# Shifted large-$N$ expansion for the energy levels of relativistic particles

M. M. Panja and R. Dutt

*Department of Physics, University of Visva-Bharati, Santiniketan 731 235, West Bengal, India*

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The shifted large-$N$ expansion method, which was developed to obtain accurate energy eigenvalues for nonrelativistic-potential problems, has been extended to deal with relativistic particle (with or without spin) bound in a spherically symmetric potential. The calculations are carried out for any arbitrary quantum state using expansion in terms of a parameter $1/\bar{k}$, where $\bar{k}$ contains the dimension of the space $N$ and the so-called shift parameter. Similar to the work of T. Imbo, A. Pagnamenta, and U. Sukhatme [Phys. Rev. D **29**, 1669 (1984)] we suggest determination of the shift parameter in such a way that the exact analytic result for the nonrelativistic Coulomb binding energy is restored. As a consequence of this choice, we obtain also a highly convergent expansion for the relativistic part of the energy eigenvalue. Although the formalism is developed for spin-zero and spin-$\frac{1}{2}$ particles in any arbitrary spherically symmetric potential, it is illustrated for the Coulomb potential as a special case. Our results are consistently better than those previously obtained by using the unshifted $1/N$ expansion technique. The shifted $1/N$ expansion is seen to be applicable to a much wider class of relativistic potentials which may have applications in atomic processes. A few interesting aspects of our approach are briefly discussed.

## I. INTRODUCTION

The application of the large-$N$ expansion technique, where $N$ is the number of spatial dimensions, has proved itself to be quite rewarding in a variety of potential problems in the context of Schrödinger quantum mechanics.[1–12] The underlying idea of this method is that if $N$ is the dimensionality of the theory, then the theory may have large-$N$ generalization which can be solved explicitly in the limit $N \rightarrow \infty$. For spherically symmetric potentials, such a solution allows one to calculate physical observables such as the energy eigenvalues and eigenfunctions in the original $N=3$ dimensions by expanding them systematically in powers of $1/k$ where $k=N+2l$. However, in certain cases the expansion suffers from slow convergence particularly for the $s$ states.

Recently, Sukhatme and co-workers[8,9] proposed a modification of this method, called the shifted $1/N$ expansion, which considerably improves the analytic structure and convergence of the perturbation series for the energy eigenvalue. The modification consists of using $1/\bar{k}$ as an expansion parameter, where $\bar{k}=(k-a)$ and $a$ is a suitable shift. The shift parameter is chosen in such a way that the first-order correction to the large-$\bar{k}$ energy vanishes. This simple modification gives rise to dramatic consequences; not only does it lead to the exact energy eigenvalues and eigenfunctions for the harmonic oscillator and Coulomb potential in the leading order of the $1/\bar{k}$ series (higher-order corrections vanish identically), but it also yields very accurate analytic results effectively putting no constraint on the potential or the quantum numbers involved in the problem. The shifted large-$N$ expansion method has recently been applied to a large number of potentials[9–12] with remarkable success.

The objective of this paper is extend the shifted large-$N$ formalism to determine the energy levels of relativistic particles trapped in a spherically symmetric potential. To our knowledge, only a few groups[13–15] have so far applied unshifted $1/N$ expansion to study the relativistic bound-state energies of spin-zero and spin-$\frac{1}{2}$ particles. These authors found that the relativistic correction to the nonrelativistic limit is nonleading in $1/N$ expansion for the energy eigenvalues. Furthermore, the rate of convergence of the expansion is very slow for the relativistic part of the energy eigenvalue as compared to the same for the nonrelativistic part.

Our purpose is to show that these deficiencies of the unshifted $1/N$ method may be removed to a large extent by employing the shifted large-$N$ expansion technique to the relativistic wave equations. Although it is not *a priori* guaranteed that this method, which gives high accuracy for the nonrelativistic binding energies, will also yield a similar effect for the relativistic part of the problem, it has been shown explicitly that appropriate choice of the shift parameter gives faster convergence to the series representing the relativistic correction terms. In this paper we have developed the shifted large-$N$ method to determine the energy eigenvalues of both Klein-Gordon (KG) and Dirac equations for radially symmetric potentials assuming the rest energy to be large compared to the relativistic correction. Though the procedure applies to any radially symmetric potential, explicit calculations have been presented for the Coulomb problem as a test case. The main aspect of our approach is that whether one starts with the KG or the Dirac equation, it is possible to convert it to a Schrödinger-like equation to which one may apply the shifted large-$N$ technique of Sukhatme and co-workers in a straightforward manner. In Sec. II, we discuss the KG problem for a general spherically symmetric potential. For the Coulomb case, we give the explicit results and restore the exact analytic expression for the nonrelativistic Coulomb binding energy and a highly convergent expansion for the relativistic correction of order $1/c^2$. The relative improvement of the relativistic part of the binding energy in comparison to that obtained from the unshifted method[15] is also discussed. In Sec. III, we discuss our method in the context of the Dirac equation and give approximate analytic results for the Dirac-Coulomb problem. In the concluding section, we mention briefly the scope of further work in this direction for relativistic screened Coulomb potentials which may have relevance in atomic physics.

