\partial D},\quad\frac{\partial}{\partial c}\rightarrow\tfrac{1}{2}D\ ,\\
&b'\rightarrow-\frac{\partial}{\partial B},\quad\frac{\partial}{\partial b'}\rightarrow B\ .
\end{aligned}\tag{3.7}$$

We thus obtain the Fokker-Planck equation

$$\begin{aligned}
\frac{\partial}{\partial t}P(\mathbf{x})=&\left[\kappa\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^+}\alpha^+ +\frac{\partial^2}{\partial\alpha\,\partial\alpha^+}2n+\frac{\partial^2}{\partial\alpha^2}m+\frac{\partial^2}{\partial\alpha^{+2}}m^*\right]+\frac{(\omega_{12}+\omega_{21})}{2}\left[\frac{\partial}{\partial\nu}\nu+\frac{\partial}{\partial\nu^+}\nu^+ +2\frac{\partial}{\partial D}D\right]\right.\\
&-(\omega_{12}-\omega_{21})\frac{\partial}{\partial D}B+g\left[2\frac{\partial}{\partial D}\alpha\nu^+-\frac{\partial}{\partial\nu}\alpha D-\frac{\partial}{\partial\alpha}\nu+\frac{\partial^2}{\partial B\,\partial\alpha}\nu-\frac{\partial^2}{\partial D\,\partial\alpha}\nu+\tfrac{1}{2}\frac{\partial^2}{\partial\nu^+\partial\alpha}(B+D)+2\frac{\partial}{\partial D}\alpha^+\nu\right.\\
&\left.\left.-\frac{\partial}{\partial\nu^+}\alpha^+D-\frac{\partial}{\partial\alpha^+}\nu^+ +\frac{\partial^2}{\partial B\,\partial\alpha^+}\nu^+-\frac{\partial^2}{\partial D\,\partial\alpha^+}\nu^+ +\tfrac{1}{2}\frac{\partial^2}{\partial\nu\,\partial\alpha^+}(B+D)\right]\right]P(\mathbf{x})\ .
\end{aligned}\tag{3.8}$$

The important thing to notice is that this Fokker-Planck equation contains only second-order derivatives, and hence does not require a truncation of the higher-order derivatives through a large-N approximation. We now rewrite the FPE by explicitly using the independent complex nature of the variables and so divide into real and imaginary parts. Thus (3.8) can be written in the general form

$$\frac{\partial}{\partial t}P(\mathbf{x})=\left[\frac{\partial}{\partial x_\mu}A_\mu(\mathbf{x})+\frac{1}{2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}D_{\mu\nu}(\mathbf{x})\right]P(\mathbf{x})\ ,$$

$$A_\mu(\mathbf{x})=A^x_\mu(\mathbf{x})+iA^y_\mu(\mathbf{x})\ ,\tag{3.9}$$

$$D_{\mu\nu}(\mathbf{x})=(dd^T)_{\mu\nu}=(d^x+id^y)_{\mu\sigma}(d^x+id^y)_{\nu\sigma}\ .$$

Now due to the analyticity of the characteristic function it can be shown that for each of the x_μ there is the equivalence relation[7]

$$\frac{\partial}{\partial x_\mu}\leftrightarrow\frac{\partial}{\partial x^x_\mu}\leftrightarrow-i\frac{\partial}{\partial x^y_\mu}\ ,\tag{3.10}$$

where $x_\mu=x^x_\mu+ix^y_\mu$. Hence (3.9) is equivalent to the following equation:

$$\begin{aligned}
\frac{\partial}{\partial t}P(\mathbf{x})=&\left[\frac{\partial}{\partial x^x_\mu}A^x_\mu+\frac{\partial}{\partial x^y_\mu}A^y_\mu+\frac{\partial}{\partial x^x_\mu}\frac{\partial}{\partial x^y_\nu}d^x_{\mu\sigma}\,d^y_{\nu\sigma}\right.\\
&+\tfrac{1}{2}\left[\frac{\partial}{\partial x^x_\mu}\frac{\partial}{\partial x^x_\nu}d^x_{\mu\sigma}\,d^x_{\nu\sigma}\right.\\
&\left.\left.+\frac{\partial}{\partial x^y_\mu}\frac{\partial}{\partial x^y_\nu}d^y_{\mu\sigma}\,d^y_{\nu\sigma}\right]\right]P(\mathbf{x})\ ,
\end{aligned}\tag{3.11}$$

which can be seen explicitly to always have positive-definite diffusion. The new FPE (3.11) is equivalent to the following set of (Ito) stochastic differential equations

$$\frac{\partial}{\partial t}x_\mu=-A_\mu(\mathbf{x})+d_{\mu\nu}(\mathbf{x})\xi_\nu(t)\ ,\tag{3.12}$$

where $\langle\xi_\mu(t)\xi_\nu(t')\rangle=\delta_{\mu\nu}\delta(t-t')$.

Thus corresponding to (3.8) we have the following set of SDE's:

$$\begin{aligned}
&\dot\alpha=-\kappa\alpha+g\nu+\Gamma_\alpha\ ,\\
&\dot\alpha^+=-\kappa\alpha^+ +g\nu^+ +\Gamma_{\alpha^+}\ ,\\
&\dot\nu=-\gamma\nu+g\alpha D+\Gamma_\nu\ ,\\
&\dot\nu^+=-\gamma\nu^+ +g\alpha^+D+\Gamma_{\nu^+}\ ,\\
&\dot D=-2\gamma D+(\omega_{12}-\omega_{21})B-2g(\alpha^+\nu+\alpha\nu^+)+\Gamma_D\ ,\\
&\dot B=\Gamma_B\ ,
\end{aligned}\tag{3.13}$$

where $\gamma=\frac{1}{2}(\omega_{21}+\omega_{12})$ and $\langle\Gamma_i(t)\Gamma_j(t')\rangle=d_{ij}\delta(t-t')$:

$$\underline{d}=\begin{bmatrix} 2\kappa m & 2\kappa n & 0 & \frac{1}{2}g(B+D) & -g\nu & g\nu \\ 2\kappa n & 2\kappa m^* & \frac{1}{2}g(B+D) & 0 & -g\nu^+ & g\nu^+ \\ 0 & \frac{1}{2}g(B+D) & 0 & 0 & 0 & 0 \\ \frac{1}{2}g(B+D) & 0 & 0 & 0 & 0 & 0 \\ -g\nu & -g\nu^+ & 0 & 0 & 0 & 0 \\ g\nu & g\nu^+ & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{3.14}$$

It is important to notice that since none of the second-order coefficients depends on its respective derivatives, it can be moved inside or outside the derivatives and hence has both an Ito and Stratonovich noise interpretation. Thus the Eqs. (3.13) can be interpreted as Ito or Stratonovich SDE's to suit different circumstances.

The drift terms are identical to the standard laser equation drift terms (i.e., the single-mode Maxwell-Bloch equations) if we interpret B as N, the number of atoms. However, the equation for B has a nonzero diffusion term indicating that B is not constant, and hence does not exactly represent the number of atoms (although all moments of B are equal to those of N). This formulation then could equally accommodate the situation where N is not precisely known, as was treated by Gordon[3] in his model of a two-level laser (with varying populations) interacting with a ground level. Indeed, making the change of variables

$$\begin{aligned} D &= N_2 - N_1 , \\ B &= N_2 + N_1 , \end{aligned} \tag{3.15}$$

the FPE which Gordon obtained can be seen to agree exactly with (3.8) when all the transitions between the upper two levels and the ground level in his model are set to zero. This equivalence is not too surprising when it is considered that the quantity Gordon defines as $\sigma_a(\chi_j)$ works in essentially the same way as our new characteristic function (2.10) in order that they both lead to Fokker-Planck equations that contain only second-order derivatives. The major difference is that Gordon works from a direct expansion of the density operator in terms of $\sigma_a(\chi_j)$ (requiring a slightly different model of an extra ground level with an effectively infinite number of atoms in addition to the atomic reservoir) whereas our method used a characteristic function to introduce our positive P function.

