Hamiltonian description of nonlinear propagation in optical fibers

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Nonlinear propagation in single-mode and multimode fibers in the presence of the optical Kerr effect is described in terms of a number of parameters (four for each propagating mode) which can be interpreted as conjugate variables of a suitable Hamiltonian system. The formal simplicity of this approach, which admittedly furnishes a limited description of nonlinear propagation because of the finiteness of the number of variables employed, is, however, very useful for gaining a straightforward physical insight into the problem. The solution of the pertinent equations, either analytical or numerical, presents a much less formidable task than the solution of the set of nonlinear equations fully describing propagation.

I. INTRODUCTION

A light pulse propagating in a fiber undergoes a self-induced change of both amplitude and phase associated with the nonlinear contribution to the refractive index of silica proportional to the instantaneous optical intensity (the optical Kerr effect). This process, whose basic features depend on the interplay between the spectral self-broadening associated with the nonlinearity of the refractive index and chromatic dispersion, can become significant, because of the low-loss interaction length provided by the fiber, for values of the pertinent parameters (such as fiber length, pulse width, and input intensity) which can occur in the framework of fiber-optic telecommunications. This accounts for the great deal of interest, during the last few years, in investigating this kind of nonlinear propagation, especially in view of the possibility of propagating, in a single-mode fiber and in the regime of negative group-velocity dispersion ($\lambda \geq 1.3$ μm for silica), a distortionless pulse of suitable amplitude and shape (fundamental envelope soliton).[1] This possibility, together with the realization of the existence of higher-order envelope solitons reproducing themselves after a certain fiber length, has been exploited for conceiving new schemes of high-rate data transmission[2,3] or pulse reshaping,[4] and more generally has promoted the study, mainly based on numerical approaches, of the set of nonlinear partial differential equations which provides the complete analytical description of the problem of pulse evolution in a single or multimode optical fiber.

The basic purpose of this paper is to present an approach to the problem which, besides being more amenable to a computational analysis than the cumbersome numerical integration of the corresponding set of nonlinear equations, is able to provide a simple physical picture. This is accomplished by limiting the description of a pulse propagating on a given mode to the introduction of four parameters, two representing the width and the velocity of the center of mass of its envelope and two accounting for shift and modulation of its instantaneous frequency, thus renouncing *a priori* any detailed information on the fine structure of the pulse itself (as, for example, that associated with higher-order solitons). Remarkably, it turns out that the $4\times N$ parameters (where N is the number of modes supported by the fiber) introduced for describing nonlinear propagation obey a Hamiltonian formalism, that is, they can be interpreted as conjugate variables of a suitable Hamiltonian system. The corresponding Hamilton equations, whose numerical integration is obviously much less involved than that pertaining to the complete problem, are simple enough to allow one to draw some general conclusions on the dynamical behavior of the system and, in particular, whenever more than one mode is present, to express in a more quantitative way the qualitative results on pulse attraction and self-confinement first obtained by Hasegawa.[5,6]

The set of $4\times N$ ordinary differential Hamilton equations which constitutes the main result of our paper has been worked out under the assumption that the pulse envelope maintains a given shape during its evolution. Our formalism finds thus its most natural application in the study of the interaction among pulses which, in the absence of mode coupling, would exhibit a distortionless solitonlike behavior, and is particularly apt for investigating the evolution of the mutual distances of their centers of mass and of their widths.

We wish to point out that an approach, which can be considered as the Lagrangian counterpart of our formalism, has been developed in the literature.[7,8] The method described there basically refers to propagation in a single-mode fiber supporting one polarization state and is appropriate for the study of the evolution of one soliton or for soliton interaction, but does not concern propagation over different modes possessing different group velocities.

II. HAMILTONIAN FORMALISM

The electromagnetic field propagating inside a multimode fiber can be written as

$$\mathbf{E}(\mathbf{r},z,t)=\sum_n \mathbf{E}_n(\mathbf{r})e^{i\omega_0 t-i\beta_n(\omega_0)z}\Phi_n(z,t) , \qquad (1)$$

where the z axis coincides with the fiber axis, $\mathbf{r}=(x,y)$, $\mathbf{E}_n(r)$ and β_n are, respectively, the spatial configuration and the propagation constant of the nth guided mode, ω_0 the mid-frequency of the carrier, and $\phi_n(z,t)$ the complex modal amplitude. The optical Kerr effect is associated with the presence of a nonlinear term proportional to the instantaneous optical intensity $I=|\mathbf{E}(\mathbf{r},z,t)|^2$, which adds up to the linear refractive index $n_1(r,\omega)$ of the propagation medium to give a refractive index of the form

$$n(\mathbf{r},\omega)=n_1(\mathbf{r},\omega)+n_2 I , \qquad (2)$$

where n_2 is called the nonlinear refractive-index coefficient [$n_2\cong 10^{-22}$ (m/V)^2 for silica].