## II. KG PROBLEM

### A. Shifted large-$N$ expansion for spin-zro particle

In this section, we formulate the shifted large-$N$ expansion for the relativistic motion of a spin-zero particle bound in spherically symmetric potential $V(r)$. For such a particle of rest mass $m$ and total energy $E$, the radial part of the KG equation in $N$-dimensional hyperspherical coordinates is[14]

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\hbar^2(k-1)(k-3)}{8mr^2}-\frac{1}{2mc^2}\{[E-V(r)]^2-m^2c^4\}\right]u(r)=0\ , \tag{1}$$

where $k=N+2l$ and $u(r)$ is the reduced radial wave function. Introducing a shift parameter $a$ through the relation[9]

$$\bar{k}=k-a\ , \tag{2}$$

Eq. (1) becomes

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\hbar^2\bar{k}^2}{8mr^2}\left[1-\frac{1-a}{\bar{k}}\right]\left[1-\frac{3-a}{\bar{k}}\right]-\frac{1}{2mc^2}\{[E-V(r)]^2-m^2c^4\}\right]u(r)=0\ . \tag{3}$$

We shall determine a convenient choice for the shift parameter later. Let us now discuss the limit $N\to\infty$ (i.e., $\bar{k}\to\infty$) of Eq. (3). In order to get a sensible result in the leading order in $1/\bar{k}$ expansion, Eq. (3) can be written as

$$\left[-\frac{1}{2m\bar{k}^2}\frac{d^2}{dr^2}+\frac{\hbar^2}{8mr^2}\left[1-\frac{1-a}{\bar{k}}\right]\left[1-\frac{3-a}{\bar{k}}\right]-\frac{1}{2mc^2Q}\{[E-V(r)]^2-m^2c^4\}\right]u(r)=0\ , \tag{4}$$

where $Q$ is a constant which rescales the potential and the rest-mass energy and will be set equal to $\bar{k}^2$ at the end of the calculation.[9]

In the large-$\bar{k}$ limit, the particle becomes effectively localized and one may represent

$$u(r)\sim\delta(r-r_0)\ . \tag{5}$$

This gives the leading-order energy

$$E_0=V(r_0)+mc^2\left[1+\frac{\hbar^2Q}{4m^2c^2r_0^2}\right]^{1/2}, \tag{6}$$

where $r_0$ satisfies the equation

$$r_0^3\frac{dV}{dr}\bigg|_{r=r_0}\left[1+\frac{\hbar^2Q}{4m^2c^2r_0^2}\right]^{1/2}=\frac{\hbar^2Q}{4m}\ . \tag{7}$$

To determine higher-order corrections to the energy eigenvalues in a manner similar to that followed in the case of nonrelativistic potentials, one is required to convert Eq. (3) into a Schrödinger-like equation. We then write Eq. (3) as

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\hbar^2\bar{k}^2}{8mr^2}\left[1-\frac{1-a}{\bar{k}}\right]\left[1-\frac{3-a}{\bar{k}}\right]-\frac{1}{2mc^2}\{[E_a-V(r)]^2-m^2c^4\}+\frac{1}{mc^2}(E-E_a)[V(r)-V(r_0)]\right]u(r)$$
$$=\frac{1}{2mc^2}\{[E-V(r_0)]^2-[E_a-V(r_0)]^2\}u(r)\ , \tag{8}$$

where $E_a$ is some approximate solution for $E$. For our purpose, we shall replace $E_a$ by $E_0$ given in Eq. (6) and neglect the term $(E-E_0)[V(r)-V(r_0)]/mc^2$, assuming it to be small. With this approximation, Eq. (8) becomes the Schrödinger-like equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\hbar^2\bar{k}^2}{8mr^2}\left[1-\frac{1-a}{\bar{k}}\right]\left[1-\frac{3-a}{\bar{k}}\right]+U(r)\right]u(r)=\mathcal{E}u(r)\ , \tag{9}$$