To solve the system (3.13)–(3.14) it is best to write $\Gamma_\alpha, \Gamma_{\alpha^+}$, etc. in terms of linear combinations of independent white noises, with the coefficients determined so that the correlation properties (3.14) are maintained. This requires 14 noises and can be written as (assuming $m, m^* = 0$),

$$\begin{aligned} &\Gamma_\alpha = \sqrt{\kappa n}\,(\xi_1+i\xi_2)+\tfrac{1}{4}\sqrt{g}\,(B+D)(\xi_3+i\xi_4)-\tfrac{1}{2}\sqrt{g}\,\nu[(\xi_7+i\xi_8)-(\xi_{11}+i\xi_{12})] , \\ &\Gamma_{\alpha^+} = \sqrt{\kappa n}\,(\xi_1-i\xi_2)+\tfrac{1}{4}\sqrt{g}\,(B+D)(\xi_5+i\xi_6)-\tfrac{1}{2}\sqrt{g}\,\nu^+[(\xi_9+i\xi_{10})-(\xi_{13}+i\xi_{14})] , \\ &\Gamma_\nu = \sqrt{g}\,(\xi_5-i\xi_6) , \Gamma_{\nu^+} = \sqrt{g}\,(\xi_3-i\xi_4) , \\ &\Gamma_D = \sqrt{g}\,[(\xi_7-i\xi_8)+(\xi_9-i\xi_{10})] , \Gamma_B = \sqrt{g}\,[(\xi_{11}-i\xi_{12})+(\xi_{13}-i\xi_{14})] , \end{aligned} \tag{3.16}$$

where the $\xi_i(t)$ are independent real white Langevin sources, with the correlation properties

$$\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t') . \tag{3.17}$$

It is, of course, possible to write the equations in terms of twelve noises corresponding to the twelve equations but then the noises would not be independent. We can see immediately that the equation for B is that of a complex Wiener process[4]

$$\dot{B} = \Gamma_B = \sqrt{g}\,[(\xi_{11}+\xi_{13})-i(\xi_{12}+\xi_{14})] \tag{3.18}$$

which can be integrated with the initial condition $B(t=0)=N$ to give

$$B = N + \int_0^t \Gamma_B(t')dt' . \tag{3.19}$$

We then have a system of differential equations involving five complex variables

$$\begin{aligned} \dot{\alpha} &= -\kappa\alpha + g\nu + \Gamma_\alpha , \\ \dot{\alpha}^+ &= -\kappa\alpha^+ + g\nu^+ + \Gamma_{\alpha^+} , \\ \dot{\nu} &= -\gamma\nu + g\alpha D + \Gamma_\nu , \\ \dot{\nu}^+ &= -\gamma\nu^+ + g\alpha^+ D + \Gamma_{\nu^+} , \\ \dot{D} &= -2\gamma(D-D_0) - 2g(\alpha^+\nu + \alpha\nu^+) \\ &\quad + 2\gamma\frac{D_0}{N}\int_0^t \Gamma_B(t')dt' + \Gamma_D , \end{aligned} \tag{3.20}$$

where we have used the usual laser definitions[8]

$$\gamma=\tfrac{1}{2}(\omega_{21}+\omega_{12}) \quad D_0=\frac{(\omega_{12}-\omega_{21})}{(\omega_{12}+\omega_{21})}N \ . \tag{3.21}$$

These equations are relatively simple although the equation for $\dot{D}$ contains a nonwhite noise term. This creates problems in solving the system as can be seen in Appendix D where techniques to be developed in Secs. V and VI are used. A more conventional and familiar method of solution then, is to eliminate the extra variable as is done in Sec. IV,

IV. ELIMINATION OF THE EXTRA VARIABLE

To eliminate the extra variable B we introduce the new characteristic function $f(i\beta,i\beta^*,x,y,z)$ via the substitution

$$\chi=b^N f(i\beta,i\beta^*,c^-/b,c^+/b,c/b) \ , \tag{4.1}$$

where N is fixed as the number of atoms. Then using relations such as

$$\frac{\partial}{\partial b}\chi=b^{N-1}\left[N-x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}-z\frac{\partial}{\partial z}\right]f \tag{4.2}$$

we can convert Eq. (2.13) into a partial differential equation for the characteristic function f. Now employing the positive P representation as in Sec. III, we use a transformation similar to (3.7) to obtain a FPE in five independent complex variables $P(\mathbf{x})$ $\mathbf{x}=(\alpha,\alpha^+,\nu,\nu^+,D)$,

$$\begin{aligned}\frac{\partial}{\partial t}P(\mathbf{x})=&\left[\kappa\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^+}\alpha^++\frac{\partial^2}{\partial\alpha\partial\alpha^+}2n+\frac{\partial^2}{\partial\alpha^2}m+\frac{\partial^2}{\partial\alpha^{+2}}m^*\right]\right.\\
&+\frac{(\omega_{12}+\omega_{21})}{2}\left[\frac{\partial}{\partial\nu}\nu+\frac{\partial}{\partial\nu^+}\nu^++2\frac{\partial}{\partial D}D\right]-(\omega_{12}-\omega_{21})\left[\frac{\partial}{\partial D}N+\frac{\partial^2}{\partial\nu\partial D}\nu+\frac{\partial^2}{\partial\nu^+\partial D}\nu^++\frac{\partial^2}{\partial D^2}D\right]\\
&+g\left[2\frac{\partial}{\partial D}(\nu^+\alpha+\nu\alpha^+)-\frac{\partial}{\partial\nu}D\alpha-\frac{\partial}{\partial\nu^+}D\alpha^+-\frac{\partial}{\partial\alpha^+}\nu^+-\frac{\partial^2}{\partial D\partial\alpha^+}\nu^++\frac{1}{2}\frac{\partial^2}{\partial\nu\partial\alpha^+}(D+N)\right.\\
&+\frac{1}{2}\frac{\partial^3}{\partial\nu^2\partial\alpha^+}\nu+\frac{1}{2}\frac{\partial^3}{\partial\nu\partial\nu^+\partial\alpha^+}\nu^++\frac{1}{2}\frac{\partial^3}{\partial\nu\partial D\partial\alpha^+}D-\frac{\partial}{\partial\alpha}\nu-\frac{\partial^2}{\partial D\partial\alpha}\nu\\
&\left.\left.+\frac{1}{2}\frac{\partial^2}{\partial\nu^+\partial\alpha}(D+N)+\frac{1}{2}\frac{\partial^3}{\partial\nu\partial\nu^+\partial\alpha}\nu+\frac{1}{2}\frac{\partial^3}{\partial\nu^{+2}\partial\alpha}\nu^++\frac{1}{2}\frac{\partial^3}{\partial\nu^+\partial D\partial\alpha}D\right]\right]P(\mathbf{x}) \ .\end{aligned} \tag{4.3}$$