The nonlinear part of the refractive index is responsible for a coupling mechanism among the various modes whose evolution is characterized by the following set of coupled differential equations:[6]

$$\left[\frac{\partial}{\partial z}+\frac{1}{v_m}\frac{\partial}{\partial t}-\frac{i}{2A_m}\frac{\partial^2}{\partial t^2}\right]\Phi_m=-2i\left[\frac{R_{mm}}{2}|\Phi_m|^2+\sum_{n=1}^{N}\sum_{\substack{m=1\\ m\neq n}}^{N}R_{mn}|\Phi_n|^2\right]\Phi_m,\quad m=1,2,\ldots,N \qquad (3)$$

where

$$v_m=\left.\frac{d\beta_m}{d\omega}\right|_{\omega_0}^{-1},\quad A_m=\left.\frac{d^2\beta_m}{d\omega^2}\right|_{\omega_0}^{-1}, \qquad (4)$$

are, respectively, the group velocity and the group-velocity dispersion of the mth mode and

$$R_{mn}=\frac{\omega_0 n_2}{c}\int_{-\infty}^{+\infty}\int dx\,dy\,|\mathbf{E}_n(\mathbf{r})|^2|\mathbf{E}_m(\mathbf{r})|^2 , \qquad (5)$$

where

$$\int_{-\infty}^{+\infty}dx\,dy\,|\mathbf{E}_n(\mathbf{r})|^2=1 . \qquad (6)$$

The set of Eqs. (3) properly describes the case of a multimode polarization-preserving optical fiber or, for $N=2$, of a single-mode high-birefringence fiber supporting two mutually orthogonal linearly polarized states [provided R_{12} is substituted in Eqs. (3) by $(1/3)R_{12}$].[9] Also, if the last case is the most relevant in practice, it is possible, without increasing too much the degree of formal complexity of our approach, to investigate the situation pertaining to a generic N.

We write the amplitude of the mth mode as

$$\Phi_m(z,t)=|\Phi_m(z,t)|e^{i\psi_m(z,t)} \qquad (7)$$

and assume that $\Phi_m(z,t)$ and $\partial\psi_m/\partial t$ can be expressed, in the spirit of our approach, as

$$|\Phi_m(z,t)|^2=N_m F[(t-\tau_m(z))/\sigma_m(z)]/\sigma_m(z) ,\qquad m=1,2,\ldots,N \qquad (8)$$

$$\frac{\partial\psi_m(z,t)}{\partial t}=-2\{\xi_m(z)+2\eta_m(z)[t-\tau_m(z)]\}/N_m ,\qquad m=1,2,\ldots,N \qquad (9)$$

where N_m, $\tau_m(z)$, and $M_m=\sigma_m^2(z)$ represent, respectively, the zeroth-, first-, and second-order moment of the function $|\Phi_m(z,t)|^2$,

$$N_m=\int_{-\infty}^{+\infty}dt\,|\Phi_m(z,t)|^2 , \qquad (10)$$

$$\tau_m(z)=\frac{1}{N_m}\int_{-\infty}^{+\infty}dt\,t\,|\Phi_m(z,t)|^2 , \qquad (11)$$

$$M_m(z)=\frac{1}{N_m}\int_{-\infty}^{+\infty}dt(t-\tau_m)^2|\Phi_m(z,t)|^2 ,\qquad m=1,2,\ldots,N \qquad (12)$$

and $F(x)$ is an arbitrary positive-definite even function of the argument normalized to 1 and possessing variance 1, that is,

$$\int_{-\infty}^{+\infty}dx\,F(x)=\int_{-\infty}^{+\infty}dx\,x^2F(x)=1 . \qquad (13)$$