in which

$$U(r)=-\frac{1}{2mc^2}\{[E-V(r)]^2-m^2c^4\}\ , \tag{10}$$

$$\mathcal{E}=\frac{1}{2mc^2}\{[E-V(r_0)]^2-[E_0-V(r_0)]^2\}\ . \tag{11}$$

Clearly, $U(r)$ and $\mathcal{E}$ play the role corresponding to the potential and the bound-state energy, respectively, in the nonrelativistic problem, and so one may apply the previous $1/\bar{k}$ expansion scheme[9] directly. However, this calculation is bound to be complicated and lengthy due to the complex structure of $U(r)$ and $\mathcal{E}$ as given in (10) and (11). Following the steps of Ref. 9, we change the variable

$$r=r_0+\frac{r_0}{(\bar{k})^{1/2}}y\ . \tag{12}$$

Substituting (12) into Eq. (9) and expanding about $y=0$ in terms of $y$ one gets

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}+\tfrac{1}{2}m\omega^2y^2+\varepsilon_0+\frac{1}{(\bar{k})^{1/2}}(\varepsilon_1 y+\varepsilon_3 y^3)+\frac{1}{\bar{k}}(\varepsilon_2 y^2+\varepsilon_4 y^4)+\frac{1}{\bar{k}^{3/2}}(\delta_1 y+\delta_3 y^3+\delta_5 y^5)+\frac{1}{\bar{k}^2}(\delta_2 y^2+\delta_4 y^4+\delta_6 y^6)+\cdots\right]\bar{u}(y)=\lambda\bar{u}(y)\ , \tag{13}$$

where

$$\omega=\left[\frac{3\hbar^2}{4m^2}+\frac{r_0^4U''(r_0)}{mQ}\right]^{1/2} \tag{14}$$

and $\varepsilon_j,\delta_j$ are the same as given in Ref. 9 except that $V(r)$ and its derivatives have to be replaced by our $U(r)$ and its derivatives, respectively. From Eq. (13) it is clear that our original three-dimensional problem has been converted into a nonrelativistic equation for the one-dimensional anharmonic-oscillator problem. Using standard perturbative results, we obtain from Eq. (13)

$$\mathcal{E}=\frac{a^{(0)}\bar{k}^2}{r_0^2}+\frac{\Lambda\bar{k}}{r_0^2}+\frac{1}{r_0^2}(b^{(0)}+b^{(1)}+b^{(2)})+\frac{1}{r_0^2\bar{k}}(c^{(1)}+c^{(2)}+c^{(3)}+c^{(4)})+\cdots\ , \tag{15}$$

in which

$$a^{(0)}=\frac{\hbar^2}{8m}+\frac{r_0^2U(r_0)}{Q}\ , \tag{16a}$$

$$\Lambda=-\frac{(2-a)\hbar^2}{4m}+(1+2n_r)\frac{\hbar\omega}{2}\ , \tag{16b}$$

$$b^{(0)}=\frac{\hbar^2}{8m}(1-a)(3-a)\ , \tag{16c}$$

$$b^{(1)}=(1+2n_r)\tilde{\varepsilon}_2+3(1+2n_r+2n_r^2)\tilde{\varepsilon}_4\ , \tag{16d}$$

$$b^{(2)}=-\frac{1}{\hbar\omega}[\tilde{\varepsilon}_1^2+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3+(11+30n_r+30n_r^2)\tilde{\varepsilon}_3^2]\ , \tag{16e}$$

$$c^{(1)}=(1+2n_r)\tilde{\delta}_2+3(1+2n_r+2n_r^2)\tilde{\delta}_4+5(3+8n_r+6n_r^2+4n_r^3)\tilde{\delta}_6\ , \tag{16f}$$

$$\begin{aligned}c^{(2)}=-\frac{1}{\hbar\omega}[&(1+2n_r)\tilde{\varepsilon}_2^2+12(1+2n_r+2n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_4+2(21+59n_r+51n_r^2+34n_r^3)\tilde{\varepsilon}_4^2+2\tilde{\varepsilon}_1\tilde{\delta}_1\\&+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\delta}_3+30(1+2n_r+2n_r^2)\tilde{\varepsilon}_1\tilde{\delta}_5+6(1+2n_r)\tilde{\varepsilon}_3\tilde{\delta}_1\\&+2(11+30n_r+30n_r^2)\tilde{\varepsilon}_3\tilde{\delta}_3+10(13+40n_r+42n_r^2+28n_r^3)\tilde{\varepsilon}_3\tilde{\delta}_5]\ ,\end{aligned} \tag{16g}$$

$$\begin{aligned}c^{(3)}=\frac{1}{(\hbar\omega)^2}[&4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2+36(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3+8(11+30n_r+30n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2+24(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_4\\&+8(31+78n_r+78n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\tilde{\varepsilon}_4+12(57+189n_r+225n_r^2+150n_r^3)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4]\ ,\end{aligned} \tag{16h}$$