This means we can write directly the SDE's corresponding to (4.3) as

$$\begin{aligned}
\dot{\alpha}&=-\kappa\alpha+g\nu+\Gamma_\alpha+X_\alpha \ ,\\
\dot{\alpha}^+&=-\kappa\alpha^++g\nu^++\Gamma_{\alpha^+}+X_{\alpha^+} \ ,\\
\dot{\nu}&=-\gamma\nu+g\alpha D+\Gamma_\nu+X_\nu \ ,\\
\dot{\nu}^+&=-\gamma\nu^++g\alpha^+D+\Gamma_{\nu^+}+X_{\nu^+} \ ,\\
\dot{D}&=-2\gamma(D-D_0)-2g(\nu^+\alpha+\nu\alpha^+)+\Gamma_D+X_D \ ,
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
&\langle\Gamma_\alpha(t)\Gamma_{\alpha^+}(t')\rangle=2\kappa n\delta(t-t') \ ,\\
&\langle\Gamma_\alpha(t)\Gamma_\alpha(t')\rangle=2\kappa m\delta(t-t')=[\langle\Gamma_{\alpha^+}(t)\Gamma_{\alpha^+}(t')\rangle]^* \ ,\\
&\langle\Gamma_\alpha(t)\Gamma_{\nu^+}(t')\rangle=\tfrac{1}{2}g(D+N)\delta(t-t')=\langle\Gamma_{\alpha^+}(t)\Gamma_\nu(t')\rangle \ ,\\
&\langle\Gamma_\alpha(t)\Gamma_D(t')\rangle=-g\nu\delta(t-t') \ ,\\
&\langle\Gamma_{\alpha^+}(t)\Gamma_D(t')\rangle=-g\nu^+\delta(t-t') \ ,\\
&\langle\Gamma_\nu(t)\Gamma_D(t')\rangle=-2\gamma\frac{D_0}{N}\nu\delta(t-t') \ ,\\
&\langle\Gamma_{\nu^+}(t)\Gamma_D(t')\rangle=-2\gamma\frac{D_0}{N}\nu^+\delta(t-t') \ ,\\
&\langle\Gamma_D(t)\Gamma_D(t')\rangle=-4\gamma\frac{D_0}{N}D\delta(t-t') \ ,
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\langle X_\alpha(t)X_\nu(t')X_{\nu^+}(t'')\rangle&=-\tfrac{1}{2}g\nu\delta(t-t')\delta(t'-t'')\\
&=\tfrac{1}{2}\langle X_{\alpha^+}(t)X_\nu(t')X_\nu(t'')\rangle \ ,\\
\langle X_\alpha(t)X_{\nu^+}(t')X_{\nu^+}(t'')\rangle&=-g\nu^+\delta(t-t')\delta(t'-t'')\\
&=2\langle X_{\alpha^+}(t)X_{\nu^+}(t')X_\nu(t'')\rangle \ ,
\end{aligned} \tag{4.6}$$

$$\langle X_\alpha(t)X_{\nu^+}(t')X_D(t'')\rangle=-\tfrac{1}{2}gD\delta(t-t')\delta(t'-t'')$$
$$=\langle X_{\alpha^+}(t)X_\nu(t')X_D(t'')\rangle \ .$$

Equations (4.3)–(4.6) are the major results of this paper, and may be easily adjusted to solve for the laser or optical bistability situations. The drift or deterministic terms are exactly those obtained by Haken and others,[9] but there are now also second- and third-order derivatives which we can interpret as second and third-order noises in the SDE's, and which have correlation properties given by (4.5) and (4.6). The use of third-order noise terms was first treated by Gardiner and Chaturvedi[10] and its possible usage in quantum optics has been recently indicated by Tombesi and Mecozzi.[11]

V. SOLUTION BY ADIABATIC ELIMINATION OF THE ATOMS AND SCALING

To solve the system (4.4)–(4.6) we assume

$$\kappa \ll \gamma \tag{5.1}$$

and hence adiabatically eliminate the "fast" atomic variables.[4] That is, we set

$$\dot{\nu}=0 \Longrightarrow \nu=\frac{g}{\gamma}D\alpha+\frac{1}{\gamma}(\Gamma_\nu+X_\nu)\ , \tag{5.2}$$

$$\dot{\nu}^+=0 \Longrightarrow \nu=\frac{g}{\gamma}D\alpha^+ +\frac{1}{\gamma}(\Gamma_{\nu^+}+X_{\nu^+})\ , \tag{5.3}$$

$$\dot{D}=0 \Longrightarrow D=D_0-\frac{g}{\gamma}(\nu\alpha^+ +\nu^+\alpha)+\frac{\Gamma_D+X_D}{2\gamma}\ , \tag{5.4}$$

and obtain the equations for α and α^+

$$\dot{\alpha}=-\kappa(1-C\Pi)\alpha+F,\quad \dot{\alpha}^+=-\kappa(1-C\Pi)\alpha^+ +F^+\ , \tag{5.5}$$

where $C=g^2D_0/\gamma\kappa$ is the cooperativity parameter,

$$\Pi=\left[1+\frac{\alpha\alpha^+}{n_0}\right]^{-1}\ ,$$

$$n_0=\tfrac{1}{2}(\gamma/g)^2$$

is the saturation photon number, and

$$F=\Gamma_\alpha+X_\alpha+\left[\frac{g}{\gamma}\right]\left[1-\left[\frac{g}{\gamma}\right]^2\alpha\alpha^+\Pi\right](\Gamma_\nu+X_\nu)$$
$$+\frac{1}{2}\left[\frac{g}{\gamma}\right]^2\alpha\Pi(\Gamma_D+X_D)-\left[\frac{g}{\gamma}\right]^3\Pi\alpha^2(\Gamma_{\nu^+}+X_{\nu^+})\ , \tag{5.6}$$

$$F^+=\Gamma_{\alpha^+}+X_{\alpha^+}+\left[\frac{g}{\gamma}\right]\left[1-\left[\frac{g}{\gamma}\right]^2\alpha\alpha^+\Pi\right](\Gamma_{\nu^+}+X_{\nu^+})$$
$$+\frac{1}{2}\left[\frac{g}{\gamma}\right]^2\alpha^+\Pi(\Gamma_D+X_D)-\left[\frac{g}{\gamma}\right]^3\Pi\alpha^{+2}(\Gamma_\nu+X_\nu)\ . \tag{5.7}$$

The deterministic equation is (one equation only since $\alpha^+=\alpha^*$ in the deterministic part)

$$\dot{\alpha}=-\kappa(1-C\Pi)\alpha\ , \tag{5.8}$$

hence the steady-state field has two solutions,

$$\alpha=0,\quad C\le 1\ ;$$
$$\frac{\alpha\alpha^+}{n_0}=C-1,\quad C>1\ . \tag{5.9}$$

Initially we assume that N is large and so follow Haken's analysis around threshold by scaling

$$D\approx D_0\approx N\ ,$$
$$g\approx N^{-1/2}\ ,$$
$$\alpha\approx N^{1/2x}\ , \tag{5.10}$$
$$\nu\approx N^{1/2x}N^{1/2}\ .$$

For $x\approx\frac{1}{2}$ this means that the dominant second-order noise terms in (4.3) are

$$\tfrac{1}{2}g\left[\frac{\partial^2}{\partial\nu\,\partial\alpha^+}+\frac{\partial^2}{\partial\nu^+\,\partial\alpha}\right](D+N) \text{ and } \frac{\partial^2}{\partial\alpha\,\partial\alpha^+}2\kappa n\ ,$$

with the third-order terms always being at least an order of N smaller than the second-order terms. Thus the SDE's are, in this scaling,

$$\dot{\alpha}=-\kappa\alpha+g\nu+\Gamma_\alpha\ ,$$
$$\dot{\nu}=-\gamma\nu+gD\alpha+\Gamma_\nu\ , \tag{5.11}$$
$$\dot{D}=-2\gamma(D-D_0)-2g(\alpha^+\nu+\alpha\nu^+)\ .$$