It is possible to prove (see the Appendix) that the set of equations describing the z evolution of the unknowns τ_m, M_m, ξ_m, and η_m corresponds to that pertaining to a Hamiltonian system with $2\times N$ degrees of freedom, ξ_m and η_m being the variables conjugate, respectively, to τ_m and M_m. Note that the functional dependence of $|\Phi_m|^2$ on σ_m and τ_m is determined by the boundary condition at the fiber input $z=0$ through Eq. (8), so that, from a physical point of view, propagation is described in terms of the average position and width of the pulse traveling on the mth mode, while Eq. (9) allows both for a shift and a modulation of its instantaneous frequency. The Hamiltonian of the system reads (see the Appendix)

$$H(\tau_1,\ldots,\tau_N,M_1,\ldots,M_N)=\sum_m \left[\frac{\xi_m}{v_m}-\frac{1}{N_m A_m}(\xi_m^2+4M_m\eta_m^2)-\frac{N_m a}{4A_m M_m}-\frac{R_{mm}N_m^2 b}{4M_m^{1/2}}\right] + V(\tau_1,\ldots,\tau_N,M_1,\ldots,M_N)\ , \quad (14)$$

where

$$V=-\sum_{\substack{m,n\\ m\neq n}}\sum \frac{1}{2}\frac{R_{mn}N_m N_n}{(M_m M_n)^{1/2}} \times \int_{-\infty}^{+\infty} dt\, F\left[\frac{t-\tau_m}{M_m^{1/2}}\right] F\left[\frac{t-\tau_n}{M_n^{1/2}}\right] \quad (15)$$

and

$$a=\int_{-\infty}^{+\infty} dx \frac{1}{F}\left[\frac{dF}{dx}\right]^2,\quad b=\int_{-\infty}^{+\infty} dx\, F^2(x)\ , \quad (16)$$

the corresponding Hamilton equations being

$$\frac{d\tau_m}{dz}=\frac{\partial H}{\partial \xi_m}=\frac{1}{v_m}-\frac{2\xi_m}{N_m A_m}\ , \quad (17)$$

$$\frac{d\xi_m}{dz}=-\frac{\partial H}{\partial \tau_m}=-\frac{\partial}{\partial \tau_m}V(\tau_1,\ldots,\tau_N,M_1,\ldots,M_N)\ , \quad (18)$$

$$\frac{dM_m}{dz}=\frac{\partial H}{\partial \eta_m}=-\frac{8\eta_m M_m}{N_m A_m}\ , \quad (19)$$

$$\frac{d\eta_m}{dz}=-\frac{\partial H}{\partial M_m}=\frac{4\eta_m^2}{N_m A_m}-\frac{N_m a}{4A_m M_m^2}-\frac{R_{mm}N_m^2 b}{8M_m^{3/2}} - \frac{\partial}{\partial M_m}V(\tau_1,\ldots,\tau_N,M_1,\ldots,M_N)\ . \quad (20)$$

Obviously, the Hamiltonian H is a first integral and besides, since it is invariant under translation, that is, under the displacement $\tau_i \rightarrow \tau_i+\tau$ for every i, the "total momentum" of the system

$$\Xi=\sum_{i=1}^{N}\xi_i\ , \quad (21)$$

is also a constant of motion.

III. PROPAGATION IN A SINGLE-MODE FIBER

In the case $N=1$ pertaining to a single-mode fiber (that is, to a single-mode polarization preserving fiber supporting one linearly polarized state), the previous equations become

$$\frac{d\tau_1}{dz}=\frac{\partial H}{\partial \xi_1}=\frac{1}{v_1}-\frac{2\xi_1}{N_1 A_1}\ , \quad (22)$$

$$\frac{d\xi_1}{dz}=-\frac{\partial H}{\partial \tau_1}=0\ , \quad (23)$$

$$\frac{dM_1}{dz}=\frac{\partial H}{\partial \eta_1}=-\frac{8\eta_1 M_1}{N_1 A_1}\ , \quad (24)$$

$$\frac{d\eta_1}{dz}=\frac{\partial H}{\partial M_1}=\frac{4\eta_1^2}{N_1 A_1}-\frac{N_1 a}{4A_1 M_1^2}-\frac{R_{11}N_1^2 b}{8M_1^{3/2}}\ , \quad (25)$$

with

$$H=\frac{\xi_1}{v_1}-\frac{\xi_1^2}{N_1 A_1}-\frac{4M_1\eta_1^2}{N_1 A_1}-\frac{N_1 a}{4A_1 M_1}-\frac{R_{11}N_1^2 b}{4M_1^{1/2}}\ . \quad (26)$$