$$c^{(4)}=-\frac{1}{(\hbar\omega)^3}[8\tilde{\varepsilon}_1^3\tilde{\varepsilon}_3+108(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2+48(11+30n_r+30n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3+30(31+109n_r+141n_r^2+94n_r^3)\tilde{\varepsilon}_3^4]\ , \tag{16i}$$

and

$$\tilde{\varepsilon}_j=\frac{\varepsilon_j}{\left[\frac{2m\omega}{\hbar}\right]^{j/2}},\quad \tilde{\delta}_j=\frac{\delta_j}{\left[\frac{2m\omega}{\hbar}\right]^{j/2}} . \tag{17}$$

Here $n_r$ stands for the radial quantum number. From (15), it is evident that the leading contribution to $\mathcal{E}$ is of the order $\bar{k}^2$. The next contribution being of the order $\bar{k}$ is given by

$$\frac{\Lambda\bar{k}}{r_0^2}=\frac{\bar{k}}{r_0^2}\left[-\frac{(2-a)\hbar^2}{4m}+(1+2n_r)\frac{\hbar\omega}{2}\right] . \tag{18}$$

It may be pointed out that our expression (18) is identical to that obtained in Ref. 9. However, in our case, $\omega$ contains nonrelativistic as well as relativistic contributions as $U(r)$ in (10) contains terms of order $1/c^2$. Expanding all quantities in powers of $1/c^2$, it is possible to split

$$\omega=\omega_{\rm Nr}+\omega_R . \tag{19}$$

Here $\omega_{\rm NR}$ stands for the nonrelativistic part independent of $c$ and $\omega_R$ contains terms of order $1/c^2$ and higher.

We propose to determine the shift parameter $a$ in such a manner that the nonrelativistic part of (18) vanishes. This gives

$$a=2-2(2n_r+1)\frac{m\omega_{\rm NR}}{\hbar} . \tag{20}$$

Consequently, the term of order $\bar{k}$ in (15) picks up only a contribution for the relativistic part of the bound-state energy. From Eqs. (11), (15), and (18)–(20) we get finally

$$E=V(r_0)+mc^2\left[1+\frac{\hbar^2\bar{k}^2}{4m^2c^2r_0^2}+\frac{(1+2n_r)\hbar\omega_R}{mc^2r_0^2}\bar{k}+\frac{2(b^{(0)}+b^{(1)}+b^{(2)})}{mc^2r_0^2}+\frac{2(c^{(1)}+c^{(2)}+c^{(3)}+c^{(4)})}{mc^2r_0^2}\frac{1}{\bar{k}}+\cdots\right]^{1/2} . \tag{21}$$

### B. Application to the Coulomb problem

Here we intend to show explicitly the expression for the $1/\bar{k}$ expansion for the binding energy of a relativistic spin-zero particle bounded by an attractive Coulomb potential

$$V(r)=-\beta/r,\quad \beta=Ze^2 . \tag{22}$$

Using (22) and expanding all terms up to order $1/c^2$, we get from Eqs. (6)–(20),

$$r_0=\frac{\hbar^2Q}{4m\beta}\left[1-\frac{4\beta^2}{Q\hbar^2c^2}\right]^{1/2}\simeq\frac{\hbar^2Q}{4m\beta}\left[1-\frac{2\beta^2}{Q\hbar^2c^2}\right] , \tag{23}$$

$$\omega=\frac{\hbar}{2m}\left[1-\frac{4\beta^2}{Q\hbar^2c^2}\right]^{1/2}\simeq\frac{\hbar}{2m}\left[1-\frac{2\beta^2}{Q\hbar^2c^2}\right] , \tag{24}$$