Here and in what follows we have not explicitly written out the equations for the variables α^+ and ν^+ (and later we do not write out the equations for F^+); they can be simply written by treating them as pseudo complex-conjugate equations to the α and ν variable equations. It is important to remember though that in the positive P representation they are equations of independent complex variables, and so are strictly not complex conjugates. Hence adiabatically eliminating ν, ν^+, and D one obtains

$$\dot{\alpha}=-\kappa(1-C\Pi)\alpha+F\ , \tag{5.12}$$

where

$$F=\Gamma_\alpha+\left[\frac{g}{\gamma}\right]\left[1-\left[\frac{g}{\gamma}\right]^2\alpha\alpha^+\Pi\right]\Gamma_\nu-\left[\frac{g}{\gamma}\right]^3\alpha^2\Pi\Gamma_{\nu^+}\ .$$

So calculating the leading noise term,

$$\langle F(t)F^+(t')\rangle=\langle\Gamma_\alpha(t)\Gamma_{\alpha^+}(t')\rangle+[\langle\Gamma_\alpha(t)\Gamma_{\nu^+}(t')\rangle+\langle\Gamma_{\alpha^+}(t)\Gamma_\nu(t')\rangle]\left[\frac{g}{\gamma}\right]\left[1-\left[\frac{g}{\gamma}\right]^2\alpha\alpha^+\Pi\right]$$
$$=\left\{2\kappa n+\frac{g^2}{\gamma}(D+N)\left[1-\left[\frac{g}{\gamma}\right]^2\alpha\alpha^+\Pi\right]\right\}\delta(t-t')\ . \tag{5.13}$$

We now make a first-order approximation for the atomic variables in this fluctuation term, by using the semiclassical values of the atomic variables in the steady state. This approximation is justified in the large-N approximation, where the fluctuations which are of lower order in N are small, and gives

$$D=D_0\Pi,\nu=\frac{g}{\gamma}\alpha D_0\Pi\ . \tag{5.14}$$

In the standard laser it is usual to think of the atoms as all being in the excited state (being completely pumped), and so in Haken's analysis the inversion is set as $D_0=N$. Equation (5.13) is of course valid for any choice of D_0, but to compare with Haken's result we now use (5.14) and set $D_0=N$ to give

$$\langle F(t)F^+(t')\rangle=\left\{2\kappa n+\frac{g^2N}{\gamma}(1+\Pi)\left[1-\left[\frac{g}{\gamma}\right]^2\alpha\alpha^+\Pi\right]\right\}\delta(t-t') . \tag{5.15}$$

Now Haken's leading noise term is

$$\langle F(t)F^+(t')\rangle=\left[2\kappa n+\frac{g^2}{\gamma}(D_0+N)(1-2\left[\frac{g}{\gamma}\right]^2\alpha\alpha^+\Pi+\left[\frac{g}{\gamma}\right]^4\alpha^2\alpha^{+2}\Pi^2)\right]\delta(t-t') , \tag{5.16}$$

so if we expand Π via [justified as $2\alpha\alpha^+(g/\gamma)^2<1$ around threshold]

$$\Pi=1-2\alpha\alpha^+\left[\frac{g}{\gamma}\right]^2+4\alpha^2\alpha^{+2}\left[\frac{g}{\gamma}\right]^4+\cdots$$

we see that (5.15) and (5.16) agree exactly to at least sixth power in the parameter g/γ. Similarly, the other noise terms $\langle F(t)F(t')\rangle$ and $\langle F^+(t)F^+(t')\rangle$ obtained from (5.12) can be seen to correspond to the large-N noises resulting from Haken's equations. Thus we have evidence of the correctness of our equations in that for the large-N situation they produce identical results to those already established by Haken. However, as will be seen in Sec. VI our equations do not require the large-N scaling, and hence results can be obtained for few-atom situations.

VI. ADIABATIC ELIMINATION WITHOUT SCALING

If we no longer assume N is large then we must include all the second- and third-order noise terms in (5.5). However it is still perfectly possible to work out the correlation functions of F and obtain general results for any particular situation, whether large or small N, whether near threshold or well above, or whether a laser or optical bistability configuration.

Defining

$$r=\frac{g}{\gamma} ,$$

it is possible to write the second and third-order noise terms in the following way in terms of the parameter r:

$$\begin{aligned}\langle F(t)F^+(t')\rangle&=\langle\Gamma_\alpha(t)\Gamma_{\alpha^+}(t')\rangle+r(1-r^2\alpha\alpha^+\Pi)[\langle\Gamma_\alpha(t)\Gamma_{\nu^+}(t')\rangle+\langle\Gamma_{\alpha^+}(t)\Gamma_\nu(t')\rangle]\\&\quad+\tfrac{1}{2}r^2\Pi[\alpha^+\langle\Gamma_\alpha(t)\Gamma_D(t')\rangle+\alpha\langle\Gamma_{\alpha^+}(t)\Gamma_D(t')\rangle]\\&\quad+\tfrac{1}{2}r^3\Pi(1-2r^2\alpha\alpha^+\Pi)[\alpha^+\langle\Gamma_\nu(t)\Gamma_D(t')\rangle+\alpha\langle\Gamma_{\nu^+}(t)\Gamma_D(t')\rangle]+\tfrac{1}{4}r^4\alpha\alpha^+\Pi^2\langle\Gamma_D(t)\Gamma_D(t')\rangle\\&=\left\{2\kappa n+rgN\left[\frac{D_0}{N}\Pi+1\right](1-r^2\alpha\alpha^+\Pi)\right.\\&\quad\left.-2r^3g\Pi^2\alpha\alpha^+D_0\left[\frac{1}{2}\left[1+\Pi\frac{D_0}{N}\right]+\frac{D_0}{N}\left[1-2r^2\alpha\alpha^+\Pi\right]\right]\right\}\delta(t-t') ,\end{aligned} \tag{6.1}$$

where we have replaced in the correlation functions the atomic variables with their steady-state values using (5.14)

$$D=D_0\Pi,\quad \nu=\frac{g}{\gamma}\alpha D=r\alpha D_0\Pi .$$

Again this first-order approximation is justified if the fluctuations are small. This will be seen to be the case if r is small, a condition which is seen later to follow from the assumptions of the adiabatic elimination. Similarly the other second- and third-order correlation functions for F can be calculated as

$$\begin{aligned}\langle F(t)F(t')\rangle&=[\langle F^+(t)F^+(t')\rangle]^+\\&=\left\{2\kappa m-2r^3g\Pi\alpha^2\left[\frac{1}{2}(D_0\Pi+N)+\Pi\frac{D_0^2}{N}(1-2r^2\alpha\alpha^+\Pi)+\tfrac{1}{2}\Pi D_0\left[1+\Pi\frac{D_0}{N}\right]\right]\right\}\delta(t-t') ,\end{aligned} \tag{6.2}$$

$$\begin{aligned}\langle F(t)F(t')F(t'')\rangle&=[\langle F^+(t)F^+(t')F^+(t'')\rangle]^+\\&=[3r^5g\Pi^2\alpha^3D_0(1-2r^2\alpha\alpha^+\Pi)+\tfrac{3}{2}r^5g\alpha^3\Pi^3D_0]\delta(t-t')\delta(t'-t'') ,\end{aligned} \tag{6.3}$$

$$\langle F(t)F(t')F^+(t'')\rangle=[\langle F(t)F^+(t')F^+(t'')\rangle]^+$$
$$=[-r^3g\Pi\alpha D_0(2-3r^2\alpha\alpha^+\Pi)(1-2r^2\alpha\alpha^+\Pi+\tfrac{1}{2}\Pi)]\delta(t-t')\delta(t'-t'') . \quad (6.4)$$

To reduce the number of equations we have to write, we have adopted the notation

$$X=(Y)^+$$

which means that the equation for Y is obtained by taking the quasi-complex-conjugate of X; that is replacing α with α^+, ν with ν^+, and vice-versa.