Equations (22) and (23) immediately yield

$$\frac{d}{dz}\frac{d\tau_1}{dz}=\frac{d}{dz}\left[\frac{1}{v_1}-\frac{2\xi_1}{N_1 A_1}\right]=0\ , \quad (27)$$

so that τ_1 and ξ_1 can be expressed in terms of their values at the fiber input $z=0$. This fact allows us to study separately the behavior of the dynamical system described by the variables M_1 and η_1 in terms of the reduced Hamiltonian $H_0=T_0+V_0$, with

$$T_0=-\frac{4M_1\eta_1^2}{N_1 A_1},\quad V_0=-\frac{N_1 a}{4A_1 M_1}-\frac{R_{11}N_1^2 b}{4M_1^{1/2}}\ . \quad (28)$$

The potential V_0 consists, in the case of negative group-velocity dispersion ($A_1<0$), of a repulsive part and of an attractive tail associated with the nonlinear interaction (see Fig. 1), the corresponding trajectories $H_0=E$ in the phase space η_1, M_1 being reported in Fig. 2.

The minimum value E_m of E is achieved for $\eta_1=0$ and $M_1=\overline{M}_1=[2(a/b)/R_{11}A_1N_1]^2$, where $\overline{M}_1$ is the value of M_1 at which the potential has a minimum (see Fig. 1). If $E=E_m$, then the solution reads

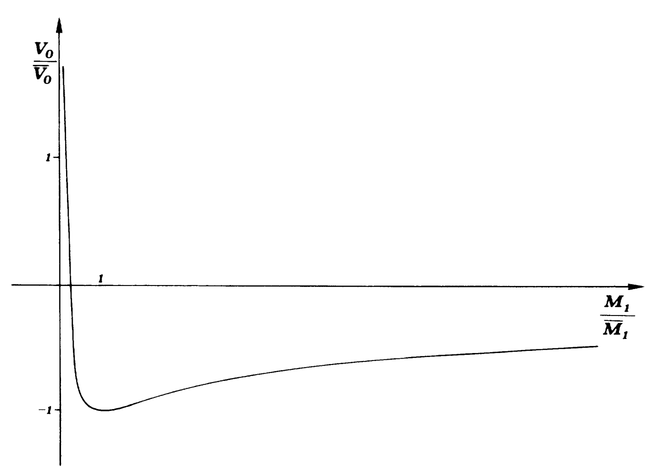

FIG. 1. Typical behavior of $V_0/\overline{V}_0$ as a function of $M_1/\overline{M}_1$, where $\overline{V}_0=-V_0(\overline{M}_1)$.

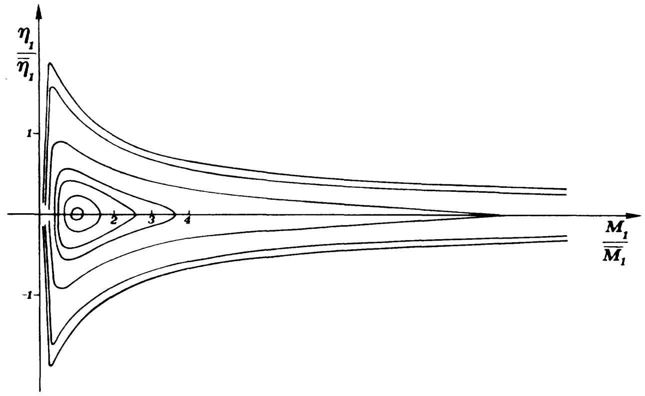

FIG. 2. Phase-space orbits for the Hamiltonian $H_0/\bar{V}_0$ in $\eta_1/\bar{\eta}_1$, $M_1/\bar{M}_1$ space, where $\bar{\eta}_1$ is implicitly defined by the relation $H_0(\bar{\eta}_1,\bar{M}_1)=0$.