$$a=1-2n_r ,$$

$$\begin{aligned}
&\tilde{\varepsilon}_1=\varepsilon_1\left[1+\frac{\beta^2}{Q\hbar^2c^2}\right],\quad \varepsilon_1=\frac{\hbar^2}{2m}(1+2n_r) ,\\
&\tilde{\varepsilon}_2=\varepsilon_2\left[1+\frac{2\beta^2}{Q\hbar^2c^2}\right],\quad \varepsilon_2=-\frac{3\hbar^2}{4m}(1+2n_r) ,\\
&\tilde{\varepsilon}_3=\varepsilon_3\left[1-\frac{\beta^2}{Q\hbar^2c^2}\right],\quad \varepsilon_3=-\frac{\hbar^2}{4m} ,\\
&\tilde{\varepsilon}_4=\varepsilon_4,\quad \varepsilon_4=\frac{3\hbar^2}{8m} ,\\
&\tilde{\delta}_1=\delta_1\left[1+\frac{\beta^2}{Q\hbar^2c^2}\right],\quad \delta_1=-\frac{\hbar^2}{m}n_r(1+n_r) ,\\
&\tilde{\delta}_2=\delta_2\left[1+\frac{2\beta^2}{Q\hbar^2c^2}\right],\quad \delta_2=\frac{3\hbar^2}{2m}n_r(1+n_r) ,\\
&\tilde{\delta}_3=\delta_3\left[1+\frac{3\beta^2}{Q\hbar^2c^2}\right],\quad \delta_3=\frac{\hbar^2}{m}(1+2n_r) ,\\
&\tilde{\delta}_4=\delta_4\left[1+\frac{4\beta^2}{Q\hbar^2c^2}\right],\quad \delta_4=-\frac{5\hbar^2}{4m}(1+2n_r) ,\\
&\tilde{\delta}_5=\delta_5\left[1+\frac{\beta^2}{Q\hbar^2c^2}\right],\quad \delta_5=-\frac{\hbar^2}{2m} ,\\
&\tilde{\delta}_6=\delta_6\left[1+\frac{2\beta^2}{Q\hbar^2c^2}\right],\quad \delta_6=\frac{5\hbar^2}{8m} .
\end{aligned} \tag{25}$$

The first and the second term on the right-hand side of (24) correspond to $\omega_{\rm NR}$ and $\omega_R$, respectively, as defined in (19). Computing $a^{(0)}$, $\Lambda$, $b^{(i)}$, $c^{(i)}$, etc. by using the expressions (23)–(25), we finally obtain the total energy for a spin-zero particle

$$E=E_{\rm KG}^{\rm Coul}=mc^2-\frac{2m\beta^2}{\hbar^2\bar{k}^2}-\frac{2m\beta^4}{\hbar^4\bar{k}^4c^2}\left[1+\frac{4(1+2n_r)}{\bar{k}}+\frac{4(1+2n_r)^2}{\bar{k}^2}+\frac{4(1+2n_r)^3}{\bar{k}^3}+\cdots\right]-O\left[\frac{1}{c^4}\right] . \tag{26}$$

The second term in (26) corresponds to the nonrelativistic Coulomb binding energy and the terms in the large parentheses contribute (to the order $1/c^2$) to the relativistic binding energy.

Our analytic result (26) may be compared with the corresponding expression (for the ground state, i.e., $n_r=0$)

$$E=mc^2-\frac{2m\beta^2}{\hbar^2k^2}\left[1+\frac{2}{k}+\frac{3}{k^2}+\cdots\right]-\frac{2m\beta^4}{\hbar^4k^4c^2}\left[1+\frac{8}{k}+\frac{34}{k^2}+\cdots\right]\cdots \tag{27}$$

obtained by Chatterjee[15] using the unshifted $1/N$ expansion scheme. There are a number of distinctive features between the expressions obtained through the shifted and unshifted formalisms.

(1) For the physical space (i.e., $N=3$), we obtain

$$\bar{k}=N+2l-a=2(n_r+l+1)=2n\ , \tag{28}$$

where $n$ is the total quantum number and thus the second term of (26) yields an exact analytic result for nonrelativistic Coulomb binding energy

$$E_{\rm NR}^{\rm Coul}=-m\beta^2/2\hbar^2n^2\ . \tag{29}$$

However, in the case of unshifted expansion (27), one obtains a series for the same and it is known[1] that the first three terms cumulatively yield about 90% of the exact ground-state energy.

(2) To test the convergence of the relativistic parts of (26) and (27), we calculate numerically the relativistic correction to the binding energy for a few $n_r=0$ states and compare the values in Table I with those computed from the exact analytic expression[16]

$$E_{\rm KG}^{\rm Coul}\ [O(1/c^2)=-\frac{m\beta^4}{2\hbar^4n^4c^2}\left[\frac{n}{l+\frac{1}{2}}-\frac{3}{4}\right]\ . \tag{30}$$

TABLE I. Relativistic correction (of order $1/c^2$) to the binding energies of $n_r=0$ states for a spin-zero particle in the Coulomb potential.

| States | Binding energies in units of $(m\beta^4/\hbar^4c^2)$ Unshifted [Eq. (27)] | Shifted [Eq. (26)] | Exact [Eq. (30)] |
|---|---|---|---|
| 1$s$ | −0.183 813 | −0.562 500 | −0.625 000 |
| 2$p$ | −0.012 672 | −0.018 066 | −0.018 229 |
| 3$d$ | −0.002 363 | −0.002 772 | −0.002 779 |
| 4$f$ | −0.000 704 | −0.000 767 | −0.000 767 |

It is observed that in contrast to the unshifted values, our results are in very good agreement with the exact numerical values.