The expressions for the noises (6.1)–(6.4) are completely general, but do not allow the system to be easily solved. However in certain regimes it is possible to make one of two scaling assumptions and hence find an easier approximate solution.

A. Scaling assumption (i)

Assume r is small, that is the coupling constant g is much less than the atomic decay rate γ. Now assuming a fully pumped laser then we may rewrite the expression for the cooperativity parameter as

$$C=\frac{r^2N}{\mu} \quad \text{where } \mu=\frac{\kappa}{\gamma} \quad (6.5)$$

is the ratio of the cavity to atomic dampings, which has been defined to be small in the adiabatic elimination limit. Close to threshold we must have $C\approx O(1)$ so that assuming r to be small is consistent with having adiabatically eliminated the atomic variables. If we are near threshold [i.e., $2r^2\alpha\alpha^+\ll 1$], we can expand Π as

$$\Pi=1-2r^2\alpha\alpha^2+4r^2\alpha^2\alpha^{+2}+\cdots . \quad (6.6)$$

So substituting (6.6) into our general expressions for the noises we obtain to $O(r^2)$

$$\langle F(t)F^+(t')\rangle=[2\kappa n+rg(D_0+N)]\delta(t-t') ,$$
$$\langle F(t)F(t')\rangle=2\kappa m\,\delta(t-t')=[\langle F^+(t)F^+(t')\rangle]^+ , \quad (6.7)$$

with all the third-order terms vanishing at this order. For the standard laser problem $D_0=N$, $m=m^*=0$ and hence the leading noise term is

$$\langle F(t)F^+(t')\rangle=(2\kappa n+2rgN)\delta(t-t') . \quad (6.8)$$

This noise is identical to (5.15) [and (5.16)] to $O(r^2)$, indicating that for this particular regime the few-atom laser has exactly the same noise properties as the many-atom laser. Assuming r is small corresponds to having a large saturation photon number. Hence these results suggest that even with only a single atom, for sufficiently large n_0 the semiclassical predictions for the laser are still valid. Thus we still have the dominant noise term arising from spontaneous emission, and existing results on photon statistics, etc. for the laser could be applied to the one-atom laser. This is in accordance with recent results obtained numerically by Savage and Carmichael[12] which showed that for large n_0 the semiclassical theory for optical bistability was valid for one atom.

B. Scaling assumption (ii)

As we go further above threshold, while we still have

$$2r^2\alpha\alpha^+\ll 1$$

then the expansion (6.6) in Π is still possible and we have the leading noise term given by

$$\langle F(t)F^+(t')\rangle=[2\kappa n+rg(D_0+N)]\delta(t-t') . \quad (6.9)$$

However, as the approximations used become less strictly correct, this noise term becomes less dominant and other terms begin to contribute. Eventually though we reach what is called the high-intensity limit where C and thus the field have increased so that

$$2r^2\alpha\alpha^+\gg 1 \quad (6.10)$$

and we may then consider the alternate expansion for Π

$$\Pi=\frac{1}{2r^2\alpha\alpha^+}\left[1-\frac{1}{2r^2\alpha\alpha^+}+\frac{1}{4r^4\alpha^2\alpha^{+2}}+\cdots\right] . \quad (6.11)$$

Now substituting this into (6.1)–(6.4), and discarding all terms of $O[(r^2\alpha\alpha^+)^{-1/2}]$ and lower, we obtain

$$\langle F(t)F^+(t')\rangle=\left[2\kappa n+\frac{rgN}{2}\right]\delta(t-t') ,$$
$$\langle F(t)F(t')\rangle=\left[2\kappa m-\frac{rgN\alpha}{2\alpha^+}\right]\delta(t-t') \quad (6.12)$$
$$=[\langle F^+(t)F^+(t'')\rangle]^+ ,$$

where all third-order-noise terms are of lower order in this approximation and hence do not contribute.

This is not the standard constant laser noise, and since the noises are independent of D_0 this indicates that the same noise limit is attained in both the laser and optical bistability configurations. These noises are in fact the same as those obtained previously in the high-intensity limit for optical bistability.[13] This can be seen to be consistent, as for very large fields the condition (6.10) must eventually be satisfied, and so regardless of the exact value of r the noise terms in (6.12) will give the right high-intensity limit. This approximation was also independent of the number of atoms, which would indicate that even for one atom, as we go far enough above threshold these noise terms will dominate. Their effect on the photon statistics for the laser and optical bistability will be discussed in forthcoming work.

VII. CONCLUSION

This paper has mainly been concerned with the method in obtaining the Eqs. (4.3)–(4.6) and hence possible uses of these equations have only been briefly examined. However, it is clear that not only are these equations equivalent to Haken's equations in the large-N situation,

but they also establish new results for single or few atom problems which were previously difficult to handle.

Applications with the one-atom laser were discussed in Sec. VI and more work will be done with these in later publications. However, by adding an extra term to the Hamiltonian

$$H_5 = i\hbar(a^\dagger \epsilon e^{-i\omega t} - a\epsilon^* e^{i\omega t}) , \tag{7.1}$$

we can easily examine the phenomenon of absorptive optical bistability using our equations. This new term only affects the cavity field (in a manner already known) and so we can carry out the transformation (3.7) and then adiabatically eliminate to obtain

$$\begin{aligned} \dot{\alpha} &= \epsilon - \kappa(1 + C'\Pi)\alpha + F , \\ \dot{\alpha}^+ &= \epsilon^* - \kappa(1 + C'\Pi)\alpha + F^+ , \end{aligned} \tag{7.2}$$

where

$$C' = -\frac{g^2 D_0}{\gamma\kappa} > 0 ,$$

as we now consider the majority of the atoms to be in the ground state, that is, we require a negative inversion to analyze optical bistability. Again it is standard to consider total inversion, $D_0 = -N$.

Now the region of interest in optical bistability is where the parameter C' has the value $C' > 8$. There are then two metastable field states for each value of the driving field ϵ. However, we can still make the scaling assumption (i), and for not too large a field [so we can expand Π via (6.6)] the leading noise terms can be calculated from (6.1)–(6.4). Notice that as in other results on optical bistability[13] the spontaneous emission term which dominates for the laser is zero and we have to drop down to the next-order terms. Thus the leading noise terms to $O(r^3)$ with m, m^* being zero are

$$\begin{aligned} \langle F(t)F^+(t')\rangle &= 2\kappa n\delta(t-t') , \\ \langle F(t)F(t')\rangle &= -2r^3 g\alpha^2 N\delta(t-t') \\ &= [\langle F^+(t)F^+(t')\rangle]^+ , \end{aligned} \tag{7.3}$$

$$\begin{aligned} \langle F(t)F(t')F(t'')\rangle &= 0 = \langle F^+(t)F^+(t')F^+(t'')\rangle , \\ \langle F(t)F(t')F^+(t'')\rangle &= 3r^3 g\alpha N\delta(t-t') \\ &= [\langle F(t)F^+(t')F^+(t'')\rangle]^+ . \end{aligned} \tag{7.4}$$

The most important thing to notice is that we can now no longer disregard the third-order terms as being necessarily small compared to the second-order terms. It is however possible to use the different α dependence in (7.3) and (7.4) to scale away either the second- or third-order terms in certain regimes. Techniques using these results to solve the optical bistability problem for arbitrary N will be dealt with in a future paper.