$$M_1(z)=\bar{M}_1, \quad \eta_1(z)=0 , \tag{29}$$

which describes a distortionless propagation of the pulse, that is

$$|\Phi_1(z,t)|^2=\frac{N_1}{\bar{\sigma}_1}F\left[\frac{t-z/v_1}{\bar{\sigma}_1}\right] , \tag{30}$$

where

$$\bar{\sigma}_1=2\frac{a}{b}\frac{1}{R_{11}|A_1|N_1} . \tag{31}$$

If we choose for F a hyperbolic-secant shape, that is, taking into account Eqs. (13),

$$F(x)=(\pi/4\sqrt{3})\mathrm{sech}^2(\pi x/2\sqrt{3}) , \tag{32}$$

then $a=\pi^2/36$, $b=\sqrt{3}\pi/18$, and Eq. (31) reproduces exactly the condition for the existence of the fundamental soliton.[1]

According to our approach, pulses of arbitrary shape F are able to propagate as distortionless solitons, whenever Eq. (31) is fulfilled. This fact follows from the intrinsic limitation of our description based on a finite number of parameters, which does not allow us to recover the result according to which pulses of suitable area give rise to asymptotically emerging solitons, after radiating the excess energy.

It is interesting to evaluate the small-oscillation period z_0 of H_0, corresponding to harmonic oscillations around the minimum of the potential. Its expression can be easily found and reads

$$z_0=(|A_1|\bar{M}_1)/\sqrt{a}=4(a^{3/2}/b^2)(|A_1|R_{11}^2N_1^2)^{-1} . \tag{33}$$

In the case of the hyperbolic-secant shape it reduces to

$$z_0=6|A_1|\bar{\sigma}_1^2/\pi , \tag{34}$$

which is identical with the so-called soliton period, that is, the fiber length after which higher-order solitons reproduce their initial shape.[3]

IV. PROPAGATION IN A TWO-MODE FIBER

In the case $N=2$, we are dealing with a dynamical system possessing four degrees of freedom and it is no longer possible to obtain simple analytical results, as in the case $N=1$. We are, however, able, under simplifying hypotheses, to draw some general conclusions. More precisely, let us assume $F(x)$ to possess a Gaussian shape, that is, after imposing Eqs. (19),

$$F(x)=\frac{1}{(2\pi)^{1/2}}e^{-x^2/2} . \tag{35}$$

With this choice, the intermodal potential can be explicitly evaluated thus getting [see Eq. (15)]

$$V(\{\tau\},\{M\})=-\frac{1}{(2\pi)^{1/2}}\frac{R_{12}N_1N_2}{(M_1+M_2)^{1/2}} \times e^{(-1/2)(\tau_1-\tau_2)^2/(M_1+M_2)} , \tag{36}$$

while [see Eq. (16)] $a=1$ and $b=1/2\sqrt{\pi}$.

We can now consider Eqs. (17)–(20) and assume $N_1=N_2$ (equal energy per mode); furthermore, we have in practice, for a single-mode birefringent fiber, $A_1\cong A_2$ and $R_{11}\cong R_{22}$, $R_{12}\cong R_{11}/3$. If we neglect, as a first approximation, the z evolution of the variables M_1 and M_2 and assume [see Eq. (29)]

$$M_1(z)=M_2(z)=M_1(0)=M_2(0)=\bar{M}_1=\bar{M}_2 , \tag{37}$$

our system is completely described by Eqs. (17) and (18). Starting from them, it is straightforward to derive, under the preceding hypotheses,

$$\frac{d}{dz}(\xi_1+\xi_2)=0 , \tag{38}$$

$$\frac{d^2}{dz^2}(\tau_1+\tau_2)=0 , \tag{39}$$

$$\frac{d}{dz}(\xi_1-\xi_2)=-2\frac{\partial}{\partial\tau_1}V(\{\tau\},\{M\}) , \tag{40}$$

$$\frac{d^2}{dz^2}(\tau_1-\tau_2)=\frac{4}{N_1A_1}\frac{\partial}{\partial\tau_1}V(\{\tau\},\{M\}) . \tag{41}$$

In particular, by recalling Eqs. (36) and (37), Eq. (41) furnishes the evolution equation of the average mutual modal delay $y\equiv\tau_1-\tau_2$ in the form

$$\frac{d^2y}{dz^2}-\mathrm{sgn}(A_1)\frac{4y}{3z_0^2}e^{-|A_1|y^2/(4z_0)}=0 , \tag{42}$$

where $z_0=A_1\bar{M}_1$ [see Eq. (33)] and $\mathrm{sgn}(x)$ is the sign of the real number x. After introducing the adimensional variables

$$\chi=\tfrac{1}{2}(|A_1|/z_0)^{1/2}y, \quad \zeta=(2/\sqrt{3})(z/z_0) , \tag{43}$$