(3) One interesting feature of our expression (26) is that not only the relativistic part corresponds to a faster convergent series as compared to the corresponding one in the unshifted expansion (27), but also it leads to the exact result (30) provided one assumes that the rest of the series follows the same pattern. This may be shown more clearly: the relativistic piece in (26) may be arranged as

$$-\frac{2m\beta^4}{\hbar^4\bar{k}^4c^2}\left[1+\frac{4(1+2n_r)}{\bar{k}}\left[1+\frac{1+2n_r}{\bar{k}}+\frac{(1+2n_r)^2}{\bar{k}^2}+\cdots\right]\right]\approx-\frac{2m\beta^4}{\hbar^4\bar{k}^4c^2}\left[1+\frac{4(1+2n_r)}{\bar{k}-(1+2n_r)}\right]$$
$$=-\frac{8m\beta^4}{\hbar^4\bar{k}^4c^2}\left[\frac{(N-3)+2n}{(N-3)+2l+1}-\frac{3}{4}\right]\ .$$

For the physical $N=3$ space, one retrieves (30) by using $\bar{k}=2n$ as given in (28).

## III. DIRAC-COULOMB PROBLEM

Our objective in this section is to convert the Dirac equation for a class of spherically symmetric potential to an equivalent KG-like equation permitting relativistic calculations utilizing the procedure developed in Sec. II. For illustration, we shall then treat the Coulomb problem as a special case. We consider the Dirac equation for a particle of rest mass $m$ moving in a spherically symmetric potential $V(r)$ with total energy $E=mc^2+W$, where $W$ is the binding energy of the relativistic particle. Using the fact that $W\ll mc^2$, the equation for the large component of the Dirac wave function correct up to order $1/c^2$ is given by[16]

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{1}{c^2\hbar^2}\{[E-V(r)]^2-m^2c^4\}\right]F(r)$$
$$=-\frac{1}{2mc^2}\frac{dV}{dr}\left[\frac{d}{dr}-\frac{\kappa}{r}\right]F(r)\ , \tag{31}$$

in which

$$\kappa=\mp(j+\tfrac{1}{2})=\begin{cases}-(l+1) & \text{for } j=+\frac{1}{2}\\ l & \text{for } j=-\frac{1}{2}\ .\end{cases}$$

To convert Eq. (31) to the form of the KG-like equation, one needs to remove the first derivative term. Defining

$$\varphi(r)=F(r)e^{V(r)/4mc^2}\ , \tag{32}$$

one obtains

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}-\frac{1}{4mc^2}\left[V''+\frac{2\kappa V'}{r}\right]+\frac{1}{c^2\hbar^2}\{[E-V(r)]^2-m^2c^4\}\right]\varphi(r)=0\ . \tag{33}$$

It may be pointed out that since $F(r)$ is a bounded function, $\varphi(r)$ will be bounded provided $V(r)\neq\infty$ for $0\leq r\leq\infty$. As a result, Eq. (33) is valid for only a class of potentials satisfying this requirement.

For the $N$-dimensional case, $\kappa$ has to be replaced by

$$\kappa=\tfrac{1}{2}s(N_j-2)\ , \tag{34}$$

where $N_j = N + 2j$ and $s$ stands for the sign of $\kappa$. This form is consistent with the $N=3$ case. Expression (34) transforms Eq. (33) as

$$\left[ -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{8mr^2}(\bar{k}+a-1)(\bar{k}+a-3) + \frac{\hbar^2}{8m^2c^2}\left[ V'' + \frac{s(\bar{k}+a+s-2)V'}{r} \right] - \frac{1}{2mc^2}\{[E-V(r)]^2 - m^2c^4\} \right]\varphi(r) = 0 \,, \tag{35}$$

where

$$\bar{k} = N + 2(j - s/2) - a \,. \tag{36}$$

Equation (35) is analogous to the KG-like Eq. (4) with one difference: that due to inclusion of spin, one gets the additional spin-orbit coupling term[16] (third term) which contributes only to the relativistic part of the binding energy. Equation (35) should then be the starting point for the shifted large-$N$ calculation for a Dirac particle in a spherically symmetric potential.