Another potential application of these equations is in analyzing the full Jaynes-Cummings atom-cavity problem where we allow for both atomic and cavity decay. This situation corresponds to a passive atom in a cavity (i.e., $D_0 = -N$) but with no classical external injected field (i.e., $\epsilon = 0$). Hence, in scaling assumption (i) this situation is describable by Eq. (7.2) together with the correlations (7.3) and (7.4). The most noticeable difference is that since $\epsilon = 0$ the steady-state solution for the cavity field is $\alpha = 0$, therefore we have no threshold. However, if r is sufficiently small we can satisfy scaling assumption (i), and hence use the correlations to obtain results.

Finally in deriving the equations we have allowed for possible squeezed inputs in introducing m and m^*. These are zero in the traditional laser and optical bistability problems but using our equations it is a simple matter to consider for example a laser where the cavity field is damped by a squeezed bath. In the recent work by Marte and Walls[14] equations were developed which corresponded to the atoms being damped by a squeezed bath. These equations lead to a laser with a definite phase, but in relation to this paper it can be shown that the equations are reached far more simply by the use of (2.10), than via Haken's characteristic function. Similarly Jaynes-Cummings problems where the cavity interacts with squeezed fields could be treated in this formalism.

To conclude, we hope we have demonstrated a simpler way to deal with atom-cavity situations such as the two-level laser. In particular our equations enable scalings to be made which do not rely on the number of atoms N being large, and hence allow theoretical consideration of the one-atom laser problem. In further works stemming from the theory in this paper we hope to expand on this and other problems to obtain more extensive results.

APPENDIX A

Taking the partial derivative with respect to time of (2.5) gives [using (2.2)]

$$\frac{\partial\chi}{\partial t} = \mathrm{Tr}\left[O^A O^F \left[\frac{1}{i\hbar}[H_2, \rho] + \left.\frac{\partial\rho}{\partial t}\right|_A + \left.\frac{\partial\rho}{\partial t}\right|_F \right] \right] . \tag{A1}$$

It is best to look at each term individually.

(i) The cavity-field part is dealt with in the same fashion as Haken[15]

$$\mathrm{Tr}\left[O \left.\frac{\partial\rho}{\partial t}\right|_F \right] = \kappa \left[-i\beta\frac{\partial}{\partial i\beta} - i\beta^*\frac{\partial}{\partial i\beta^*} + 2ni\beta i\beta^* + m(i\beta)^2 + m^*(i\beta^*)^2 \right] \chi . \tag{A2}$$

(ii) To determine the atomic contribution we write O as

$$O = Q^{A,\mu} O^{A,\mu} , \tag{A3}$$

where

$$Q^{A,\mu} = O^F \prod_{i\,(\neq\mu)} O^{A,i} \quad \text{and} \quad O^{A,\mu} = b + \mathbf{c}\cdot\boldsymbol{\sigma}_\mu ,$$

Hence

$$\mathrm{Tr}\left[O\frac{\partial\rho}{\partial t}\bigg|_A\right] = \mathrm{Tr}\left[\sum_\mu Q^{A,\mu}O^{A,\mu} \times[\tfrac{1}{2}\omega_{21}(2\sigma_\mu^-\rho\sigma_\mu^+ - \sigma_\mu^+\sigma_\mu^-\rho - \rho\sigma_\mu^+\sigma_\mu^-) + \tfrac{1}{2}\omega_{12}(2\sigma_\mu^+\rho\sigma_\mu^- - \sigma_\mu^-\sigma_\mu^+\rho - \rho\sigma_\mu^+\sigma_\mu^+)]\right], \tag{A4}$$

so using the properties of the trace and of the Pauli matrices we deal with each term individually, e.g.,

$$\mathrm{Tr}\left[\sum_\mu Q^{A,\mu}(b+\mathbf{c}\cdot\boldsymbol{\sigma}_\mu)\omega_{21}\sigma_\mu^-\rho\sigma_\mu^+\right] = \mathrm{Tr}\left[\sum_\mu Q^{A,\mu}\omega_{21}\sigma_\mu^+(b+\mathbf{c}\cdot\boldsymbol{\sigma}_\mu)\sigma_\mu^-\rho\right] \tag{A5}$$

$$= \mathrm{Tr}\left[\sum_\mu Q^{A,\mu}\omega_{21}(b-\tfrac{1}{2}c)(\sigma_{z,\mu}+\tfrac{1}{2})\rho\right] \tag{A6}$$

$$= \mathrm{Tr}\left[\sum_\mu Q^{A,\mu}\omega_{21}(b-\tfrac{1}{2}c)\left[\frac{\partial}{\partial c}+\tfrac{1}{2}\frac{\partial}{\partial b}\right]O^{A,\mu}\rho\right]. \tag{A7}$$

Following a similar method for all the other terms we then substitute into (A4) to obtain

$$\mathrm{Tr}\left[O\frac{\partial\rho}{\partial t}\bigg|_A\right] = \mathrm{Tr}\left[\sum_\mu Q^{A,\mu}\left[-\tfrac{1}{2}(\omega_{21}+\omega_{12})\left[c^-\frac{\partial}{\partial c^-}+c^+\frac{\partial}{\partial c^+}+2c\frac{\partial}{\partial c}\right]+\tfrac{1}{2}(\omega_{12}-\omega_{21})c\frac{\partial}{\partial b}\right]O^{A,\mu}\rho\right]. \tag{A8}$$

Now using the product rule for differentiation we can turn the sum of derivatives of $O^{A,\mu}$ into one derivative of the product O,

$$\mathrm{Tr}\left[O\frac{\partial\rho}{\partial t}\bigg|_A\right] = \mathrm{Tr}\left[\left[-\tfrac{1}{2}(\omega_{21}+\omega_{12})\left[c^-\frac{\partial}{\partial c^-}+c^+\frac{\partial}{\partial c^+}+2c\frac{\partial}{\partial c}\right]+\tfrac{1}{2}(\omega_{12}-\omega_{21})c\frac{\partial}{\partial b}\right]O\rho\right] \tag{A9}$$

$$= \left[-\tfrac{1}{2}(\omega_{21}+\omega_{12})\left[c^-\frac{\partial}{\partial c^-}+c^+\frac{\partial}{\partial c^+}+2c\frac{\partial}{\partial c}\right]+\tfrac{1}{2}(\omega_{12}-\omega_{21})c\frac{\partial}{\partial b}\right]\chi. \tag{A10}$$

(iii) Expanding out the commutator the interaction term can be evaluated as

$$g\,\mathrm{Tr}[O(a^\dagger S^-\rho - \rho a^\dagger S^- + \rho a S^+ - a S^+\rho)],$$

where

$$S^- = \sum_\mu \sigma_\mu^- \text{ and } S^+ = \sum_\mu \sigma_\mu^+,$$

$$g\,\mathrm{Tr}[O(a^\dagger S^-\rho - \rho a^\dagger S^- + \rho a S^+ - a S^+\rho)] = g\,\mathrm{Tr}[O^A S^- O^F a^\dagger\rho - S^- O^A a^\dagger O^F\rho + S^+ O^A a O^F\rho - O^A S^+ O^F a\rho] \tag{A11}$$

$$= g\,\mathrm{Tr}\left[\left[\frac{\partial}{\partial i\beta^*}+i\beta\right]O^A S^- O^F\rho - \frac{\partial}{\partial i\beta^*}S^- O^A O^F\rho + \left[\frac{\partial}{\partial i\beta}+i\beta^*\right]S^+ O^A O^F\rho - \frac{\partial}{\partial i\beta}O^A S^+ O^F\rho\right]. \tag{A12}$$

Now following a procedure similar to (A5)–(A10) for the atomic operators gives Eq. (A12) equal to

$$g\,\mathrm{Tr}\left[(b-\tfrac{1}{2}c)\left[i\beta\frac{\partial}{\partial c^-}+i\beta^*\frac{\partial}{\partial c^+}\right]+(i\beta c^+ + i\beta^* c^-)\left[\frac{\partial}{\partial c}+\frac{1}{2}\frac{\partial}{\partial b}\right] + c^+\frac{\partial}{\partial i\beta^*}\left[2\frac{\partial}{\partial c}-\frac{\partial}{\partial c^-}\right]+c^-\frac{\partial}{\partial i\beta}\left[2\frac{\partial}{\partial c}-\frac{\partial}{\partial c^+}\right]\right]\chi. \tag{A13}$$

Combining (A2), (A10), and (A13) we obtain (2.13).