Eq. (42) can be rewritten as (we refer for definiteness to the case of negative group-velocity dispersion $A_1<0$)

$$\frac{d^2\chi}{d\zeta^2}+\chi e^{-\chi^2}=0 . \tag{44}$$

It can be also recast, after imposing the boundary conditions

$$y(0)=0, \quad \left.\frac{dy}{dz}\right|_{z=0} = \frac{1}{v_1} - \frac{1}{v_2} \equiv \frac{1}{v} , \tag{45}$$

that is

$$\chi(0)=0, \quad \left.\frac{d\chi}{d\zeta}\right|_{\zeta=0} = \delta , \tag{46}$$

with $\delta=\sqrt{3}(z_0 \mid A_1 \mid)^{1/2}/4v$, in the form

$$(d\chi/d\zeta)^2 = \delta^2 - 1 + e^{-\chi^2} . \tag{47}$$

By inspecting Eq. (47), it is not difficult to show that the temporal interval y between the two pulses traveling on modes 1 and 2 increases without bound or possesses a periodic behavior as a function of the fiber length z according to whether $\delta^2 > 1$ or $\delta^2 < 1$, respectively [a fact which provides a simple analytical confirm of the results obtained numerically in Ref. (10)]. In practice, the fact that z_0 scales as the inverse of the pulse peak power implies the existence of a critical injected power above which the splitting between the two pulses started together at the fiber input never exceeds a certain value $\bar{\chi}$. More precisely, for $\delta^2 < 1$, the period ζ_p of the solution of Eq. (47) is given by the expression

$$\zeta_p = 4 \int_0^{\bar{\chi}} (\delta^2 - 1 + e^{-\chi^2})^{-1/2} d\chi , \tag{48}$$

where $\bar{\chi}$ is the (positive) root of the equation $\delta^2 - 1 + \exp(-\chi^2) = 0$, that is,

$$\bar{\chi} = \{ -\ln(1-\delta^2) \}^{1/2} . \tag{49}$$

For $\delta^2 \ll 1$ the preceding integration can be explicitly performed, thus getting $\zeta_p = 2\pi$, while a numerical evaluation of the behavior of ζ_p as a function of δ^2 is reported in Fig. 3 (ζ_p obviously diverges when $\delta^2 \to 1$). Typical values of the physical parameters involved are $A_1 = 10^{23}$ km/sec^2 (around $\lambda = 1.3$ μm) and z_0 (km)$\cong 1/2P$ (where P is the peak power launched into the fiber, expressed in watts) for a core size of about 4 μm, while $1/v$ ranges in the interval $10^{-9}-10^{-12}$ sec/km.

It is worthwhile to note that the limit $\delta \to 0$ cannot be considered in the frame of our approach, since this would imply a low-birefringence fiber, while the set of Eqs. (3) correctly describes only the case of a high-birefringence fiber (see, e.g., Ref. 11). However, the analytic description provided by Eqs. (3) still applies to nonlinear propagation in a highly twisted single-mode birefringent fiber, provided that Φ_1 and Φ_2 are substituted by the corresponding right and left circularly polarized states Φ_r and Φ_l and $R_{ii} \to (2/3)R_{ii}$, $R_{12} \to 2R_{12}$;[12] in this case, in fact, $1/v \to 0$ and the case $\delta = 0$ is included in our treatment.

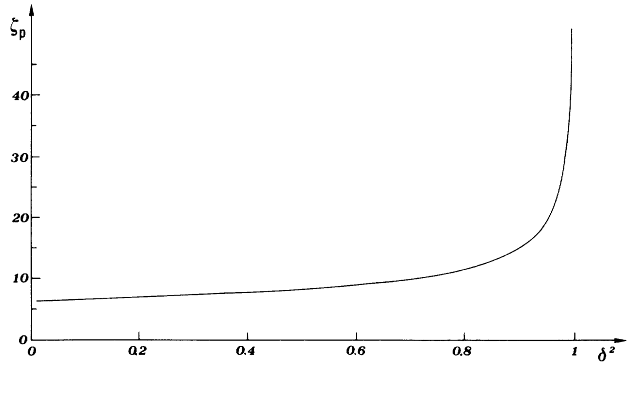

FIG. 3. Plot of ζ_p as a function of δ^2.