For the illustration, we present here the results for the Coulomb problem. Following the steps elucidated in Sec. II we calculate all relevant quantities such as $\omega$, $r_0$, $\delta$'s, and $\varepsilon$'s correct up to order $1/c^2$. The detailed expressions are presented in the Appendix. The final analytic expression for the $1/\bar{k}$ expansion of the energy eigenvalue appropriate to the Dirac-Coulomb problem is

$$E = E_{\rm Dirac}^{\rm Coul} = mc^2 - \frac{2m\beta^2}{\hbar^2\bar{k}^2} - \frac{2m\beta^4}{\hbar^4\bar{k}^4c^2}\left[ 1 + \frac{4(1+2n_r-s)}{\bar{k}} + \frac{4(1+2n_r-s)^2}{\bar{k}^2} + \frac{4(1+2n_r-s)^3}{\bar{k}^3} + \cdots \right] + O\left[\frac{1}{c^4}\right] . \tag{37}$$

It is easy to verify that the exact analytic expression for the nonrelativistic binding energy (24) may be restored if one sets $s=+1$. For this choice, (37) reduces to

$$E = mc^2 - \frac{2m\beta^2}{\hbar^2\bar{k}^2} - \frac{2m\beta^4}{\hbar^4\bar{k}^4c^2}\left[ 1 + \frac{4(2n_r)}{\bar{k}} + \frac{4(2n_r)^2}{\bar{k}^2} + \frac{4(2n_r)^3}{\bar{k}^3} + \cdots \right] + O\left[\frac{1}{c^4}\right] , \tag{38}$$

in which

$$\bar{k} = N - 3 + 2(n_r + j + \tfrac{1}{2}) \,. \tag{39}$$

Proceeding in the same manner discussed in Sec. II B, it is found that the terms in the first pair of large parentheses in (38) sum up to the exact analytic result for the relativistic part of the binding energy[16]

$$E_{\rm Dirac}^{\rm Coul}\,[O(1/c^2)] = -\frac{m\beta^4}{8\hbar^4n^4c^2}\left[ \frac{n}{j+\frac{1}{2}} - \frac{3}{4} \right] . \tag{40}$$

However, one may also be interested to test how well the relativistic part of the binding energy is predicted by the first four terms as presented in (38). We list our energy eigenvalues for various quantum states in Table II and compare them with numerical results obtained from (40). Except for the states with higher values of the radial quantum number $n_r$, our results are in excellent agreement with the exact results. As expected, the agreement is best when $n_r = 0$. Thus the shifted $1/N$ expansion scheme works well for the Dirac-Coulomb problem.

## IV. CONCLUDING REMARKS

We have extended in this paper the shifted large-$N$ method originally developed for the Schrödinger equation to obtain the energy levels of a relativistic spin-zero or spin-$\frac{1}{2}$ particle moving in a spherically symmetric po-

TABLE II. Relativistic correction (or order $1/c^2$) to the binding energies of a spin-$\frac{1}{2}$ particle in the Coulomb field.

| States $n(l)_j$ | $\kappa$ | Binding energies in units of $(m\beta^4/\hbar^4c^2)$ Shifted [Eq. (38)] | Exact [Eq. (40)] |
|---|---|---|---|
| $1s_{1/2}$ | 1 | −0.125 000 | −0.125 000 |
| $2p_{3/2}$ | 2 | −0.007 813 | −0.007 813 |
| $2p_{1/2}$ | −1 | −0.035 156 | −0.039 063 |
| $2s_{1/2}$ | 1 | | |
| $3d_{5/2}$ | 3 | −0.001 543 | −0.001 543 |
| $3d_{3/2}$ | −2 | −0.004 515 | −0.004 630 |
| $3p_{3/2}$ | 2 | | |
| $3p_{1/2}$ | −1 | −0.010 231 | −0.013 889 |
| $3s_{1/2}$ | 1 | | |

tential. The main trick lies in the conversion of the KG or the Dirac equation to an effective Schrödinger-like equation. The formalism has been developed without sacrificing the accuracy of nonrelativistic binding energies achieved previously by a suitable choice of the shift parameter. The interesting aspect of our work is that this choice has attributed faster convergence to the relativistic part of the $1/N$ expansion of the energy levels as compared to that obtained in the unshifted large-$N$ expansion scheme.[15] For the Coulomb potential in particular our method looks quite impressive for it yields the exact energy at least to the order $1/c^2$ considered in this paper.

It is realized that there is scope for extension of the present method to more realistic potentials such as the screened Coulomb potentials which have wide applications to atomic phenomena.[17] Recently, Pratt and coworkers[18,19] have studied the relativistic screened Coulomb radial wave functions, normalizations, bound-state energies, and various bound-bound transitions from atomic inner shells using an analytic perturbation theory. However, this approach has a limitation in the sense that the results do not converge for large values of the screening parameter. Such a problem does not arise in our scheme as the large-dimension expansion is basically a nonperturbative approach. Furthermore, highly improved wave functions accurate over a wide range of $r$ and any choice of the quantum numbers $n_r$ and $l$ are available in this framework.[20] All these aspects of screened Coulomb potentials are presently under study and will be reported later.