APPENDIX B

In addition to (2.11) and (2.12), by making use of the properties of the Pauli matrices and of our new characteristic function, we can easily calculate higher moments of the collective atomic operator averages—we treat only the atomic parts as the field part is dealt with in the standard way. For example,

$$\frac{\partial^2\chi}{\partial c^{-2}} = \mathrm{Tr}\left[\frac{\partial^2}{\partial c^{-2}}\prod_{\mu=1}^N(b+\mathbf{c}\cdot\boldsymbol{\sigma}_\mu)\rho\right] = \mathrm{Tr}\left[\sum_{\mu=1}^N\sigma_\mu^-\sum_{k\neq\mu}^N\sigma_k^-\prod_{j\neq k,\mu}^N(b+\mathbf{c}\cdot\boldsymbol{\sigma}_j)\rho\right]. \tag{B1}$$

Now evaluating the derivative at $b=1$, $\mathbf{c}=0$ (with $\beta,\beta^*=0$ for the field part) gives

$$\left.\frac{\partial^2\chi}{\partial c^{-2}}\right|_{b=1,\ c=0}=\mathrm{Tr}\left[\sum_{\mu=1}^{N}\sigma_\mu^-\sum_{k\neq\mu}^{N}\sigma_k^-\rho\right] \tag{B2}$$

$$=\mathrm{Tr}\left[\sum_{\mu=1}^{N}\sigma_\mu^-\left[\sum_{k=1}^{N}\sigma_k^--\sigma_\mu^-\right]\rho\right] \tag{B3}$$

$$=\mathrm{Tr}\left[\left[\sum_{\mu=1}^{N}\sigma_\mu^-\right]^2\rho\right], \tag{B4}$$

where we have used $\sigma_\mu^-\sigma_\mu^-=0$. Hence, defining

$$S^-=\sum_{\mu=1}^{N}\sigma_\mu^- \tag{B5}$$

$$\langle S^-S^-\rangle=\left.\frac{\partial^2\chi}{\partial c^{-2}}\right|_{b=1,\ c=0}. \tag{B6}$$

Similarly, defining

$$S^+=\sum_{\mu=1}^{N}\sigma_\mu^+,\quad S_z=\sum_{\mu=1}^{N}\sigma_{\mu,z}, \tag{B7}$$

then it is easily shown that

$$\begin{aligned}\langle S^+S^+\rangle&=\left.\frac{\partial^2\chi}{\partial c^{+2}}\right|_{b=1,\ c=0},\\ \langle S_zS_z\rangle&=\left.\frac{\partial^2\chi}{\partial c^2}\right|_{b=1,\ c=0}+\frac{N}{4}.\end{aligned} \tag{B8}$$

Cross correlations can also be simply worked out using the operator rules, e.g.,

$$\langle S^+S^-\rangle=\left.\frac{\partial^2\chi}{\partial c^-\partial c^+}\right|_{b=1,\ c=0}+\langle S_z\rangle-\frac{N}{2}, \tag{B9}$$

$$\langle S^-S^+\rangle=\left.\frac{\partial^2\chi}{\partial c^+\partial c^-}\right|_{b=1,\ c=0}+\frac{N}{2}-\langle S_z\rangle. \tag{B10}$$

Higher moments and correlations cannot always be immediately specified as with the usual characteristic function, except in the special cases,

$$\begin{aligned}\left.\frac{\partial^i\chi}{\partial c^{-i}}\right|_{b=1,\ c=0}&=\langle(S^-)^i\rangle,\\ \left.\frac{\partial^j\chi}{\partial c^{+j}}\right|_{b=1,\ c=0}&=\langle(S^+)^j\rangle;\end{aligned} \tag{B11}$$

however, in all cases by applying the operator rules, any correlation may be calculated up to Nth order, and there is the advantage over the previous characteristic function that specifying non-normally ordered operator averages is as simple as specifying normally ordered averages.

In a similar vein to (2.12) the extra variable b may be utilized to find the higher-order moments of N, i.e.,

$$N(N-1)=\left.\frac{\partial^2\chi}{\partial b^2}\right|_{b=1,\ c=0} \tag{B12}$$

and, in general,

$$N(N-1)\cdots[N-(i-1)]=\left.\frac{\partial^i\chi}{\partial b^i}\right|_{b=1,\ c=0}. \tag{B13}$$

APPENDIX C

Multitime averages of the collective atomic operators can also be calculated (at least theoretically) and these also illustrate what is called quantum classical correspondence. Now suppose we wish to calculate general multitime correlations of the form

$$\langle A_0(s_0)A_1(s_1)\cdots A_m(s_m)B_n(t_n)B_{n-1}(t_{n-1})\cdots B_0(t_0)\rangle, \tag{C1}$$

where $t_n\geq t_{n-1}\geq\cdots\geq t_0$ and $s_m\geq s_{m-1}\geq\cdots\geq s_0$ then we order the times t and s in sequence and rename them τ, so that

$$\tau_0\leq\tau_1\leq\cdots\leq\tau_{r-1}\leq\tau_r\quad(r=n+m+1).$$

Letting F_i be the Schrödinger operator which is evaluated at τ_i, and defining the operator f_i by

$$\begin{aligned}f_i\rho&=F_i\rho\quad\text{if } F_i \text{ is one of the } B\text{'s},\\ f_i\rho&=\rho F_i\quad\text{if } F_i \text{ is one of the } A\text{'s},\end{aligned} \tag{C2}$$

then for a master equation of the form

$$\frac{\partial\rho}{\partial t}=L\rho \tag{C3}$$

with evolution operator defined by

$$V(\tau_r,\tau_{r-1})=\exp\left[\int_{\tau_{r-1}}^{\tau_r}L(t)dt\right], \tag{C4}$$

Gardiner and Collett[16] have shown that

$$\langle A_0(s_0)A_1(s_1)\cdots B_1(t_1)B_0(t_0)\rangle=\mathrm{Tr}[f_rV(\tau_r,\tau_{r-1})f_{r-1}V(\tau_{r-1},\tau_{r-2})f_{r-2}\cdots V(\tau_1,\tau_0)f_0\rho(t_0)]. \tag{C5}$$