V. CONCLUSION

The problem of nonlinear propagation in an optical fiber (in the presence of modal and chromatic dispersion and optical Kerr effect), which requires the solution (most often numerical, see, e.g., Refs. 10 and 13) of a nonlinear set of partial differential equations, has been approached through a simplified formalism which greatly reduces its complexity. It basically consists in describing the pulse propagating in each mode by means of four quantities, connected in a simple way to its amplitude and phase, which can be interpreted as conjugate variables of a suitable Hamiltonian system. The associated set of equations proves manageable enough to provide in some case exact solutions and to allow the drawing of general analytical conclusions on the nonlinear pulse evolution.

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APPENDIX

By inserting Eq. (7) into Eq. (3) and taking the real and imaginary part of the resulting equation, we obtain, respectively,

$$\frac{\partial |\Phi_m|}{\partial z} + \frac{1}{v_m}\frac{\partial |\Phi_m|}{\partial t} + \frac{1}{A_m}\frac{\partial |\Phi_m|}{\partial t}\frac{\partial \psi_m}{\partial t} + \frac{1}{2A_m}|\Phi_m|\frac{\partial^2 \psi_m}{\partial t^2} = 0 , \tag{A1}$$

$$|\Phi_m|\frac{\partial \psi_m}{\partial z} + \frac{1}{v_m}|\Phi_m|\frac{\partial \psi_m}{\partial t} + \frac{1}{2A_m}|\Phi_m|\left[\frac{\partial \psi_m}{\partial t}\right]^2 - \frac{1}{2A_m}\frac{\partial^2 |\Phi_m|}{\partial t^2} + 2|\Phi_m|\left[\frac{R_{mm}}{2}|\Phi_m|^2 + V_m(\{\Phi\})\right] = 0 , \tag{A2}$$

where we have set

$$V_m(\{\Phi\}) = \sum_{\substack{n=1 \\ n\neq m}}^{N} \sum_{m=1}^{N} R_{mn} |\Phi_n|^2 . \tag{A3}$$

After multiplying Eqs. (A1) and (A2) by $|\Phi_m|$ we have

$$I_m^{(1)}(z,t) \equiv \frac{\partial |\Phi_m|^2}{\partial z} + \frac{1}{v_m}\frac{\partial |\Phi_m|^2}{\partial t} + \frac{1}{A_m}\frac{\partial}{\partial t}\left[|\Phi_m|^2 \frac{\partial \psi_m}{\partial t}\right] = 0 \tag{A4}$$

and

$$I_m^{(2)}(z,t) \equiv |\Phi_m|^2 \frac{\partial \psi_m}{\partial z} + \frac{1}{v_m}|\Phi_m|^2\frac{\partial \psi_m}{\partial t} + \frac{1}{2A_m}|\Phi_m|^2\left[\frac{\partial \psi_m}{\partial t}\right]^2 - \frac{1}{2A_m}|\Phi_m|\frac{\partial^2 |\Phi_m|}{\partial t^2} + 2|\Phi_m|^2\left[\frac{R_{mm}}{2}|\Phi_m|^2 + V_m(\{\Phi\})\right] = 0 , \tag{A5}$$

while multiplying Eq. (A2) by $\partial |\Phi_m| / \partial t$ we get

$$I_m^{(3)}(z,t) \equiv \frac{\partial |\Phi_m|^2}{\partial t}\frac{\partial \psi_m}{\partial z} + \frac{1}{v_m}\frac{\partial |\Phi_m|^2}{\partial t}\frac{\partial \psi_m}{\partial t} + \frac{1}{2A_m}\frac{\partial |\Phi_m|^2}{\partial t}\left[\frac{\partial \psi_m}{\partial t}\right]^2 - \frac{1}{A_m}\frac{\partial |\Phi_m|}{\partial t}\frac{\partial^2 |\Phi_m|}{\partial t^2} + 2\frac{\partial |\Phi_m|^2}{\partial t}\left[\frac{R_{mm}}{2}|\Phi_m|^2 + V_m(\{\Phi\})\right] = 0 . \tag{A6}$$

Equation (A4) furnishes

$$\int_{-\infty}^{+\infty} dt\, I_m^{(1)}(z,t) = 0 , \tag{A7}$$

which, provided the vanishing of the $|\Phi_m|$'s for $t \rightarrow \pm\infty$ is sufficiently rapid, yields

$$\frac{d}{dz}\int_{-\infty}^{+\infty} dt\, |\Phi_m(z,t)|^2 = \frac{d}{dz}N_m = 0 . \tag{A8}$$