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## APPENDIX

For interested readers, we list here the analytic expressions of $\tilde{\varepsilon}_j$, $\tilde{\delta}_j$, $\varepsilon_j$, $\delta_j$, etc. appropriate to the Dirac-Coulomb problem:

$$\tilde{\varepsilon}_1=\varepsilon_1\left[1+\frac{\beta^2}{Q\hbar^2c^2}\left[1-\frac{3s}{1+2n_r}\right]\right],$$

$$\tilde{\varepsilon}_2=\varepsilon_2\left[1+\frac{2\beta^2}{Q\hbar^2c^2}\left[1-\frac{2s}{1+2n_r}\right]\right]$$

$$\tilde{\varepsilon}_3=\varepsilon_3\left[1-\frac{\beta^2}{Q\hbar^2c^2}\right],$$

$$\tilde{\varepsilon}_4=\varepsilon_4\ ,$$

$$\tilde{\delta}_1=\delta_1\left[1+\frac{\beta^2}{Q\hbar^2c^2}\left[1-\frac{3s(1+2n_r+s)}{2n_r(1+n_r)}\right]\right],$$

$$\tilde{\delta}_2=\delta_2\left[1+\frac{2\beta^2}{Q\hbar^2c^2}\left[1-\frac{s(1+2n_r+s)}{n_r(1+n_r)}\right]\right],$$

$$\tilde{\delta}_3=\delta_3\left[1+\frac{\beta^2}{Q\hbar^2c^2}\left[3-\frac{5s}{1+2n_r}\right]\right],$$

$$\tilde{\delta}_4=\delta_4\left[1+\frac{2\beta^2}{Q\hbar^2c^2}\left[2-\frac{3s}{1+2n_r}\right]\right].$$

The values of other quantities such as $r_0$, $\omega$, $a$, $\tilde{\delta}_5$, etc. remain the same as given in Sec. II.

[1] E. Witten, Nucl. Phys. B **160**, 57 (1979); in *Recent Developments in Gauge Theories,* edited by X. t' Hooft *et al.* (Plenum, New York, 1980).

[2] L. Mlodinow and N. Papanicolaou, Ann. Phys. (N.Y.) **128**, 314 (1980); **131**, 1 (1981).

[3] S. Kalara, University of Rochester Report No. UR-812, Rochester, NY, 1982; L. G. Yaffe, Phys. Today **36**, 50 (1983).

[4] C. Bender, L. Mlodinow, and N. Papanicolaou, Phys. Rev. A **25**, 1305 (1982).

[5] L. Mlodinow and M. Shatz, J. Math. Phys. **25**, 943 (1984).

[6] G. Moreno and A. Zepeda, J. Phys. B **17**, 21 (1984).

[7] A. Chatterjee, J. Phys. A **18**, 735 (1985); **18**, 1193 (1985); **18**, 2403 (1985).

[8] U. Sukhatme and T. Imbo, Phys. Rev. D **28**, 418 (1983).

[9] T. Imbo, A. Pagnamenta, and U. Sukhatme, Phys. Rev. D **29**, 1669 (1984); Phys. Lett. **105A**, 183 (1984).

[10] R. Dutt, U. Mukherji, and Y. P. Varshni, J. Phys. B **18**, 3311 (1985); **19**, 3411 (1986).

[11] R. Dutt and Y. P. Varshni, Z. Phys. D **2**, 207 (1986).

[12] A. Pagnamenta and U. P. Sukhatme, Phys. Rev. D **34**, 3528 (1986); E. Papp, Phys. Rev. A **36**, 3550 (1987); S. A. Maluendes, F. M. Fernandez, and E. A. Castro, Phys. Lett. A **124**, 215 (1987).

[13] M. M. Nieto, Am. J. Phys. **47**, 1067 (1979).

[14] J. L. Miramontes and C. Pajares, Nuovo Cimento B **64**, 10 (1984).

[15] A. Chatterjee, J. Math. Phys. **27**, 2331 (1986).

[16] A. S. Davydov, *Quantum Mechanics* (Pergamon, New York, 1976), pp. 230, 274–276.

[17] R. K. Roychoudhury and Y. P. Varshni, J. Phys. A **20**, L1083 (1987).

[18] J. Mc Ennan, D. J. Botto, and R. H. Pratt, Phys. Rev. A **16**, 1768 (1977).

[19] O. V. Gabriel, S. Chaudhuri, and R. H. Pratt, Phys. Rev. A **24**, 3088 (1981).

[20] T. Imbo and U. Sukhatme, Phys. Rev. D **31**, 2655 (1985).