For the usual two-time correlation function this gives the standard result

$$\langle A(t+\tau)B(t)\rangle=\mathrm{Tr}[AV(t+\tau,t)B\rho(t)]. \tag{C6}$$

We now use our characteristic function to write the master equation for the density operator in terms of a generalized FPE in the positive P representation. Defining $W(\tau_1,\tau_0)$ to be the evolution operator of the FPE corresponding to the evolution operator $V(\tau_1,\tau_0)$ of the master equation, etc., it can be shown that (considering only the atomic part)

$$\langle A_0(s_0)A_1(s_1)\cdots B_1(t_i)B_0(t_0)\rangle=\int d^2\nu\int d^2\nu^+\int d^2D\int d^2B\,f_r(\mathbf{x})W(\tau_r,\tau_{r-1})f_{r-1}(\mathbf{x})\cdots W(\tau_1,\tau_0)f_0(\mathbf{x})P(\mathbf{x},t), \tag{C7}$$

where $\mathbf{x}=(v,v^+,D,B)$. The $f_r(\mathbf{x})$'s give the correspondence between the quantum operator acting on the density operator, and the stochastic variable which represents it in the FPE, and may be calculated from the characteristic function in the following manner. Consider in the master equation terms of the form

$$\mathrm{Tr}\left[O\rho\sum_\mu \sigma_\mu^+\right]=\mathrm{Tr}\left[\sum_\mu Q^{A,\mu}(b+\mathbf{c}\cdot\boldsymbol{\sigma}_\mu)\rho\sigma_\mu^+\right], \tag{C8}$$

where we have utilized the definition of the characteristic function employed in Appendix A. Now using the properties of the Pauli matrices and the trace this can be written as

$$\begin{aligned}\mathrm{Tr}\left[\sum_\mu Q^{A,\mu}(b+\mathbf{c}\cdot\boldsymbol{\sigma}_\mu)\rho\sigma_\mu^+\right]&=\mathrm{Tr}\left[\sum_\mu Q^{A,\mu}[(b-\tfrac{1}{2}c)\sigma_\mu^+ +c^-(\sigma_{\mu,z}+\tfrac{1}{2})]\rho\right]\\&=\mathrm{Tr}\left[\sum_\mu Q^{A,\mu}\left[(b-\tfrac{1}{2}c)\frac{\partial}{\partial c^+}+c^-\left[\frac{\partial}{\partial c}+\tfrac{1}{2}\frac{\partial}{\partial b}\right]\right]O^{A,\mu}\rho\right]\\&=\left[(b-\tfrac{1}{2}c)\frac{\partial}{\partial c^+}+c^-\left[\frac{\partial}{\partial c}+\tfrac{1}{2}\frac{\partial}{\partial b}\right]\right]\chi ,\end{aligned} \tag{C9}$$

again using the product rule to change a sum of derivatives into a derivative of a product as in Appendix A. Hence a term ρS^+ in the master equation has an effect

$$(b-\tfrac{1}{2}c)\frac{\partial}{\partial c^+}+c^-\left[\frac{\partial}{\partial c}+\tfrac{1}{2}\frac{\partial}{\partial b}\right] \tag{C10}$$

in the characteristic function equation. Now if we use the transformation (3.7) to go from the characteristic function to the positive P function then we have the relation that ρS^+ in the master equation has an effect

$$v^+\left[1+\frac{\partial}{\partial D}-\frac{\partial}{\partial B}\right]-\tfrac{1}{2}(B+D)\frac{\partial}{\partial v} \tag{C11}$$

in the FPE. This process may then be repeated for all the collective operators acting both before and after the density operator, to completely specify the $f_r(\mathbf{x})$ as

$$f_r(\mathbf{x})=\begin{cases} v^+\left[1+\dfrac{\partial}{\partial D}-\dfrac{\partial}{\partial B}\right]-\tfrac{1}{2}(D+B)\dfrac{\partial}{\partial v}\leftrightarrow\rho S^+ ,\\ v^+\left[1-\dfrac{\partial}{\partial D}-\dfrac{\partial}{\partial B}\right]+\tfrac{1}{2}(D-B)\dfrac{\partial}{\partial v}\leftrightarrow S^+\rho ,\\ v\left[1-\dfrac{\partial}{\partial D}-\dfrac{\partial}{\partial B}\right]+\tfrac{1}{2}(D-B)\dfrac{\partial}{\partial v^+}\leftrightarrow\rho S^- ,\\ v\left[1+\dfrac{\partial}{\partial D}-\dfrac{\partial}{\partial B}\right]-\tfrac{1}{2}(D+B)\dfrac{\partial}{\partial v^+}\leftrightarrow S^-\rho ,\\ \dfrac{1}{2}\left[D-D\dfrac{\partial}{\partial B}-B\dfrac{\partial}{\partial D}+\dfrac{\partial}{\partial v}v-\dfrac{\partial}{\partial v^+}v^+\right]\leftrightarrow\rho S_z ,\\ \dfrac{1}{2}\left[D-D\dfrac{\partial}{\partial B}-B\dfrac{\partial}{\partial D}-\dfrac{\partial}{\partial v}v+\dfrac{\partial}{\partial v^+}v^+\right]\leftrightarrow S_z\rho . \end{cases} \tag{C12}$$

If we now add the field operators and integrate over the complex plane for the independent variables α and α^+ together with the known result[17]

$$f_r(\alpha)=\begin{cases}\alpha\leftrightarrow\rho a ,\\ \alpha-\dfrac{\partial}{\partial\alpha^+}\leftrightarrow a\rho ,\\ \alpha^+-\dfrac{\partial}{\partial\alpha}\leftrightarrow\rho a^\dagger ,\\ \alpha^+\leftrightarrow a^\dagger\rho ,\end{cases} \tag{C13}$$

then we can specify any multitime correlation, provided of course we have managed to solve for the operator $W(t_1,t_2)$ in some way.

There is thus a quantum classical correspondence in that quantum-mechanical averages of the operators can be related to averages of the classical Fokker-Planck variables.

APPENDIX D

We proceed in the same manner as in Sec. V and adiabatically eliminate the atoms by assuming

$$\kappa \ll \gamma \ . \tag{D1}$$

Hence from (3.11) we set $\dot{\nu}$ and $\dot{D}$ to be zero, so that

$$\nu = \frac{g\alpha D}{\gamma} + \frac{\Gamma_\nu}{\gamma} \ , \tag{D2}$$

$$D = D_0 - \frac{g}{\gamma}(\alpha^+ \nu + \alpha \nu^+) + \frac{D_0}{N}\int_0^t \Gamma_B(t')dt' + \frac{\Gamma_D}{2\gamma} \ . \tag{D3}$$

Substituting into the equation for α we obtain after eliminating ν and D

$$\dot{\alpha} = -\kappa(1 - C\Pi)\alpha + F \tag{D4}$$

where

$$F = \Gamma_\alpha + \left[\frac{g}{\gamma}\right]\left[1 - \left[\frac{g}{\gamma}\right]^2 \alpha\alpha^+\Pi\right]\Gamma_\nu + \frac{1}{2}\left[\frac{g}{\gamma}\right]^2 \alpha\Pi\Gamma_D + \frac{g^2}{\gamma}\alpha\Pi\frac{D_0}{N}\int_0^t \Gamma_B(t')dt' - \left[\frac{g}{\gamma}\right]^3 \alpha^2\Pi\Gamma_{\nu^+} \ .$$

The F equation has exactly the same second-order noise terms as obtained in Eq. (5.5) except for the extra term

$$\int_0^t \Gamma_B(t')dt' \ , \tag{D5}$$

which must in some way represent the third-order noise terms present in (5.5). However, the presence of (D5) makes it difficult to solve (D4) by calculating the appropriate correlations of F as in (6.1)–(6.4) as we have integrals of white noises turning up at various powers of r. Nevertheless, in the scaling assumptions (i) and (ii) of Sec. VI it was found that in certain regimes the third order contributions are of lower order, and if these scaling assumptions are applied to (D4) the terms arising from correlations with (D5) are also of lower order and hence we can ignore (D5). In this case solving (D4) leads to identical results as obtained in Sec. VI. To solve the system (D4) exactly will probably require the technique of simulation of the SDE's (3.13) which will be looked at in a future paper.

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