Equation (A8) states that the normalization coefficients N_m are independent from z and expresses energy conservation in each mode. By multiplying Eq. (A4) by t and integrating, we have

$$\int_{-\infty}^{+\infty} dt\, t I_m^{(1)}(z,t) = 0 , \tag{A9}$$

from which it is not difficult to derive, after recalling Eq. (11),

$$\frac{d}{dz}\tau_m(z) = \frac{1}{v_m} + \frac{S_m^{(0)}(z)}{A_m} , \tag{A10}$$

where we have defined

$$S_m^{(0)}(z) = \frac{1}{N_m}\int_{-\infty}^{+\infty} dt\, |\Phi_m|^2 \frac{\partial \psi_m}{\partial t} = -\frac{2\xi_m}{N_m} , \tag{A11}$$

the last equality following from Eq. (9). In a similar way, starting from

$$\int_{-\infty}^{+\infty} dt (t-\tau_m)^2 I_m^{(1)}(z,t) = 0 , \tag{A12}$$

it is possible to obtain, recalling Eq. (12),

$$\frac{d}{dz}M_m(z) = \frac{2}{A_m}S_m^{(1)}(z) , \tag{A13}$$

where

$$S_m^{(1)}(z) = \frac{1}{N_m}\int_{-\infty}^{+\infty} dt (t-\tau_m)|\Phi_m|^2\frac{\partial \psi_m}{\partial t} = -\frac{4\eta_m M_m}{N_m} , \tag{A14}$$

the last equality following from Eq. (9). In order to evaluate $d\xi_m/dz$ and $d\eta_m/dz$, we consider Eqs. (A4), (A5), and (A6) and write

$$\int_{-\infty}^{+\infty} dt \left[I_m^{(3)}(z,t) - I_m^{(1)}(z,t)\frac{\partial \psi_m}{\partial t}\right] = 0 \tag{A15}$$

and

$$\int_{-\infty}^{+\infty} dt \left[I_m^{(3)}(z,t) - I_m^{(1)}(z,t)\frac{\partial \psi_m}{\partial t}\right](t-\tau_m) + \int_{-\infty}^{+\infty} dt\, I_m^{(2)}(z,t) = 0 . \tag{A16}$$

From Eqs. (A15) and (A16) it, respectively, follows, after some tedious, if not straightforward, algebra,

$$\frac{dS_m^{(0)}}{dz} = -\frac{2}{N_m}\int_{-\infty}^{+\infty} dt\, |\Phi_m|^2 \frac{\partial}{\partial t}V_m(\{\Phi\}) \tag{A17}$$

and

$$\frac{dS_m^{(1)}}{dz} = \frac{1}{v_m}S_m^{(0)} - \frac{d\tau_m}{dz}S_m^{(0)} + \frac{1}{N_m A_m}\int_{-\infty}^{+\infty}\left[\frac{\partial |\Phi_m|}{\partial t}\right]^2 + \frac{R_{mm}}{2N_m}\int_{-\infty}^{+\infty} dt\, |\Phi_m|^4 + \frac{1}{N_m A_m}\int_{-\infty}^{+\infty} dt\, |\Phi_m|^2\left[\frac{\partial \psi_m}{\partial t}\right]^2 - \frac{2}{N_m}\int_{-\infty}^{+\infty} dt\, |\Phi_m|^2 (t-\tau_m)\frac{\partial V_m(\{\Phi\})}{\partial t} . \tag{A18}$$

Finally, by taking the z derivative of Eqs. (A11) and (A14) and keeping in mind Eqs. (A10), (A13), (A17), and (A18), it is possible to deduce two equations for $d\xi_m/dz$ and $d\eta_m/dz$ which, together with [see Eqs. (A10) and (A11) and Eqs. (A13) and (A14)]

$$\frac{d}{dz}\tau_m = \frac{1}{v_m} + \frac{S_m^{(0)}}{A_m} = \frac{1}{v_m} - \frac{2\xi_m}{N_m A_m} \tag{A19}$$

and

$$\frac{dM_m}{dz} = \frac{2}{A_m}S_m^{(1)} = -\frac{8\eta_m M_m}{N_m A_m} , \tag{A20}$$

reproduce Eqs. (17)–(20) of the paper.

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